The Universal Coefficient Theorem

Renzo's math 571

The Universal Coefficient Theorem relates homology and cohomology. It describes the k-th cohomology group with coefficients in a(n abelian) group G in terms of the, k-th, $(k-1)$-th homology groups and of the group G. The precise formulation is:

$$H^k(X,G) = Hom(H_k(X),G) \oplus Ext^1(H_{k-1}(X),G)$$

When $G = \mathbb{Z}$, then this takes the simple form:

$$H^k(X,\mathbb{Z}) = Free(H_k(X)) \oplus Torsion(H_{k-1}(X))$$

This worksheet leads you to prove these facts. We have to start from some algebra background.

1 Split Sequences

Prove that the following definitions are equivalent.

Definition 1. *A short exact sequence (SES) of abelian groups*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is **split** *if and only if one of the equivalent conditions below holds:*

1. *$B \cong A \oplus C$ in such a way that the natural diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & A \oplus C & \xrightarrow{\pi} & C & \to & 0 \\ & & \downarrow Id. & & \downarrow & & \downarrow Id. & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \end{array}$$

2. *f has a left inverse: there is a homomorphism $f' : B \to A$ such that $f'f$ is the identity on A.*

3. *g has a right inverse: there is $g' : C \to B$ such that gg' is the identity on C.*

Problem 1. *Give an example of a short exact sequence that is NOT split.*

Problem 2. *Show that if C is a free abelian group, than any SES is split.*

2 Hom functor

Let $\mathfrak{Ab}$ be the category of abelian groups (complexes of abelian groups, R-modules, etc...). Then for a fixed abelian group G the functor:

$$Hom(-,G) : \mathfrak{Ab} \to \mathfrak{Ab}$$

is a contravariant functor. If this is not completely clear to you, make sure to spend five minutes pondering on what exactly this means!

Problem 3. *Hom(-, G) is a* **left exact** *functor. This means that given a SES:*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

then the following dualized sequence is exact:

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G)$$

Note that I did not forget to finish the sequence. The last map needs not be surjective!

Problem 4. *Find an example of a SES where the dualized sequence has last map not surjective.*

Problem 5. *If C is free abelian, then show that*

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to 0$$

is in fact exact.

3 Ext functors

Let C be an abelian group. A **free resolution** $F_\bullet$ of C is an exact sequence:

$$\ldots \to F_2 \to F_1 \to F_0 \to C \to 0,$$

where all the F_i's are free abelian groups.

Definition 2. *The group $Ext^i(C,G)$ is defined as follows:*

1. *Take a free resolution of C.*
2. *Dualize it:*

$$0 \to Hom(C,G) \to Hom(F_0,G) \to Hom(F_1,G) \to \ldots$$

3. *Drop the first term:*

$$0 \to Hom(F_0,G) \to Hom(F_1,G) \to \ldots$$

4. *$Ext^i(C,G)$ is now defined to be the i-th cohomology group of this complex.*

Ok, before this definition even makes sense, we need to check that the end result is independent of the choice of free resolution. This is achieved through the following lemmas. Do not prove them here in class, but try to do it in the quiet of your room one of these nights!

Lemma 1. *Let $A \xrightarrow{\varphi} B$ be a homomorphism of abelian groups, and $F_\bullet^A, F_\bullet^B$ free resolutions of A and B. Then*

1. *φ can be extended to a morphism of complexes between the two free resolutions $F_\bullet^A \xrightarrow{\tilde{\varphi}} F_\bullet^B$.*
2. *any two such extensions of φ are chain homotopic.*

Lemma 2. *The functor Hom(-,G), applied to the category of complexes of abelian groups, preserves chain homotopies. I.e. if $f \sim g : C_\bullet \to D_\bullet$, then $f^* \sim g^* : Hom(D_\bullet, G) \to Hom(C_\bullet, G)$.*

Problem 6. *Use Lemma* **??** *and Lemma* **??** *to prove that $Ext^i(C,G)$ is well defined (i.e. it is independent of the choice of free resolution).*

Problem 7. *Show that $Ext^0(C,G) = Hom(C,G)$.*

Every abelian group C has a two term free resolution (why?):

$$0 \to F_1 \to F_0 \to C \to 0$$

This means in particular that for abelian groups, the only possibly non-zero Ext^i are when $i = 0, 1$. The reason the definition was framed in a more general context is that the Ext functors can be applied to other objects (sheaves, complexes of sheaves, etc), in which case higher Ext can appear.

Problem 8. *Ok, now let us get ourselves familiar with this Ext^1 thingies. In the case of finitely generated abelian groups, you can do anything you want using the following three facts, that you should prove!*

1. *$Ext^1(C \oplus D, G) \cong Ext^1(C,G) \oplus Ext^1(D,G)$.*
2. *$Ext^1(C,G) = 0$ if C is free (abelian).*
3. *$Ext^1(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.*

Here are two interesting facts about Ext^1 groups that are interesting. The proofs, as fas as I can tell, are not particularly enlightening, so we won't worry about them.

Facts 1 (Extensions)**.** *A short exact sequence of the form:*

$$0 \to G \to E \to C \to 0$$

is called an extension of C by G. Two extensions are isomorphic if we have three vertical isomorphisms making all squares commute. Then $Ext^1(C,G)$ can be naturally identified with the set of isomorphism classes of extensions of C by G.

The main ingredient required to prove this correspondence is the notion of pushout of two maps: given a two term free resolution of C and a map $f : F_1 \to G$, one can fill in the diagram:

$$\begin{array}{ccccccccc} 0 & \to & G & \to & E & \to & C & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F_1 & \to & F_0 & \to & C & \to & 0 \end{array},$$

where the rightmost vertical arrow is just the identity, and E is defined to be an appropriate quotient of $G \oplus F_0$.

Facts 2 (Long Exact Sequence)**.** *Given a short exact sequence:*

$$0 \to A \to B \to C \to 0,$$

we get a long exact sequence

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to Ext^1(C,G) \to$$

$$\to Ext^1(B,G) \to Ext^1(A,G) \to Ext^2(C,G) \to \dots$$

Of course, in the case of abelian groups the sequence is only 6 terms long (why?):

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to Ext^1(C,G) \to Ext^1(B,G) \to Ext^1(A,G) \to 0$$

4 The Universal Coefficient Theorem (finally!)

Throughout this section denote $C_\bullet$ your favorite chain complex associated to a space X, and $C^\bullet$ the dual cochain complex. We also denote $Z_k \subseteq C_k$ (resp. $Z^k \subseteq C^k$) the subgroup of cycles, i.e. $Ker\partial_k$ (resp. cocycles, $ker\delta^k$) and $B_k \subseteq C_k$ (resp. $B^k \subseteq C^k$) the subgroup of boundaries, i.e. $Im\partial_{k+1}$ (resp. $Im\delta^{k-1}$).

Problem 9. *Define a natural map $h : H^k(C^\bullet; G) \to Hom(H_k(C_\bullet), G)$.*

To achieve this goal you have to carefully unravel the definitions.

Problem 10. *Show that h is a surjective homomoprhism.*

To do so, consider the split (why?) exact sequence:

$$0 \to Z_k \to C_k \to B_{k-1} \to 0$$

By the exercise on split sequences there is a projection map $C_k \to Z_k$ that realizes the splitting. Use this fact to construct a preimage via h of an element $\phi \in Hom(H_k(C_\bullet), G)$. Note that in the process of showing surjectivity of h you have actually constructed a splitting for the exact sequence:

$$0 \to Ker h \to H^k(C^\bullet; G) \to Hom(H_k(C_\bullet), G) \to 0.$$

Consider the SES of complexes,

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0,$$

where the Z_k's and the B_k's are put into complexes by making all the differentials 0. Note that if you apply the Hom functor to the above SES, you still get a SES (why?):

$$0 \to B^{\bullet-1} \to C^\bullet \to Z^\bullet \to 0,$$

Consider the long exact sequence in cohomology. First of all, what are the cohomology groups associated to the complexes $Z^\bullet$ and $B^{\bullet-1}$?

Problem 11. *Show (by chasing through the snake lemma definition) that the connecting homomorphism $Z^k \to B^k$ is just the dual map i_k^* to the inclusion*

$$i_k : B_k \to Z_k$$

Problem 12. *Break the long exact sequence in cohomology above into SES, and notice they have the form:*

$$0 \to Coker\ i_{k-1}^* \to H^k(C^\bullet, G) \to Ker\ i_k^* \to 0.$$

Observe that $Ker\ i_k^* = Hom(H_k(C_\bullet), G)$ and the second to last map on the right is just h.

We are therefore left to understand $Coker\ i_{k-1}^*$.

Problem 13. *Use the short exact sequence*

$$0 \to B_{k-1} \to Z_{k-1} \to H_{k-1}(C_\bullet) \to 0$$

and Fact **??** *to deduce*

$$Coker\ i_{k-1}^* \cong Ext^1(H_{k-1}(C_\bullet), G).$$