Midterm

Renzo's math 571

1 Exterior Products

Let V be a (finite dimensional) real vector space of dimension n, with basis $e_1, \ldots, e_n$.

Definition 1. *The k-th exterior product $\Lambda^k(V)$ is a quotient of the k-th tensor product $V^{\otimes k}$ by the subspace spanned by elements of the form $v_1 \otimes v_2 \otimes \ldots \otimes v_k$ where any two of the vectors are the same.*

The equivalence class of an element of the form $v_1 \otimes v_2 \otimes \ldots \otimes v_k$ is denoted:

$$[v_1 \otimes v_2 \otimes \ldots \otimes v_k] = v_1 \wedge v_2 \wedge \ldots \wedge v_k$$

Facts 1.

1. $\Lambda^0(V) = \mathbb{R}$.
2. $\Lambda^1(V) = V$.
3. $v_1 \wedge v_2 = -v_2 \wedge v_1 \in \Lambda^2(V)$.
4. *For any k, permuting the k vectors in $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ has the effect of changing the element by the sign of the permutation.*
5. $\Lambda^k(V)$ *is a vector space of dimension $\binom{n}{k}$ with a basis given by :*

$$\{e_{i_1} \wedge \ldots e_{i_k} | i_1 < \ldots < i_k\}.$$

Now we of course take a more sophisticated point of view.

Definition 2. *The above construction determines a (covariant) functor*

$$\Lambda^k : \mathfrak{Vect} \to \mathfrak{Vect}$$

(where $\mathfrak{Vect}$ is the category of (finite dimensional) vector spaces, with functions being linear functions). We have described how this functor acts on objects. You should convince yourself that there is a natural way that Λ^k acts on linear functions, and that all axioms of a functor are satisfied.

Finally, here comes the problem for you.

Problem 1. *Given a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, prove that $\Lambda^3(A) : \mathbb{R} \to \mathbb{R}$ is multiplication by $det(A)$.*

Of course this fact is not only true in dimension 3. In fact you might want to warm up to the proof by showing it for dimension 2.

2 Homology and Cohomology

From the point of view of algebra, homology and cohomology are essentially the same functor. The only difference consists in how we index the groups in our chain or cochain complexes. What distinguishes the two theories geometrically is that a dualization occurs in the construction of the cochain complex and makes cohomology a contravariant functor.

However, it should come as no big surprise that any property of homology that depends exclusively on commutative algebra (i.e. where we started the proof with having a complex of groups, and it was of no importance how that complex was cooked up), should carry through for cohomology.

Problem 2. *Choose ONE of the following properties, and prove it. You are not allowed to use the Universal coefficient theorem!*

2.1 Long exact Sequence for a Pair

For $A \subseteq X$, define relative cohomology as follows. Define relative cochains groups $C^k(X, A; G) = Hom(C_k(X, A), G)$. You can think of relative cochains as functions that vanish on chains supported in A. Show that there is an exact sequence:

$$0 \to C^k(X, A; G) \to C^k(X; G) \to C^k(A; G) \to 0$$

Understand relative coboundary maps, and finally use the short exact sequence to deduce a long exact sequence in cohomology. You don't need to re-prove the snake lemma!

2.2 Homotopy Invariance

If $f \sim g : X \to Y$, then $f^* = g^* : H^k(Y; G) \to H^k(X; G)$. Deduce that homotopic spaces have isomorphic cohomology.

2.3 Equivalence of simplicial and singular cohomology

If X is a CW complex define the cellular cochain complex by dualizing the cellular chain complex. Define cellular cohomology and show it is isomorphic to singular cohomology.

2.4 Mayer-Vietoris

For $A, B \subseteq X$ such that the union of the interiors of A and B cover X, there is a long exact sequence:

$$\ldots \to H^k(X; G) \to H^k(A; G) \oplus H^k(B, G) \to H^k(A \cap B; G) \to H^{k+1}(X; G) \to \ldots$$