Project

Renzo's math 571

1 De Rham and singular cohomology

Throughout this worksheet our coefficient ring is $\mathbb{R}$. The goal is to understand that, for any manifold M

$$H^i_{dR}(M) \cong H^i(M)$$

Let us briefly recall the construction of de Rham cohomology. Given a manifold M, $\Omega^i(M)$ is the group of differential i-forms on M. The differential map $d : \Omega^i(M) \to \Omega^{i+1}(M)$ fits all these groups together into a complex, and the de Rham cohomology of M is defined to be the cohomology of this complex.

Note: the fact that the de Rham functor is obtained as the composition of a geometric functor associating a cochain complex to M with the commutative algebra (co)-homology functor, makes de Rham cohomology automatically a cohomology theory in the sense of axiomatic cohomology.

In order to show that it "coincides" with singular cohomology, we must exhibit a natural transformation T that induces an isomorphism for the cohomologies of the point.

Step 1. *Use the Universal coefficient theorem to identify $H^i(M)$ with $Hom(H_i(M), \mathbb{R})$.*

Note that here we are using essentially the fact that we have real coefficients!

Step 2. *Construct the natural transformation T using integration. Define it first at the level of cycles and cocycles, then show it is well defined at the level of cohomology.*

Recall from your days of analysis:

Stokes' Theorem:

$$\int_\alpha d\omega = \int_{\partial\alpha} \omega$$

Change of variables formula:

$$\int_\alpha f^*\omega = \int_{f(\alpha)} \omega$$

Step 3. *Check that T induces an isomorphism between the cohomologies of a point.*

2 Compactly Supported Cohomology

This is an (almost) cohomology theory that coincides with ordinary cohomology when our manifold M is compact, but it is somewhat better behaved when M is a non compact space. We introduce it is several different equivalent ways:

2.1 De Rham CS Cohomology

I think that the most natural way to make compactly supported cohomology arise is from de Rham theory. Define $\Omega^i_{cs}(M)$ to be the subgroup of differential i-forms with compact support. The differential sends a compactly supported differential form to a compactly supported differential form, thus giving rise to the **compactly supported de Rham complex**. Compactly supported cohomology is now simply the cohomology of this complex.

Exercise 1. *Compute* $H^i_{dR,cs}(\mathbb{R})$.

2.2 CW CS Cohomology

If M can be given the structure of a CW complex (possibly with infinitely many cells...without compactness it's hard to assume otherwise... but we do assume local compactness, i.e. that any point has a neighborhood that meets only finitely many cells), define $C^i_{CW}(M)$ to be the subgroup of cochains that are compactly supported in the sense that they take non-zero values only on finitely many cells. The coboundary of a compactly supported cochain is compactly supprted and therefore we have a complex, whose cohomology is our compactly supported cohomology.

Exercise 2. *Compute* $H^i_{CW,cs}(\mathbb{R})$. *Spend at least 30 minutes on this problem, before looking for help on Hatcher...*

2.3 Singular CS cohomology

This feels kind of weird at first, but if you think about it for a second it is just the translation to the singular world of the two setups mentioned above.

A compactly supported singular cochain is a map $\varphi : C_i(M) \to \mathbb{R}$ such that there is a compact set K_φ with the property that for any $\sigma \in C_i(M)$ with $Im(\sigma) \subseteq M - K$, $\varphi(\sigma) = 0$.

The following characterization will come in handy for our purposes.

Theorem 1. *Given a nested sequence K_i of compact sets whose union is all of M,*

$$H^i_{cs}(M) = \lim_{\rightarrow} H^i(M, M \setminus K_i).$$

Exercise 3. *Show that* $H^i_{cs}(\mathbb{R}^n) \cong H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \{pt\})$*. Consider a nested sequence of compact sets whose union is all of* $\mathbb{R}^n$ *and apply the characterization of compactly supported cohomology.*

Note that this result tells us why compactly supported cohomology is not quite a cohomology theory. We have just shown that it is not homotopy invariant. In general, the problem is that it is not really a well defined functor if we allow morphism between spaces to be arbitrary continuous functions, as the pullback becomes undefined (the pullback of a compactly supported class might not be compactly supported, as we know that the inverse image of a compact set needs not be compact). Things become good again if we restrict our morphisms to just proper morphisms!

3 Poincare' duality

Well, we are really going to skip all of the hard work here. We focus on just one basic aspect.

Exercise 4. *Prove that Poincare' duality holds for* $\mathbb{R}^n$*.*

Given the exercise, one would then be able to show Poicare' duality (for compact manifolds) by showing the following inductive procedure, that you are not asked to prove, but just to meditate on a little.

Fact 1. *Let* $X = U \cup V$ *with* $U, V, U \cap V$ *connected manifolds that can be obtained by gluing together finitely many open discs. If Poincare' duality holds for* $U, V, U \cap V$*, then it holds for* X*.*