Project

Renzo's math 571

The goal of this project is to familiarize ourselves with cohomology by exploring in depth the example of projective spaces. We compute the cohomology of $\mathbb{RP}^2$ in several different ways, explore the ring structure, and finally generalize the results to higher dimensional prohective spaces and to complex projective spaces.

Problem 1. *Compute the cohomology* **groups** $H^i(\mathbb{RP}^2, \mathbb{Z})$ *in three different ways:*

1. *Giving the projective plane a simplicial structure and using the definitions.*
2. *Using the CW structure and cellular cohomology.*
3. *With the Universal Coefficient Theorem.*

Problem 2. *Do exactly the same but with coefficient ring $\mathbb{Z}/2\mathbb{Z}$.*

Now we want to explore the ring structure. To avoid orientation issues, we work with coefficients $\mathbb{Z}/2\mathbb{Z}$.

Problem 3. *Compute the cup product*

$$\cup : H^1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) \times H^1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$$

in two different ways:

1. *Using the simplicial structure developed above and the definitions.*
2. *Cheating. I.e. assuming that you can use Poincare' duality.*

Use this result to describe the ring structure of $H^*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$. In particular exibit it as the quotient of some polynomial ring.

Problem 4. *Generalize what you have done so far to describe $H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\mathbb{CP}^n, \mathbb{Z})$. You don't need to do everything two or three ways. Just find the most effective.*

Finally, we compute the cup product on the projective plane by neither cheating, nor using the definitions. But this will involve a long sequence of yoga moves with all the technology we have developed so far. Let us start by observing the master commutative diagram. Here we again assume coefficients to be $\mathbb{Z}/2\mathbb{Z}$, and we omit them from the notation so that things fit in a page (we also omit the $\mathbb{R}$ from the projective plane notation, but we are still just working with the real projective plane). We give $\mathbb{RP}^2$ homogeneous coordinates $(x_0 : x_1 : x_2)$ and denote U_{x_i} the open affine chart $\{x_i \neq 0\}$

$$\begin{array}{ccccc}
H^1(\mathbb{P}^2) & \times & H^1(\mathbb{P}^2) & \xrightarrow{\cup} & H^2(\mathbb{P}^2) \\
& \uparrow c & & & \uparrow b \\
H^1(\mathbb{P}^2, U_{x_1}) & \times & H^1(\mathbb{P}^2, U_{x_2}) & \xrightarrow{\cup} & H^2(\mathbb{P}^2, \mathbb{P}^2 - \{(1:0:0)\}) \\
& \downarrow d & & & \downarrow a \\
H^1(\mathbb{R}^2, \mathbb{R}^2 - \{x_1 = 0\}) & \times & H^1(\mathbb{R}^2, \mathbb{R}^2 - \{x_2 = 0\}) & \xrightarrow{\cup} & H^2(\mathbb{R}^2, \mathbb{R}^2 - \{(0,0)\})
\end{array}$$

Step 1. *The very first step is to understand the above diagram. Where do all those maps come from?*

Once we have all of this down, the strategy is the following:

1. *prove that all vertical maps are isomorphisms.*

2. *prove that the bottom cup product sends two generators to a generator.*

Step 2. *Prove that a is an isomorphism.*

Step 3. *Prove that b is an isomorphism. To do so, recall that cohomology (and cohomology of a pair as well) is invariant under homotopy equivalence (of pairs)...what does $\mathbb{P}^2$ minus a point deformation retract to? Then use cellular cohomology.*

Too see that c and d are isomorphisms we get ourselves a new diagram:

$$\begin{array}{ccccccc}
H^1(\mathbb{P}^2) & \overset{t_1}{\leftarrow} & H^1(\mathbb{P}^2, pt.) & \overset{t_2}{\leftarrow} & H^1(\mathbb{P}^2, U_{x_1}) & \overset{t_3}{\rightarrow} & H^1(\mathbb{R}^2, \mathbb{R}) \\
\downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 & & \downarrow v_4 \\
H^1(\mathbb{P}^1) & \overset{b_1}{\leftarrow} & H^1(\mathbb{P}^1, pt.) & \overset{b_2}{\leftarrow} & H^1(\mathbb{P}^1, U_{x_1}) & \overset{b_3}{\rightarrow} & H^1(\mathbb{R}, \mathbb{R} - 0)
\end{array}$$

Step 4. *Understand where all the maps in the diagram come from, and why, if we show that all these maps are isomorphisms, we are done with the proof.*

Step 5. *Prove that t_2, b_2 and v_3 are isomorphisms.*

Step 6. *Prove that* b_3 *is an isomorphism.*

Step 7. *Prove that* t_1, b_1 *are isomorphisms.*

Step 8. *Prove that* v_1 *is an isomorphism*

Step 9. *Show that you are done!*