Homework 3

Renzo's math 570

Third homework set. Have fun!

1 Product Spaces, Connectedness, Compactness

Remember that the product topology is induced on the cartesian product of topological spaces to be the coarsest topology that makes the projections continuous.

Exercise 1. *Consider the cartesian product of infinitely many topological spaces:*

$$X = \prod_{i \in \mathbb{N}} X_i$$

Consider the following two topologies on X:

- τ_1=$\{$*topology generated by box sets, i.e. an open set is* $U = \prod U_i$*, with each* U_i *open in* X_i $\}$.
- τ_2=$\{$*topology generated by sets of the form* $U = \prod_{i=1}^{n} U_i \times \prod_{i>n} X_i\}$.

1. *Are these two topologies different? Is one finer than the other?*
2. *Is either of these topologies the product topology? Explain why.*

Exercise 2. *Prove that $X \times Y$ is Hausdorff if and only if both X and Y are.*

Exercise 3. *Prove that the projection $\pi_X : X \times Y \to X$ is an open function (i.e. if U is open, then $\pi_X(U)$ is open). Give an example to show that the projections in general are not closed functions. However show that if X is Hausdorff and $X \times Y$ compact, then the projection to X is a closed function.*

Exercise 4. *What sets are connected topological spaces when they are given the discrete topology? And the stupid topology?*

What sets are compact topological spaces when they are given the discrete topology? And the stupid topology?

Exercise 5. *Prove that $X \times Y$ is path connected if and only if both X and Y are.*

Exercise 6. *Prove that a product space is compact if and only if it is "compact for box sets covers"(i.e. if any open cover by box sets can be refined to a finite subcover).*

Exercise 7. *Prove that the projective plane is a compact topological space.*

Exercise 8. *Prove the easy direction of Heine Borel. If X is a compact subset in $\mathbb{R}^n$, then X is closed and bounded.*