Making New Spaces from Old Ones - Part 1

Renzo's math 570

In this set of problems we want to illustrate, with some hopefully interesting and useful examples, the strategy discussed in class about how to induce a topology on a set X, given the datum of a topological space (Y, τ_Y) and a set function either from Y to X or from X to Y. The resulting topology on X is called the **induced topology** (induced by the pair (Y, τ_Y)).

1 The philosophy

1.1 $f : X \to (Y, \tau_Y)$

In this case we define the induced topology to be the coarsest topology on X that makes f continuous. Equivalently, the induced topology is composed of all preimages of open sets of Y.

1.2 $X \leftarrow (Y, \tau_Y) : f$

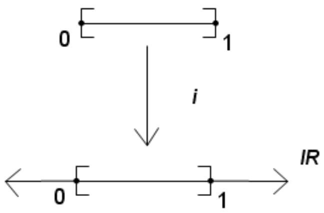
In this case the induced topology is the finest topology on X that makes f continuous. Or the induced topology is generated by all subsets of Y whose preimage is open in X.

2 The subspace topology

If X is a subset of (Y, τ_Y), we can use the inclusion function $i : X \to Y$ to induce a topology on X. In this case open sets of Y are obtained by intersecting open sets in X with Y.

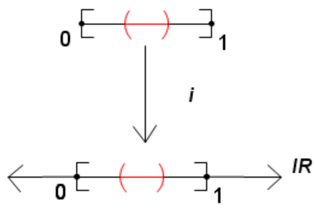
Problem 1. *Consider the closed interval* $[0, 1] \subset (\mathbb{R}, \tau_{eucl})$*. Describe the open sets in the subspace topology.*

Solution: For U to be an opem set in X, there must exist an open set V in $\mathbb{R}$ s.t. $i^{-1}(V) = U$.

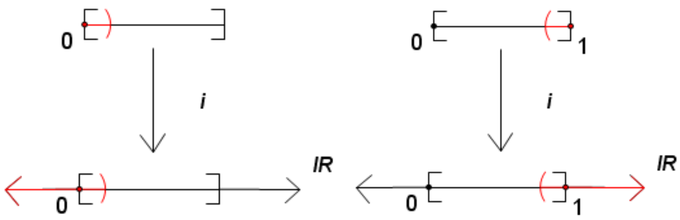


The open sets in X.

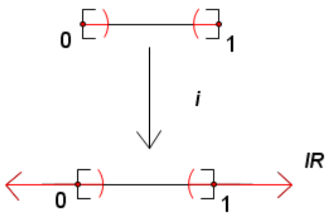
- Interior open sets



- Contain an endpoint with some "fuzz"



- Contains both endpoints with "fuzz"



3 The product topology

Let (X, τ) and (Y, σ) be two topological spaces. We define a new topological space $X \times Y$, called the **product (space) of X and Y**. We define a natural topology on $X \times Y$, starting from τ and σ, called the **product topology**.

$X \times Y$ **as a set:** the points of $X \times Y$ correspond to ordered pairs (x, y), where $x \in X$ and $y \in Y$,

Recall that you have two natural maps from a product space called the **projections**:

$$\begin{array}{rccc} \pi_X : & X \times Y & \longrightarrow & X \\ & (x, y) & \mapsto & x \end{array}$$

$$\begin{array}{rccc} \pi_Y : & X \times Y & \longrightarrow & Y \\ & (x, y) & \mapsto & y \end{array}$$

According to our philosophy, the **product topology** is the coarsest topology that makes both projection maps continuous.

Problem 2. *Show that the following definition of the product topology is equivalent.*

A basis for the product topology: *a* ***basic open set*** $B \in \beta$ *is a set of the form* $U \times V$*, with* $U \in \tau$ *and* $V \in \sigma$*).* β *is a base for the* ***product topology****.*

($\Rightarrow$)

If the product topology, γ, is the coarsest topology that makes both π_X and π_Y continuous, then we have fore every U $\in \tau$ and for every V $\in \sigma$,

$$(\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)) \in \gamma$$
$$\Rightarrow (U \times Y \cap C \times V) \in \gamma$$
$$\Rightarrow U \times B \in \gamma$$

This means elements like $U \times V$ are in the product topology which means the unions and intersections of such elements are in the product topology. So basic open sets in the form $U \times V$, form the set of basic open sets, β which form a basis for the product topology, γ.

($\Leftarrow$)

If $U \in \tau$ and $V \in \sigma$,

$$\pi_X^{-1}(U) = U \times Y$$
$$\pi_Y^{-1}(V) = X \times V$$

$U \times Y \in \gamma$, $X \times V \in \gamma$ since $U \times V$, $X \times Y \in \gamma$. So both projection maps are continuous.

Now to show that γ is indeed the coarsest topology that makes both projection maps continuous.

Say there exists a coarser topology such that both projection maps are continous, then some open set in γ will not be in the new, coarser topology which means that there is a basic open set that is in γ but not in this new topology. This would mean that the projections are not continuous.

Problem 3. *Familiarize yourself with the product topology. Draw some pictures, make some examples, understand what the basic open sets look like, and find some examples of open sets in the product topology that are not basic opens.*

Problem 4. *Show that the cylinder is a product space. Show that the product topology coincides with the induced topology from the euclidean topology in* $\mathbb{R}^3$.

Theorem 1. *Let* (X, τ_x) *be a topological space, where* $X = (0, h)$. *Let* (Y, τ_d) *be a topological space, where* Y *is the open disc of radius* r. *As a set, the cylinder* $C = X \times Y = \{(x, y) : x \in X$ *and* $(y \in Y\}$. *Let* $\pi_x(x, y) = x$ *and* $\pi_y(x, y) = y$ *be continuous. Then*

(1)the cylinder is a product space, and
(2) the product topology is equivalent to the induced Euclidean topology.

Proof. (1) Consider arbitrary open sets $U \in X$ and $V \in Y$. Because π_x^{-1} and π_y^{-1} are continuous, we know that $\pi_X^{-1}(U) \bigcap \pi_Y^{-1}(V)$ is open in C. The basic open sets in C are thus of the form $U_\alpha \times V_\alpha = \pi^{-1}(U) \bigcap \pi^{-1}(V)$ with $U \in X$ and $V \in Y$. Then the candidate for the set of all open sets of the cylinder is

$$\tau_{X \times Y} = \left\{ \bigcup_{\alpha}^{\infty} \{U_\alpha \times V_\alpha\} \right\}.$$

We see that the empty set $\emptyset_C \in \tau_{X\times Y}$ by setting $U_1 = \emptyset_X$ and $V_1 = \emptyset_Y$. Similarly, we see that the cylinder is in $\tau_{X\times Y}$ by taking $U_1 = X$ and $V_1 = Y$.

The infinite union of open basic sets is in $\tau_{X\times Y}$ by its definition.

We must show that the finite intersection $U_1 \times V_1 \cap \cdots \cap U_n \times V_n$ is in $\tau_{X\times Y}$ for $\{U_1 \cdots U_n\} \in X$ and $\{V_1 \cdots V_n\} \in Y$. Consider $W = U_1 \times V_1 \cap U_2 \times V_2$ where $\{U_1, U_2\} \in X$ and $\{V_1, V_2\} \in Y$. If the intersection is empty, then the two sets are in $\tau_{X\times Y}$ separately. If the intersection is not empty, we can rewrite W as $W = U_w \times V_w = \pi_X(W) \times \pi_Y(W) \subseteq \tau_{X\times Y}$.

(2) A basic open set in τ_E, the induced Euclidean topology on the cylinder C, can be represented as a small cylinder $X' \times Y'$ where X' is a small interval (a, b) and Y' is an open disk of some radius r. Then by step (1), the basis of τ_E can be composed of basic open sets of $\tau_{X\times Y}$. So any open set in τ_E can be so represented.

A basic open set of $\tau_{X\times Y}$ can be realized as the union of perhaps infinitely many small boxes as in calculus. These boxes form an alternate basis for τ_E. So any open set in $\tau_{X\times Y}$ can be formed from the basic open sets from τ_E.

Thus the topologies τ_E and $\tau_{X\times Y}$ are equivalent.

$\square$

4 The quotient topology

If you can define an equivalence relation $\sim$ on the points of a topological space Y, then you get a surjective map to the quotient set:

$$\pi : Y \to Y/\sim$$

The **quotient topology** is the finest topology on the quotient set that makes π continuous. We now discuss a few typical ways to encounter quotient spaces.

4.1 Identification spaces

We can define an equivalence relation on some points of Y just by "declaring" some subsets of points to be equivalent. This has the effect of identifying such points.

Problem 5. *Carefully show that the identification space* $[0,1]/\{0 \sim 1\}$ *is homeomorphic to the circle with euclidean topology.* ***Note:*** *the notation above means the following: on the closed segment* $[0,1]$ *define an equivalence relation where reflexivity and symmetry are implicitely assumed, where the point* 0 *is equivalent to* 1 *and no other relation is imposed.*

Solution: Let $Y = [0,1]$ and S^1 be the unit circle. Note that every point on the unit circle is associated to an angle, θ, measured counter-clockwise from the poxitive x-axis. Define an equivalence relation $\sim$ such that 0 and 1 are identified together i.e. $\sim = \{0 \sim 1\}$. Consider the map

$$f : Y/\sim \to S^1$$
$$f(x) = 2\pi x$$

In order to show that f is indeed a homeomorphism, we must show that f is continuous, one-to-one, onto, and that f^{-1} is also continuous. Clearly, f is continuous as it is a linear function. To show f is one-to-one, assume there exists θ_1, θ_2 in S^1 and x_1, x_2 in $Y/\sim$. Suppose

$$\begin{aligned} \theta_1 &= \theta_2 \\ \implies 2\pi x_1 &= 2\pi x_2 \\ \implies x_1 &= x_2 \end{aligned} \qquad (1)$$

proving that f is injective. Note that the only potential problem of injectivity might be at 0 or 1, but since they are identified, we have alleviated this problem. To show f is onto, assume θ_1 is in S^1. Then

$$\begin{aligned} \theta_1 &= 2\pi x_1 \\ \Longrightarrow \frac{\theta_1}{2\pi} &= x_1 \end{aligned} \tag{2}$$

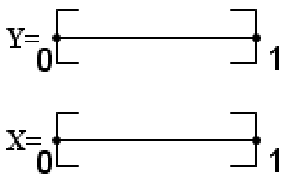
Thus, for every $\theta \in S^1$ we have a corresponding $x \in Y/\sim$ which proves that f is onto. Now, $f^{-1}(\theta) = \frac{\theta}{2\pi}$ which is clearly continuous since it is a linear function as well. Thus, f is a homeomorphism, so

$$[0,1]/\{0 \sim 1\} \cong S^1$$

4.2 Attaching spaces along a map

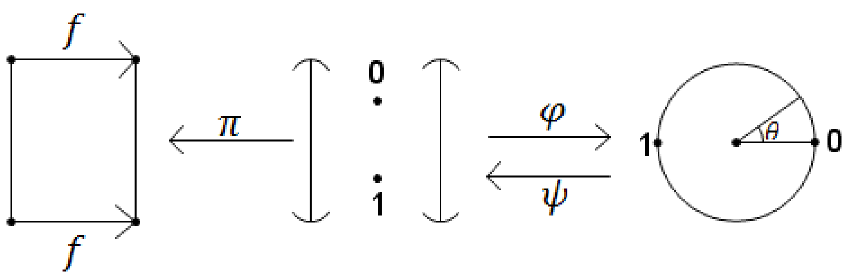
Let Y and Z be topological spaces, $W \subset Y$ a subset of Y and suppose you have a continuous function $f : W \to Z$. Then you can generate an equivalence relation on the disjoint union $Y \coprod Z$ by declaring equivalent all points of the form $(w, f(w))$. This has the effect of gluing together points of Y with their corresponding images in Z.

Problem 6. *Let Y and Z be two copies of the unit closed interval, $W = \{0,1\}$ and $f : \{0,1\} \to Z$ be the natural inclusion of the two points into the unit interval ($f(0) = 0$ and $f(1) = 1$). Show that again the space obtained by attaching the two intervals along f is homeomorphic to a circle.*



and $W = \{0,1\}$

Define $f : \{0,1\} \to X$ by $f(0_Y) = 0_X$ and $f(1_Y) = 1_X$.



Define φ by:

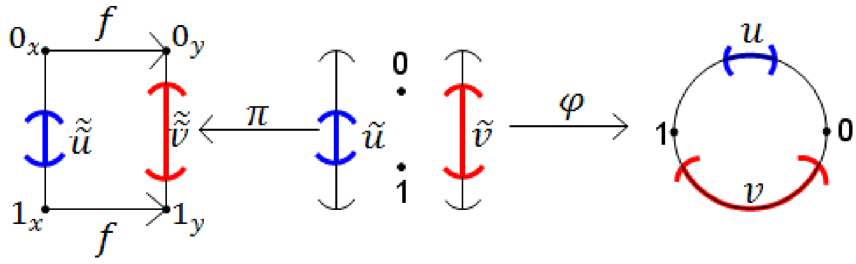
$x \to \theta = \pi x$
$y \to \theta = -\pi y$
$0 \to \theta = 0$

$1 \to \theta = 1$
Note that this is clearly a bijection.
Then define $\psi = \varphi^{-1}$

First prove that φ is continuous by showing that any open set on the circle maps to an open set on the two open intervals with points 0 and 1 added. There are three types of open sets on the circle:

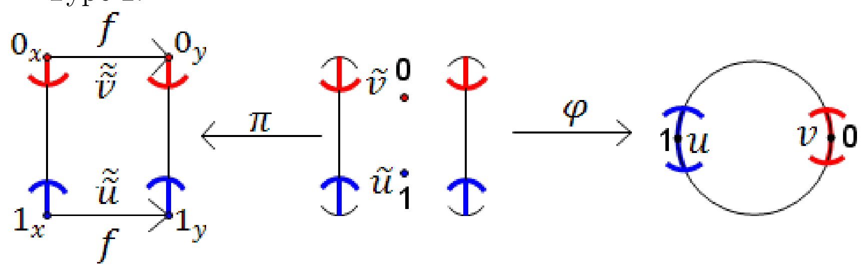
1. open sets that do not contain 0 or 1
2. open sets that contain 0 or 1
3. open set that contain 0 and 1

Type 1:



Let u be the open set on the circle shown above. Then $\varphi^{-1}(u) = \tilde{u}$. $\tilde{u}$ is open since $\pi(\tilde{u}) = \tilde{\tilde{u}}$ and $\tilde{\tilde{u}}$ is open. Similarly v is open.

Type 2:



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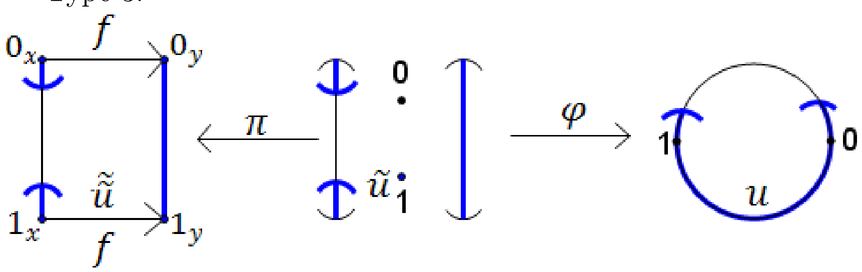
Type 3:



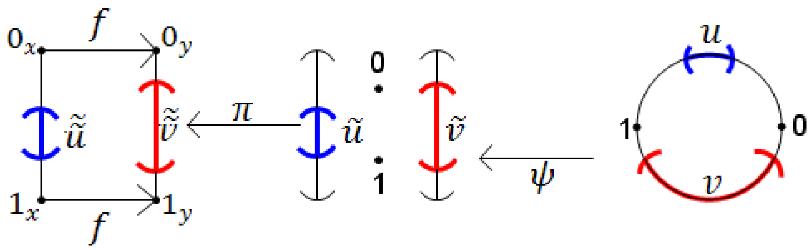
Let u be the open set on the circle shown above. Then $\varphi^{-1}(u) = \tilde{u}$. $\tilde{u}$ is open since $\pi(\tilde{u}) = \tilde{\tilde{u}}$ and $\tilde{\tilde{u}}$ is open.

Therefore φ is continuous.

Next prove that ψ is continuous by showing that any open set on the two open intervals with points 0 and 1 added maps to an open set on the circle. Again there the same three types of open sets on the two open intervals with points 0 and 1 added:

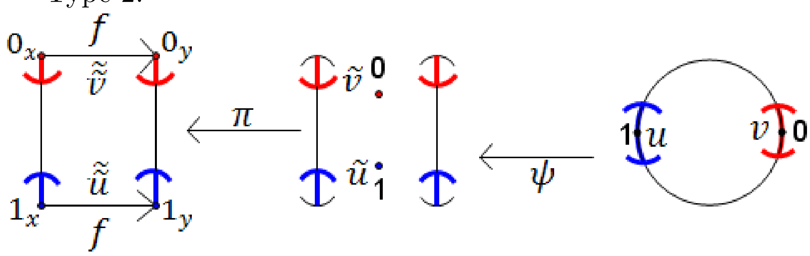
1. open sets that do not contain 0 or 1
2. open sets that contain 0 or 1
3. open set that contain 0 and 1

Type 1:



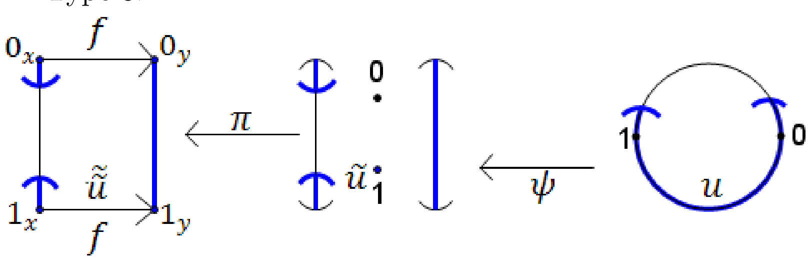
Let $\tilde{u}$ be the open set shown above. $\tilde{u}$ is open since $\tilde{\tilde{u}}$ is open and $\pi(\tilde{u}) = \tilde{\tilde{u}}$. Then $\psi^{-1}(\tilde{u})$ maps to the open set u. Similarly $\psi^{-1}(\tilde{v})$ maps to the open set v.

Type 2:

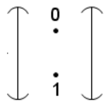


Let $\tilde{u}$ be the open set shown above. $\tilde{u}$ is open since $\tilde{\tilde{u}}$ is open and $\pi(\tilde{u}) = \tilde{\tilde{u}}$. Then $\psi^{-1}(\tilde{u})$ maps to the open set u. Similarly $\psi^{-1}(\tilde{v})$ maps to the open set v.

Type 3:



Let $\tilde{u}$ be the open set shown above. $\tilde{u}$ is open since $\tilde{\tilde{u}}$ is open and $\pi(\tilde{u}) = \tilde{\tilde{u}}$. Then $\psi^{-1}(\tilde{u})$ maps to the open set u. Therefore ψ is continuous.

So φ and ψ are continuous bijective functions. Thus  is homeomorphic to the circle.

Note: If your are wondering why we had to prove both ψ is continuous when φ was a continuous bijection, Consider the following example. Let X be nonemply and 1_a, 1_b be identity functions.

$$(X, \tau_{discrete}) \underset{1_b}{\overset{1_a}{\rightleftarrows}} (X, \tau_{stupid})$$

It is easy to see that 1_b is not continuous.

4.3 Orbit spaces

We start here with a simple scenario. Let G be a finite or a countable group (with multiplication read from right to left), and consider it a topological space by giving it the discrete topology (we will call such a creature a **discrete group**). Y is a topological space. A **group action** is a continuous function:

$$\varphi : G \times Y \to Y$$

such that:

1. for any point $y \in Y$, the identity element acts trivially on y: $\varphi(1, y) := 1 \cdot y = y$.

2. for g_1, g_2 any pair of elements of G:

$$g_2 \cdot (g_1 \cdot y) = (g_2 g_1) \cdot y$$

Then you can declare an equivalence relation on Y by declaring $y_1 \sim y_2$ whenever there is a group element that takes one into the other (i.e. there exists $g \in G$ such that $g \cdot y_1 = y_2$. Equivalence classes are called **orbits**, and the topological space obtained by inducing a topology on the quotient set is called the **orbit space**.

Problem 7. *Let $G = \mathbb{Z}$ act on the real line by the action*

$$n \cdot x = x + n$$

Check carefully that this is a kosher group action, and verify that the orbit space is (yet again) homeomorphic to the circle.

Proof: We will first prove that the function $\phi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ defined as $\phi((n, x)) = x + n$ is a group action and then we will show the orbit space which this group action defines is homeomorphic to the unit circle by demonstrating a homeomorphism between the two spaces.

To verify that ϕ is a group action we must show:

1. $\forall x \in \mathbb{R}, \phi(e, x) = x$ **where** e **is the identity element of the group.**

The group $(\mathbb{Z}, +)$ where $+$ is simply addition, has $e = 0$ as an identity element. Thus for any real number $x, \phi(0, x) = x + 0 = x$.

2. $\forall z_1, z_2 \in \mathbb{Z}$ **and** $x \in \mathbb{R}, \phi(z_2, \phi(z_1, x)) = \phi(z_1 z_2, x)$**.**

$\phi(z_2, \phi(z_1, x)) = \phi(z_1, x) + z_2 = (z_1 + x) + z_2 = (z_1 + z_2) + x = \phi(z_1 z_2, x)$.

3. ϕ **is continuous.**

Let $\hat{U}$ be an open set in $\mathbb{R}$. Then $\hat{U} = \bigcup_n (a_n, b_n)$ ($\hat{U}$ is the union of non-degenerate open intervals).

We aim to show that ϕ is continuous by demonstrating that

$$\hat{V} = \phi^{-1}(\hat{U}) = \{(z, x) \in \mathbb{Z} \times \mathbb{R} : \phi(z, x) = x + z \in \hat{U}\}$$

is open in the product topology. (We save our detailed description of this set for later).

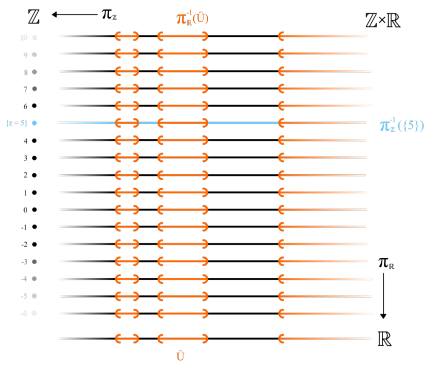
Observe that for any integer z the set $\{z\} \times \hat{U}$ is open in $\mathbb{Z} \times \mathbb{R}$. To see this recall that the topology on $\mathbb{Z} \times \mathbb{R}$ is the coarsest topology which makes both projection functions, $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{Z}}$, continuous. Consider the open set $\{z\} \subset \mathbb{Z}$ (this is open since $\mathbb{Z}$ is equipped with the discrete topology).

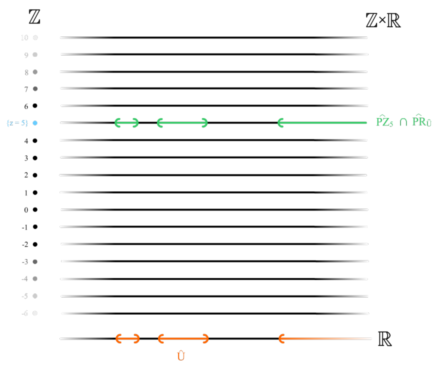
$$\hat{PZ}_z = \pi_{\mathbb{Z}}^{-1}(\{z\}) = \{(z, x) \in \mathbb{Z} \times \mathbb{R} : x \in \mathbb{R}\}$$

Now consider the open set $\hat{U}$ already identifed and take its preimage under the projection function.

$$\hat{PR}_{\hat{U}} = \pi_{\mathbb{R}}^{-1}(\hat{U}) = \{(z, u) \in \mathbb{Z} \times \mathbb{R} : z \in \mathbb{Z} \text{ and } u \in \hat{U}\}$$

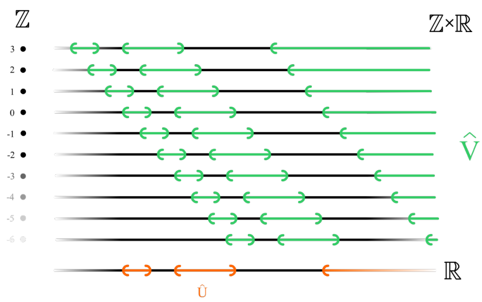
If we think of the product topology as countably infinitely many copies of the real line at various integer heights, z, then $\hat{PZ}_z$ is a copy of the real line at height z. $\hat{PR}_{\hat{U}}$ is countably infinitely many copies of the set $\hat{U}$, one at each height $z \in \mathbb{Z}$. Both of these sets must be open in the product topology and therefore there intersection is an open set in the product topology. For example $P\hat{Z}_5 \cap P\hat{R}_{\hat{U}}$ is one copy of $\hat{U}$ at height $z = 5$ as depicted in the figure below.





Now return to the set $\phi^{-1}(\hat{U})$. Observe that it is infinitely many copies of the set $\hat{U} = \bigcup_n (a_n, b_n)$, one at each height $z \in \mathbb{Z}$, each shifted by exactly z. Explicitly:

$$\hat{V} = \bigcup_{z \in \mathbb{Z}} \{z\} \times \hat{U}_z, \text{ where } \hat{U}_z = \bigcup_n (a_n - z, b_n - z)$$



The care in discussing the openness of $P\hat{Z}_z \cap P\hat{R}_{\hat{U}}$ was to show by construction that $\hat{V}$ is actually the countably infinite union of open sets in the product topology, namely those sets $P\hat{Z}_z \cap P\hat{R}_{\hat{U}_z}$ unioned over all integers z and thus is open in the product topology which means ϕ is continuous and therefore it is a well defined group action.

We now construct a new topological space, one of equivalence classes, by creating an equivalence relation using this group action.

For all $x, y \in \mathbb{R}$ we say $x \sim y$ (read x is equivalent to y) if and only if there exists an integer $z \in \mathbb{Z}$ such that $\phi(z,x) = y$. Said another way: $y - x = z$. This relation says two real numbers are equivalent if they differ by an integer.

Using this relation we construct the space of equivalence classes $\mathbb{R}/\mathbb{Z} = \mathbb{R}/\sim$, and we denote a point in this space $[x]$, knowing this to mean the set of all points equivalent to x.

Our first observation is that this space is homeomorphic to $[0,1]/\{1 \sim 0\} = (0,1) \cup \{[0]\}$ as described in problem 5. We can explicitly define a continuous and open bijection between $\mathbb{R}/\mathbb{Z}$ and $[0,1]/\{1 \sim 0\}$. First denote the equivalence class of x in $[0,1]/\{1 \sim 0\}$ by $\langle x \rangle$.

Let $f : [0,1]/\{1 \sim 0\} \to \mathbb{R}/\mathbb{Z}$ be the following function: $f(\langle x \rangle) = [x]$.

Consider $\langle x \rangle, \langle y \rangle \in [0,1]/\{1 \sim 0\}$, the domain of f. Assume that $f(\langle x \rangle) = f(\langle y \rangle)$. Then $[x] = [y]$ and so it certainly must be true that $x \sim y$ or in terms of ϕ, there exists an integer z such that $\phi(z,x) = y \iff y - x = z$. Assume first w.l.o.g. that $x = 0$. Then $y = 1$ or $y = 0$ since they differ by an integer and $x, y \in [0,1]$. But then $\langle x \rangle = \langle y \rangle$. Now assume $0 < x, y < 1$, then it is clear that $-1 < y - x < 1$. The only integer strictly between -1 and 1 is 0 and therefore $y - x = 0 \implies y = x \implies \langle y \rangle = \langle x \rangle$. Therefore f is injective.

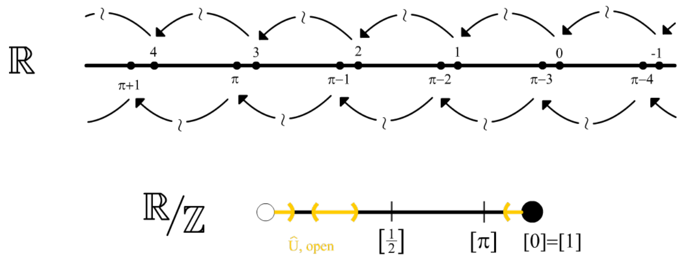
Now consider any point $[x] \in \mathbb{R}/\mathbb{Z}$. Let $y = x - INT(x)$, where $INT : \mathbb{R} \to \mathbb{Z}$ is the integer function which takes a real number to its integer portion. If $x = INT(x)$ then x is an integer and $[0] = [x]$ and so $y = 0 \implies f(\langle y \rangle) = [x]$. If $x \neq INT(x)$ then $0 < y < 1$ and $f(\langle y \rangle) = [x]$. Therefore f is surjective. We do this step of defining y because $[x] = [x+z], \forall z \in \mathbb{Z}$. We could have simply stated that x is the smallest non-negative member of the equivalence class defined by ϕ.

So the inverse exists and, incidentally, you can write it down.

$$f^{-1}([x]) = \langle x - INT(x) \rangle$$

We now show that f is open and continuous. Let $\hat{U} \subset \mathbb{R}/\mathbb{Z}$ be open in the quotient topology. A set of equivalence classes is open in $\mathbb{R}/\mathbb{Z}$ if and only if their union is open in $\mathbb{R}$. So $\hat{U} = \bigcup_n ([a_n],[b_n]) \cup \bigcup_m [0,[c_m]) \cup ([d_m],1)$, the union of open intervals of equivalence classes and sets of the form $[[0],[c_m]) \cup ([d_m],[1])$, illustrated below.

The necessity for sets of the form $[[0],[c_m]) \cup ([d_m],[1])$ can be illustrated with an example: Consider $\hat{W} = \bigcup_{z \in \mathbb{Z}} (-\frac{1}{4} + z, \frac{1}{4} + z)$. Then $\hat{W}$



is open in $\mathbb{R}$, but it is also the union of points of the equivalence classes $\hat{Y} = [[0], [\frac{1}{4}]) \cup ([\frac{1}{4}], [1])$ and therefore $\hat{Y}$ must be open in the quotient topology. Next observe that:

$$f^{-1}(\hat{U}) = \bigcup_n (\langle a_n \rangle, \langle b_n \rangle) \cup \bigcup_m [\langle 0 \rangle, \langle c_m \rangle) \cup (\langle d_m \rangle, \langle 1 \rangle).$$

which is exactly the description of open sets in $[0, 1]/\{1 \sim 0\}$.

A nearly identical argument shows that f is open by taking any open subset of $[0, 1]/\{1 \sim 0\}$ and showing that its image under f is an open set of equivalence classes. The crucial observation being that any open set in $[0, 1]/\{1 \sim 0\}$ which contains $\langle 0 \rangle$ must also contain an interval of equivalence classes of the form $(\langle a \rangle, \langle 1 \rangle)$ where $a < 1$ and $(\langle 0 \rangle, \langle b \rangle)$ where $b > 0$ which will be also open in $\mathbb{R}/\mathbb{Z}$ as we have already discussed.

So f is a homeomorphism and $\mathbb{R}/\mathbb{Z} \cong [0, 1]/\{1 \sim 0\}$. By the result of problem 5 we conclude that $\mathbb{R}/\mathbb{Z} \cong S^1$.

We can prove the equivalence directly using another homeomorphism.

$$\text{Consider } g : \mathbb{R} \to S^1, g(x) = e^{i2\pi x}$$

Whenever x and y differ by an integer we know that $g(x) = g(y)$. But this is the definition of our equivalence relation and so the map

$$h : \mathbb{R}/\mathbb{Z} \to S^1 \text{ defined by } h([x]) = e^{i2\pi x}$$

is well defined. It turns out that it is also a homeomorphism.

It is injective because if we assume that $h([x]) = h([y])$ then by definition $e^{i2\pi x} = e^{i2\pi y}$ which implies that x and y differ by an integer and therefore are equivalent, $[x] = [y]$.

It is surjective because the map $[x] \mapsto x$ is surjective and given any point on the circle $y \in S^1$ there exists some angle $0 \leq x < 1$ such that $y = e^{i2\pi x}$.

Now observe that the projection map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ which is defined as $\pi(x) = [x]$ is continuous (in fact we define the quotient topology to be the finest topology which makes this map continuous) and g is continuous. Since $h \circ \pi = g$ it follows that h must be continuous.

Finally let $\hat{U}$ be an open subset of $\mathbb{R}/\mathbb{Z}$ and observe that g is an open mapping. By the definition of the quotient topology $\pi^{-1}(\hat{U}) \subset \mathbb{R}$ is open. But then $h(\hat{U}) = g(\pi^{-1}(\hat{U}))$ is open since g is an open mapping. Therefore h is a homeomorphism. $\diamond$

5 Problems for everyone

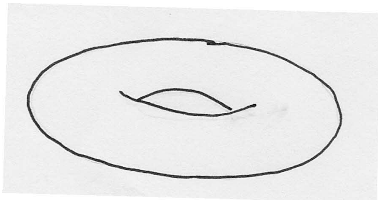
5.1 The torus

A **torus** is (informally) the topological space corresponding to the surface of a bagel, with topology induced by the euclidean topology.

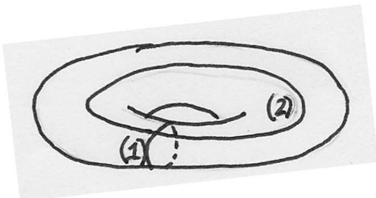
Problem 8. *Realize the torus as a product space.*

Proof. The torus can be realized as a circle cross a circle. That is $T = S^1 \times S^1$. Visually we think about it as in the pictures below.

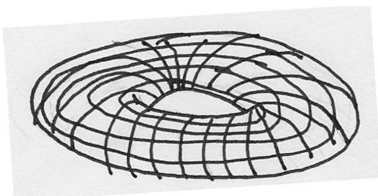
First consider the torus.



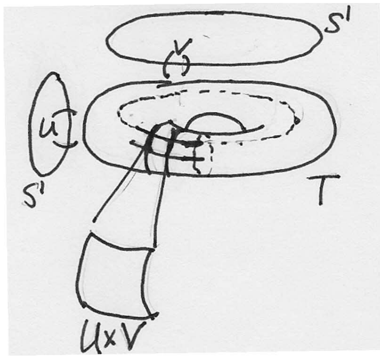
Now consider circles on the torus. There are two types in particular to consider (1) those that go around the torus through the genus, and (2) those that go around the genus of the torus.



From these two types of circles it is clear $T = S^1 \times S^1$.



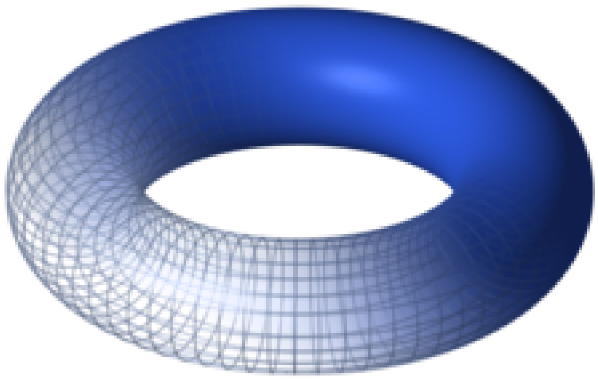
As the previous sketch suggests the open sets on the torus are small 'rounded' squares.



□

Problem 9. *Realize the torus as an identification space starting from a square.*

The torus is the surface of revolution that one gets as a result of revolving a circle around an axis that is co-planar with and not touching the circle (Wikipedia!). What do we mean?

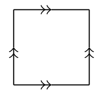


Ta da! Easy, right? Well, we want to be thinking of what happens if we were to be an ant that is walking on the surface of this doughnut. What sort of world do we live in? If we keep traveling in one direction, what sort of paths can we make that come back to our starting points? Obviously, we can 'see' how such a surface works, and how living on it would be (assuming we are constrained only to that surface – it's our entire universe!) because we're in a higher dimension.

We can see that the surface of the torus can be thought of as $S_1 x\ S_1$. Why is that? Any two angles will be able to tell us specifically where we are located at on our surface!

Proof : Mental Exercise (or someone else's problem ...).

Now, we want to show that a square can be thought of as the torus where we glue some points together, namely opposite edges with the arrows matching when we glue. Like so ...



Simple stuff, right? So, let's say $S = \{(x, y) : 0 \leq x, y \leq 1\}$. Now, let $\mathbb{T} = S_1 \ x \ S_1$. Let the equivalance relationship $\sim$ be defined as the points such that $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$. We claim that $S \ / \sim \cong \mathbb{T}$.

Proof : It will suffice to show that we can find homeomorphic functions f, g such that $f : S \ / \sim \mapsto \mathbb{T}$ and $g : \mathbb{T} \mapsto S \ / \sim$ with $f \circ g = id_{S/\sim}$ and $g \circ f = id_{\mathbb{T}}$.

So, S_1 is the unit circle, so we can identify a point on the unit circle by a single angle θ, with $0 \leq \theta < 2\pi$, such that our point in $\mathbb{C}$ is $e^{i\theta}$. Now, let $f(x, y) = (x/2\pi, y/2\pi)$ and $g(\theta, \phi) = (2\theta\pi, 2\phi\pi)$.

Let's make sure that these functions are well-defined.

Let $(x_1, y_1) = (x_2, y_2)$. Then, $f(x_1, y_1) = (2x_1\pi, 2y_1\pi)$ and $f(x_2, y_2) = (2x_2\pi, 2y_2\pi) = f(x_1, x_2)$.

So, f is well-defined. Let's make sure g is, too! Take $(\theta_1, \phi_1) = (\theta_2, \phi_2)$. Now, $g(\theta_1, \phi_1) = (\theta_1/2\pi, \phi_1/2\pi)$ and $g(\theta_2, \phi_2) = (\theta_2/2\pi, \phi_2/2\pi) = (\theta_1/2\pi, \phi_1/2\pi) = g(\theta_1, \phi_1)$. g is also well-defined.

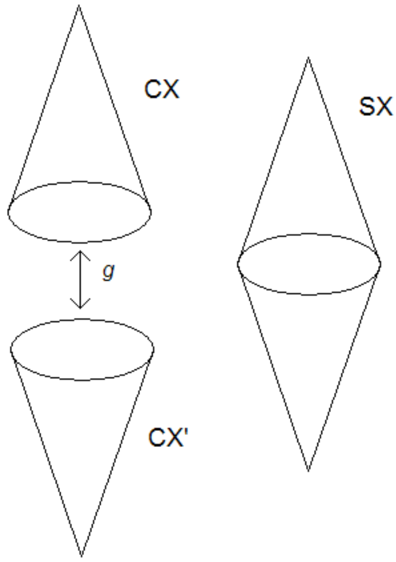
It's easy to see that these are inverses of one another and that the functions are continuous, so we have a homemeomorphism between the $S \ / \sim$ and $\mathbb{T}$.

5.2 Suspensions

The **suspension** SX of X is the identification space obtained from $X \times [-1, 1]$ by the rule:

- all points of the form $(x, -1)$ are identified;
- all points of the form $(x, +1)$ are identified;
- all other points are left alone.

Problem 10. *Show that the suspension SX can be obtained by gluing two cones over X along a function.*



Let X' be disjoint from X, and let $f : X \longrightarrow X'$ be a homeomorphism. Let $g : X \times \{0\} \longrightarrow X' \times \{0\}$ be given by $g(x, 0) = (f(x), 0)$.

We want to show that $(CX \cup CX')/g$ is homeomorphic to SX. To do this, define a map on $(X \times [0, 1]) \cup (X' \times [0, 1])$ by

$$(x, y) \mapsto \begin{cases} (x, y) & \text{if } x \in X \\ (f^{-1}(x), -y) & \text{if } x \in X' \end{cases}$$

Now, if x and x' are either both in X or both in X', the images of $(x, 1)$ and $(x', 1)$ under the above map have second component either both 1 or both -1, and therefore represent equivalent points in SX. Also, if we consider $x \in X$, then $(x, 0)$ and $(f(x), 0)$ have the same image. Therefore the above function induces a well-defined map from $(CX \cup CX')/g$ to SX. Call this map ϕ. We will show that ϕ is a homeomorphism.

To show that ϕ is one-to-one, consider two distinct points in the domain of ϕ with representatives (x, y) and (x', y'). If $y \neq y'$, then plainly the images of the points have distinct second components. If $x \neq x'$, then we break into cases. If x and x' are either both in X or both in X', then clearly $\phi(x, y)$ and $\phi(x', y')$ have distinct first component. However, if one of x and x' is in X and the other is in X', then the second components of $\phi(x, y)$ and $\phi(x', y')$ are different unless $y = y' = 0$. If then the latter case holds, x and x' cannot be equivalent by f since $(x, 0)$ and $(x', 0)$ are distinct points of the domain of ϕ. But then again the first components of $\phi(x, y)$ and $\phi(x', y')$ are distinct. We have covered all cases and thus have shown that ϕ is one-to-one.

To show that ϕ is onto, note that the point of SX represented by $(x, y) \in X \times [-1, 1]$ has preimage represented by (x, y) if $y \geq 0$ and has preimage represented by $(f(x), -y)$ if $y \leq 0$.

A basis for $[-1,1]$ consists of sets of the form $(a,b) \cap [-1,1]$. Call this basis $\mathcal{B}$. Then a basis for SX is the collection of sets whose preimages in $X \times [-1,1]$ are of the form $U \times V$ with U an open subset of X and $V \in \mathcal{B}$. To show that ϕ is continuous, it suffices to consider the above basis for SX. By definition, a set in $(CX \cup CX')/g$ is open iff its preimage in $CX \cup CX'$ is open. This preimage in turn is open iff its preimage in $(X \times [0,1]) \cup (X' \times [0,1])$ is open. Now, then, consider an open set in SX represented by $U \times ((a,b) \cap [-1,1])$. This set's preimage in $(X \times [0,1]) \cup (X' \times [0,1])$ is

$$(U \times ((a,b) \cap [0,1])) \cup (f(U) \times ((-b,-a) \cap [0,1])).$$

We see that this is a union of a basic open set in $X \times [0,1]$ and a basic open set in $X' \times [0,1]$. Thus ϕ is continuous.

Last, to show that ϕ^{-1} is continuous (that is, that ϕ is an open map), consider first basic open sets in $(CX \cup CX')/g$ that do not contain points with second component 0. Some of these sets are represented by $U \times ((a,b) \cap (0,1])$ with $U \subseteq X$ open. Then the image of the set is represented by $U \times ((a,b) \cap (0,1])$, which is open. Others have representations in which $U \subseteq X'$. Then the image of the set is represented by $f^{-1}(U) \times ((-b,-a) \cap [-1,0))$, which is open. It remains to consider basic open sets represented by $(U \times [0,a)) \cup (f(U) \times [0,b))$ with $U \subseteq X$. The image of this set is represented by $U \times (-b,a)$, which is open. Therefore ϕ is an open map. Therefore ϕ is a homeomorphism.

Problem 11. *What is the suspension of a circle homeomorphic to?*

We start by taking the product of the cirlce and an interval $[-1,1]$. This product space is a cylinder.

If we identify the points $(x,-1)$ and $(x,1)$ then this closes the ends and the space becomes homeomorphic to a sphere. We can create a bijection from the closed cylinder to the sphere via radial projection. Any open set around the pole in the sphere will pull back to an open set in the point of the suspension.

5.3 Balls and Spheres

Recall the definition of balls and spheres in euclidean space:

- the n-dimensional (closed) ball $\overline{B}^n \subset \mathbb{R}^n$ is the set of $x \in \mathbb{R}^n$ with $||x|| \leq 1$.

- the $(n-1)$-dimensional sphere $S^{n-1} \subset \mathbb{R}^n$ is the set of $x \in \mathbb{R}^n$ with $||x|| = 1$.

Problem 12. *Show that the sphere S^n can be obtained as (i.e. is homeomorphic to) an identification space from the ball $\overline{B}^n$, provided $n > 0$.*

Construct an identification space on B^n by identifying the entire boundary to be a single point and leaving all interior points alone.

Let S denote S^n with one point deleted. Let B denote the identification space with the boundary point deleted (so B is now an open ball). Then S is homeomorphic to B because both are homeomorphic to $\mathbb{R}^n$; S can be mapped to $\mathbb{R}^n$ via stereographic projection and B can be mapped to $\mathbb{R}^n$ via stretching with *arctan* in each direction.

If we put the one point back into B and the one point back into S and extend the map, we get a homeomorphism through the one point compactification of $\mathbb{R}^n$.

Problem 13. *Show that the ball $\overline{B}^n$ can be obtained as (i.e. is homeomorphic to) the cone CS^{n-1}. Look at project 1 for the definition of a cone.*

<u>Solution:</u> Define $S_r^{n-1} := \{x \in \mathbb{R}^n : ||x|| = r\}$ then we can consider $\overline{B}^n$ as

$$\overline{B}^n = \bigcup_{0<r\leq 1} S_r^{n-1} \cup \{0\}$$

where $\{0\}$ is the set consisting of the point at the origin in $\mathbb{R}^n$. The cone CS^{n-1} can be written as

$$CS^{n-1} = \bigcup_{0\leq h<1} S^{n-1} \times h \cup \{1\}$$

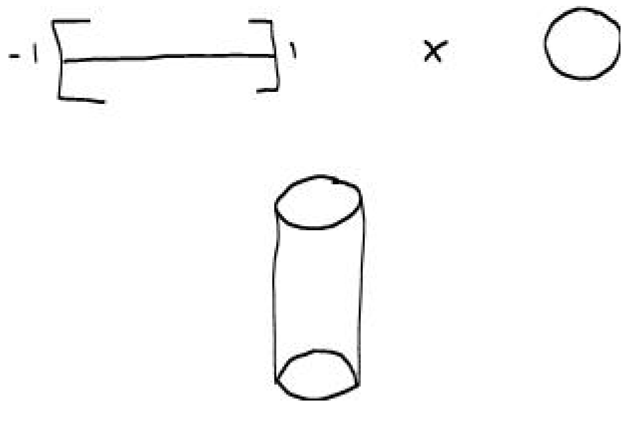


Figure 1: Product Space

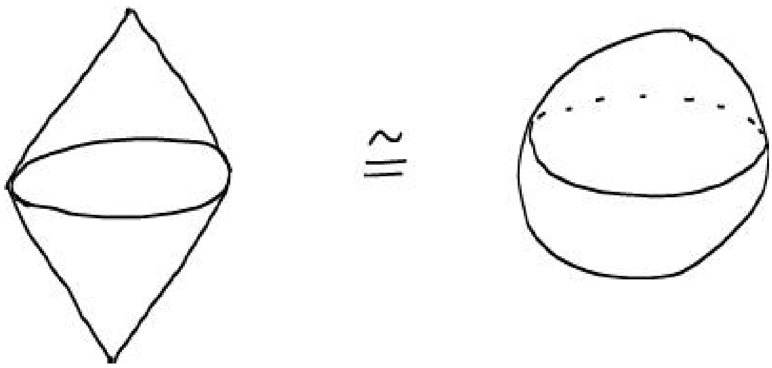


Figure 2: Identification space

where $\{1\}$ is the set covering the identification of all points of the form $(x, 1)$ together. To prove these are homeomorphic we will create continuous, bijective functions between them. Consider $f : \overline{B}^n \to CS^{n-1}$ defined by sending the $n-1$ dimensional sphere at radius r to the sphere at "height" $1-r$ in our binary pair of our cone by expanding the sphere to a radius 1 and mapping the origin to the point at "height" 1.

$$f : \overline{B}^n \to CS^{n-1}; \quad S_r^{n-1} \mapsto S^{n-1} \times (1-r),\ \{0\} \mapsto \{1\}.$$

The inverse map, g, is similarly defined by sending the sphere at "height" h to the sphere of radius $1-h$ by shrinking the sphere and mapping the point at the "top" of the cone to the point at the origin of our closed ball.

$$g : CS^{n-1} \to \overline{B}^n; \quad S^{n-1} \times h \mapsto S_{1-h}^{n-1},\ \{1\} \mapsto \{0\}.$$

(Clearly $fg = 1_{CS^{n-1}}$ and $gf = 1_{\overline{B}^n}$ *Clearly has it's usual meaning that I don't want to prove this.*) To prove continuity we consider an open set in $V \subseteq \overline{B}^n$. We see

$$V = V \cap \big(\bigcup_{0<r\leq 1} S_r^{n-1} \cup \{0\}\big) = \bigcup_{0<r\leq 1} V \cap S_r^{n-1} \cup V \cap \{0\}.$$

The intersection of V with each S_r^{n-1} is either a union of open sets in that sphere or the entire sphere and mapping under $g^{-1} = f$ to the sphere $S^{n-1} \times (1-r)$ maintains both of these properties so $f(V \cap S_r^{n-1})$ is open and the union of open sets is open. Therefore g is continuous. A nearly identical argument shoes that f is also continuous. For example consider n=2. The closed ball is then a solid disc and the cone is a tower of circles with a point at the top. Open sets on the disc are unions of open segments of the circles at radius r, and possibly the point at the origin, and each open segment maps to an open segment of the circles at the appropriate height and possibly the point at the top. This is a union of open sets so it is also open.

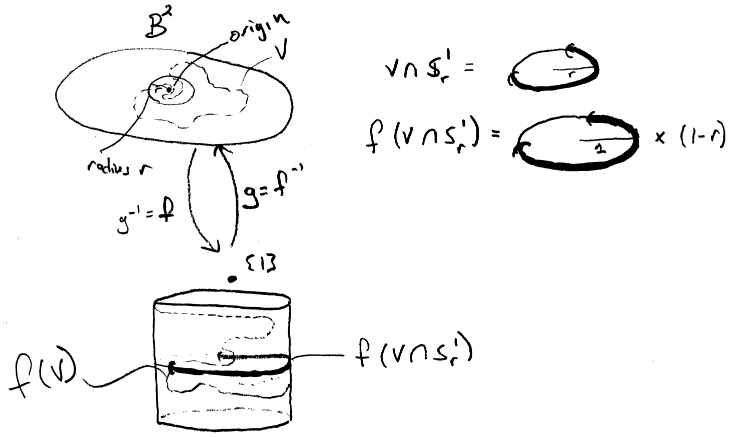
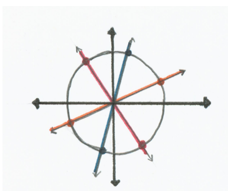


Figure 3: Example of homeomorphism of $\overline{B}^2$ to CS^1

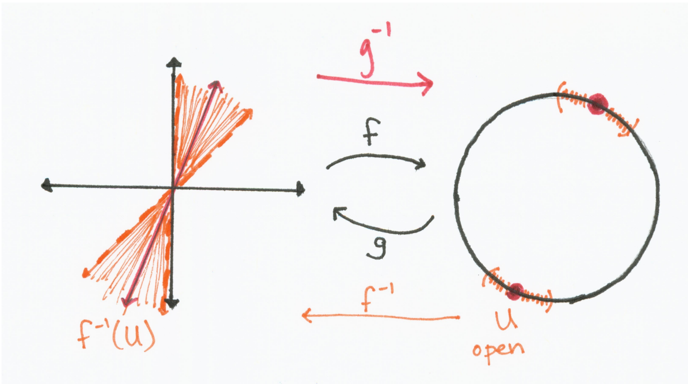
5.4 The Projective Line

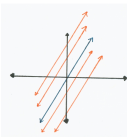
The projective line is a space parameterizing all lines through the origin in $\mathbb{R}^2$. This means that there is a natural bijection between the set of points of the projective plane and the set of lines through the origin in three dimensional euclidean space.

Problem 14. *Define a natural bijection between the projective line and the quotient space of a circle by the action of a finite group. Show that this gives a homeomorphism between the projective line and a circle.*

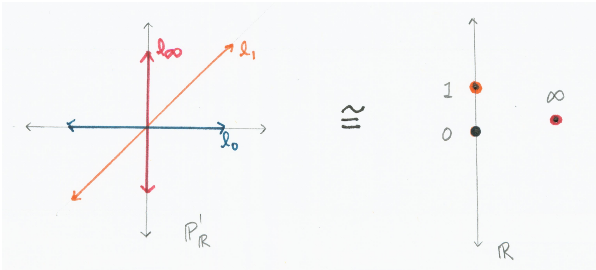
Solution: Let's start with the bijection, and let us think of it in the most natural way. Think of the real projective line ($\mathbb{P}^1_{\mathbb{R}}$) as the set of lines in $\mathbb{R}^2$ that pass through the origin. Notice that each line through the origin will intersect S^1 in two points.  Define a group action on S^1 via $\mathbb{Z}/2\mathbb{Z}$ (thought of as the multiplicative group on the elements $\{-1, 1\}$) as multiplication by the group element. Then this group action creates an equivalence class containing the points (x, y) and $(-x, -y)$, which are the antipodal points that a line through the origin passes through. Then the bijection between $\mathbb{P}^1_{\mathbb{R}}$ and $S^1/(\mathbb{Z}/2\mathbb{Z})$ is to identify a line through the origin with the equivalence class of points that the line passes through.

This bijection leads to a nice homeomorphism between these two spaces. It should be reasonably intuitive that the map $f : \mathbb{P}^1_{\mathbb{R}} \longrightarrow S^1/(\mathbb{Z}/2\mathbb{Z})$ is continuous: Take an open set of points on $S^1/(\mathbb{Z}/2\mathbb{Z})$. The inverse image of these points is a set of lines through the origin which intuitively would seem like an open set in $\mathbb{P}^1_{\mathbb{R}}$. Similarly, the map $g : S^1/(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ is also intuitively continuous: Take a single line through the origin. This is a closed set as it is a single element of $\mathbb{P}^1_{\mathbb{R}}$, and this line passes through a single equivalence class of points on $S^1/(\mathbb{Z}/2\mathbb{Z})$, which is also a closed set. So it would seem we have continuous bijective maps between these spaces, and so we can intuitively declare them to be homeomorphic.

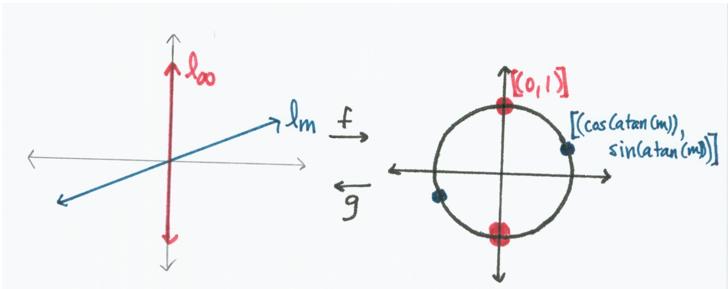


To formalize this intuition takes some more work. One way to do this is to start by noticing that $\mathbb{P}^1_{\mathbb{R}}$ is homeomorphic to $\mathbb{R} \cup \{\infty\}$ as a quotient topology. Take $\mathbb{R}^2$ and define an equivalence relation on the lines in $\mathbb{R}^2$ by identifying all lines with the same slope. Then $\mathbb{R}^2/\sim\, \cong \mathbb{P}^1_{\mathbb{R}}$, with each equivalence class of lines being represented by the line with that slope through the origin.  Now we can easily see that $\mathbb{P}^1_{\mathbb{R}} \cong \mathbb{R} \cup \{\infty\}$ by associating each line through the origin with its slope, and associating ∞ with the vertical line. Notice that these associations (homeomorphisms even) give us that the open sets of $\mathbb{P}^1_{\mathbb{R}}$ correspond with open sets of $\mathbb{R} \cup \{\infty\}$; the open sets of $\mathbb{R}$ are either genuine open sets of $\mathbb{R}$, or an open set around ∞ that includes both large positive and large negative numbers. Then the inverse images of these sets are sets of lines that have slopes that are open sets in $\mathbb{R} \cup \{\infty\}$.

Now, to go back to the homeomorphism between $\mathbb{P}^1_{\mathbb{R}}$ and $S^1/(\mathbb{Z}/2\mathbb{Z})$, we can formalize this using the fact that $\mathbb{P}^1_{\mathbb{R}} \cong \mathbb{R} \cup \{\infty\}$. Let's define our functions f and g from above more explicitly. Let ℓ_m be the line in $\mathbb{P}^1_{\mathbb{R}}$ with slope $m \in \mathbb{R} \cup \{\infty\}$, and define $f : \ell_m \mapsto [(\cos(\arctan(m)), \sin(\arctan(m)))]$ where $\arctan(\infty) = \pi/2$. Now it is easy to see that the map f is continuous because of the continuity of $\arctan(x)$ on $\mathbb{R} \cup \{\infty\}$. Additionally, take a point $(x, y) \in S^1/(\mathbb{Z}/2\mathbb{Z})$, and then define $g : [(x, y)] \mapsto \ell_{\frac{y}{x}}$ where you define



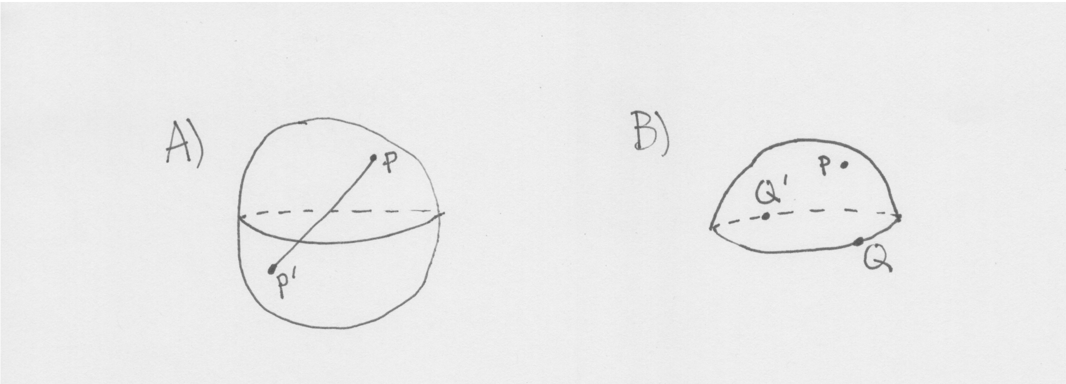
$\frac{y}{x}$ to be ∞ when $x = 0$. This is also more obviously a continuous map with this explicit definition.



Certainly for any point that is not $(0, 1)$, $f \circ g$ and $g \circ f$ are both identity maps. Consider the point $[(0, 1)]$; $f(g((0, 1))) = f(\ell_\infty) = [(\cos(\arctan(\infty)), \sin(\arctan(\infty)))] = [(\cos(\pi/2), \sin(\pi/2))] = [(0, 1)]$. Also, if you take the line ℓ_∞, then $g(f(\ell_\infty)) = g([(\cos(\arctan(\infty)), \sin(\arctan(\infty))]) = g([(0, 1)]) = \ell_\infty$. Then in every case, the compositions are the identity and the maps are continuous, so we can say we have shown that the projective line is homeomorphic to the circle.

Problem 15. *Realize the projective plane as an identification space from a disc.*

To realize the projective plane from a disc, first imagine the projective plane as a quotient space of the sphere. (Figure A) To get the projective plane from the sphere, we look at antipodal points on the sphere. Once we quotient out by this action, we get the projective plane! Now, cut the sphere in half so that all we have is a hemisphere. (Figure B) Notice that the point P has lost its antipodal partner, but the point Q has not! So we need to identify Q with Q' for all points along the base circle to get the projective plane.


A)
P
P'
B)
Q'
P
Q