

LECTURE PLAN FOR MATH 472
INTRODUCTION TO TOPOLOGY
Fall 2008

1 Aug 25

Discuss class time.

Draw pictures about things that are equal or different in different geometries. Explore:

- euclidean 3d (LENGTH, ANGLES, AREA, NUMBER OF SIDES)
- euclidean 2d (CLOCKWISE-COUNTERCLOCKWISE ORDERING)
- topography (ANGLES, PROPORTIONS)
- topology (NUMBER OF PIECES, NUMBER OF HOLES, EULER, DIMENSION)
- homotopy theory

Have them develop the notion of invariants - ask properties - make sense/don't make sense.

History: Felix Klein *Erlangen Programme, 1872*

Plan of class:

1. Continuity. Generalizing our old notion. The concept of equality in topology.
2. First invariants: connectedness, compactness.
3. Ways of constructing new spaces from old ones. Products, Quotients.
4. Study of compact surfaces. Some more sophisticated invariants: Euler char., orientability, genus.
5. Some algebraic topology. The fundamental group. The concept of homotopy.
6. Advanced topics if time permits.

2 Aug 27

Life with a metric; define a distance function.

Example 1. 1. *euclidean*.

2. *taxicab*.
3. *discrete*.
4. $\max |f - g|$.
5. $\min |f - g|$ (*NOT*)
6. $\max |x_i - y_i|$
7. $\max |x_1 - y_1|$ (*NOT*)
8. $(x-y)$ (*NOT-symmetric, positive*)
9. *do examples with two or three points*

Define continuity. Do some examples.
Define open balls, limit points.

Exercise 1. *Draw open balls for all the above metrics.*

Exercise 2. *Euclidean metric. Which are limit points:*

- *fuzzy blob and a point on the boundary.*
- *point in the interior of a blob.*
- *$A =$ isolated point outside a blob.*

Exercise 3. *What are limit points for sets in the discrete metric?*

Open and closed sets. Examples in the euclidean metric.

Exercise 4. *Prove that a set is open/closed in the euclidean metric iff it is open/closed in taxi-cab or max*

Exercise 5. *What are open and closed sets in the discrete metric?*

Properties:

- ϕ, X are open/closed.
- arbitrary union of opens is open.
- finite intersection of open is open. (Q: how about arbitrary intersection?)
- complement of an open is closed and viceversa

Theorem 1. *f is continuous if and only if $f^{-1}(\text{open})$ is open.
Same with closed.*

Exercise 6. *Any relation between continuity and the image of open sets?*

3 Aug 29

Definition of a topological space and a topology (X, τ) . Show two ways. Can define topology by opens and define complements of opens to be closed or viceversa.

Example 2. • *stupid (trivial) topology. Discrete topology.*

- *topology induced by a metric.*
- *topologies on a set with two points.*
- *the topology that loves one point.*
- $[a, b]$ on $\mathbb{R}$ (NOT)
- *induced topology on a subset.*

Remark: Notice that the induced topology can be tricky to digest. $[0, a)$ is not open in $\mathbb{R}$, but it is in $[0, 1]$ with the induced topology!

Exercise 7. Which of these are topologies on $\mathbb{R}$?

- $A = (-\infty, a)$
- $B = (a, +\infty)$
- $A \cup B$
- $(a, b]$

3.1 basic concepts

1. neighborhood.
2. interior point/interior of a set.
3. limit points/closure of a set.

Can give two equivalent definitions for interior and closure.

Exercise 8. Let X be a topological space with the stupid topology, and $x \in X$ a point of X . For what subsets of X is x :

- *an interior point?*
- *a limit point?*

What is the smallest neighborhood of x ?

Define concept of fineness of a topology.

Exercise 9. What are the coarsest and the finest topology you can define on a set X ?

Exercise 10. Partial order the following topologies on $\mathbb{R}$ by fineness:

- *stupid topology*
- *discrete topology*
- *euclidean topology*
- *finite complement topology*
- (∞, a) topology

If there is time, can introduce the notion of a base for a topology.

4 Sep 3

Define continuity.

Example 3. Any function to a space with the stupid topology is continuous.
Any function from a space with the discrete topology is continuous.
 $\sin(x)$ on $\mathbb{R}$ with the finite complement topology is not continuous.
But $1/x$ arbitrarily defined at 0 is!

Theorem 2. The composition of continuous functions is continuous

Theorem 3. f is continuous iff the preimage of any element of a base for the topology on Y is open in X .

Define a homeomorphism in two equivalent ways:

1. f continuous with inverse continuous.
2. f a bijection inducing a bijection on the topologies.

Note that a function being a homeomorphism depends on the topology on the set! Even the identity function on a set is not necessarily a homeomorphism.

Example 4. Open interval, $\mathbb{R}$. Open disc, plane, sphere minus a point.

Theorem 4. A homeo between the boundary of a disc to itself can always be extended to the whole disc.

Exercise 11. Subdivide in homeomorphism classes the letters of the alphabet.

5 Sep 5

Given a function

$$f : X \longrightarrow Y,$$

where one is a topological space and the other is not, how to induce a topology!

5.1 Y has a topology

Define τ_X to be the coarsest topology that makes f continuous. This corresponds to throwing in all preimages of open sets of Y .

Q: have we seen this already? Yes, with the subspace topology!!

5.2 X has a topology

Define τ_Y to be the finest topology that makes f continuous. I.e. define all sets whose preimage is continuous.

Exercise 12. *Step function. Use euclidean on domain and codomain and see what topologies happen!*

- In one case get basis $[n, m)$
- In the other get basis given by $\{x \notin \mathbb{Z}\} \cup \mathbb{Z}$

Example 5. *Glue the segment $[0,1]$ to get a circle.*

Need to find some good examples! Product topology...

6 Sep 8-10-12

Projects in class.

7 Sep 15

Finished up discussing projects. Assigned them as homework.

8 Sep 17-19

8.1 Connectedness and path connectedness

Idea: want to model the notion of being “only one piece”. Two ideas arise:

1. Joining points by paths.
2. Idea motivated by subspace topology.

Anticipate that the two notions will not be equivalent!

8.1.1 Connectedness

Equivalent definitions. X is **disconnected** if

1. There is a proper subset of X which is open and closed.
2. X can be decomposed into the disjoint union of two nonempty open (and hence closed) subsets.
3. You can decompose $X = A \cup B$ with $\bar{A} \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$
4. X admits a continuous non-constant map to a discrete space.

Then you define connected if not disconnected.

Example 6. 1. $\mathbb{R}$ is connected. This is tricky to prove. Assume a disjoint decomposition. Take a point in each set (call them a and b and assume $a < b$) and consider the sup of the points of A that are less than b . Look at all possible cases.

2. $\mathbb{Q}$ is NOT connected.

Theorem 5. The continuous image of a connected is connected.

Theorem 6. Connectedness is a topological invariant.

Theorem 7. If Z is dense in X and Z is connected, then X is connected. As a corollary any subset between Z and its closure is connected.

Define **connected component**: a connected subset of X maximal with respect to being connected. Note that by the theorem above a connected component is always closed. Is it open as well?

Theorem 8. $X \times Y$ connected iff X and Y are.

The difficult direction is handled this way:

Theorem 9. X is the union of a collection of connected spaces, such that any two spaces in the collection intersect, then X is connected.

8.2 Path Connectedness

Define path connectedness.

Theorem 10. path connected implies connected.

Theorem 11. connected open in $\mathbb{R}^n$ is path connected.

Note: for all not too pathological spaces the two notions are equivalent.

Exercise 13. Give an example of a connected space which is not path connected.

Typical example is the comb with teeth accumulating to one side, and a single dot on that side.

Theorem 12. $X \times Y$ path conn iff X and Y are.

9 Sep 22-24

9.1 Compactness

Intuitive idea: compactness measures the fact a set is “solid. In $\mathbb{R}^n$, compact is equivalent to closed and bounded. You can extend this definition to all metric spaces, but of course we have to be able to define things in such a way that they work even when there is no metric!!!

Define compactness: from any open cover, you can extract a finite subcover.

Exercise 14. $\mathbb{R}$ not compact. $(0, 1)$ not compact.

Exercise 15. What subsets of X are compact if X has the stupid topology? Discrete? Finite complement topology?

As usual, the first step is checking that we have defined something that generalizes our notion in metric spaces. hence the

Theorem 13 (Heine-Borel). *Cpt in $\mathbb{R}^n$ iff closed and bounded.*

Proof: Easy direction: If cpt then closed and bounded. Prove by contradiction.

Hard direction: If closed and bounded then cpt. Take arbitrary cover. Subdivide in squares of radius 1. Find one where you can't find finite subcover. Iterate. Get succession of nested closed sets with radius going to 0. Must converge to a point, which is a limit point, and therefore in the set. There is an open set in the collection containing that point. Therefore it contains one of the subdivided boxes (if radius small enough). Contradiction.

9.2 Theorems about compactness

Theorem 14. *Image of cpt is cpt.*

Corollary. Compactness is a topological invariant.

Theorem 15. *Closed subset of cpt is cpt.*

Theorem 16. *Compact in Hausdorff is closed.*

Theorem 17 (Bolzano-Weierstrass). *An infinite subset of a compact space has a limit point.*

Theorem 18. *$X \times Y$ is cpt iff X and Y are.*

10 Sep 26

Do a project on compactification. This ended up taking three lectures. One point compactifications were discussed, and then the projective plane was introduced. This led to the first exam, on October 3rd.

11 Oct 6

11.1 Identification Spaces

Define an identification space as the topological space obtained by giving a quotient set the topology induced by the quotient map. Discuss some examples, starting from very simple ones. Glue a “T. Other possible examples:

- Glue an interval to a circle;
- $\mathbb{R}/\mathbb{Z}$
- Glue a square to a cylinder and to a mobius band.
- Glue a square to a torus.
- $\mathbb{R}^2/\mathbb{Z}^2$.
- Glue a bigon to a projective plane

Exercise 16. Consider a closed n -dimensional unit ball D_n :

$$D_n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

The boundary of D_n is the $n - 1$ dimensional sphere:

$$S_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$$

Consider the following equivalence relation on D_n :

$$x_1 \sim x_2 \text{ iff } x_1 = x_2 \text{ or both } x_1 \text{ and } x_2 \in S_{n-1}$$

(basically all points of the sphere are equivalent, and each point in the interior is equivalent only to itself).

What is the identification space $D_n / \sim$?

Introduce cone construction. Can do example $C_{S_n} = D_{n+1}$. Here the homeomorphism is $(x, t) \mapsto x(1 - t)$ and $X \mapsto (X/|X|, |X|)$

12 October 8

Introduce attaching via a function. Give example of a cylinder glued to a disc via the function that maps boundary to boundary and via the constant function 0.

Theorem 19 (gluing lemma). *Let $X, Y \subseteq Z$, such that X and Y are (locally) closed in $X \cup Y$, and f, g continuous functions agreeing on $X \cap Y$. Then $f \cup g$ is continuous.*

Give counter-example:

$$X = \mathbb{Q}, Y = \mathbb{R} - \mathbb{Q}, f = 0, g = 1$$

Exercise 17. *Show that the one point compactification of the (open) Mobius strip is the projective plane.*

Exercise 18. *Consider the closed Mobius strip:*

$$M = [0, 1] \times [0, 1] / (0, y) \sim (1, 1 - y)$$

and the closed unit disc:

$$\bar{D}_2 = \{(\rho, \theta) \in \mathbb{R}^2 : \rho \leq 1\}$$

Let f be the following map, from the boundary circle of the unit disc to the boundary of the Mobius strip:

$$f(1, \theta) := \begin{cases} (\theta/\pi, 1) & 0 \leq \theta \leq \pi \\ (\theta/\pi - 1, 0) & \pi \leq \theta \leq 2\pi \end{cases}$$

What is the identification space

$$\bar{D}_2 \cup_f M?$$

13 Oct 10

13.1 Group Actions

Let G be a group, X a topological space. A group action:

$$G \times X \rightarrow X$$

such that

1. $1x = x$;
2. $g(hx) = (gh)x$.

Define orbits = equivalence classes wrt to the equivalence relation $x \sim y$ if there exists $g \in G$ st $y = gx$.

Quotient space is obtained in the usual way, using the finest topology that makes the projection continuous.

Examples we have already seen - to be framed in this language: $\mathbb{R}/\mathbb{Z}$, $\mathbb{R}^2/\mathbb{Z}^2$, $S^2/\mathbb{Z}_2$. Do some non-standard examples: circle mod reflections = interval. Sphere mod reflections = disc.

Give an overview of what's to come: classification of compact surfaces. Describe strategy as if we were discovering the result. Show that if surfaces are not compact then the thing is kind of hopeless.

1. List surfaces we understand. Develop ways to make surfaces: identification polygons, connected sums.
2. Explore the relationship between polygons and connected sums.
3. Get comfortable with manipulating polygons.
4. Can I describe all surfaces with polygons? (A= yes!)
5. When do two polygons represent the same surface? (A= topological invariants: Euler characteristic and orientability).

Should I do this all in project format??

14 Oct 13 - 15 - 17

Today let us start by giving definitions:

1. Surface.
2. Identification polygons.
3. Connected sum.

Point out things to do/ worry about:

- Is connected sum well defined (answer:NO - only if surface is connected)?
- Start understanding identification polygons.
- Do all identification polygons represent surfaces?
- Do all surfaces come from some identification polygon?

Monday carefully do identification polygons for polygons and quadrilaterals.

The whole week was about right for this program. Towards the end of the week we got to start talking about Euler characteristic.

15 Oct 20

Continue with Euler characteristic. Follow the guidelines of the worksheet "euler

16 Oct 22

Orientability.

We discussed in class 5 different ways to tell apart cylinder from Mobius:

1. Number of boundary components
2. all circles are separating
3. parity of intersection of circles
4. spaceman spiff's property
5. number of faces

Pointed out that last property is extrinsic, all others are intrinsic. Defined orientability for surfaces. Connected this to polygons.

17 Oct 24

Wrapped up the theorem. What was missing at this point was:

1. Using triangulation to go from surface to polygon
2. Showing inductively that all polygons are connected sums of tori and proj planes

18 Oct 27

Introduction to homotopy theory and algebraic topology.

Key idea: topological invariant is an algebraic object (e.g. a group).

How to guarantee that such a thing is a topological invariant?

Introduce the notion of a functor.

Fundamental group: a way to look for holes.

1. Look at loops on spaces.
2. Up to wiggling (notion of homotopy).
3. Composition and inverse.

Generalizations: point out π_0 and π_n

Plan of action:

1. Make notion of wiggling formal (homotopy).
2. Construct the fundamental group.
3. Be able to compute the fundamental group. Seifert Van-Kampen's theorem.

19 Oct 29

Define homotopy of maps.

Examples:

- Any two maps $X \rightarrow \mathbb{R}^n$ are homotopic.
- Any two maps $pt. \rightarrow X$ are homotopic only if X is path connected.
- Question: are any two maps $X \rightarrow S^n$
- $f : e^{ix}, g : e^{i(x+\pi)}$ are homotopic.
- embedding of meridian circle on torus and constant map are not homotopic.
- 4 sides of polygon for torus and constant map are homotopic.

Define homotopy of maps relative to a subset A of X .

Results about homotopies:

1. Homotopy is an equivalence relation.
2. composition of homotopic maps are homotopic.

Exercises: 1,2,5, 6,7

20 Nov 3

Construct the fundamental group. Prove that if X is path connected, then fundamental groups based at different points are isomorphic. Show continuous maps induce homomorphisms on fundamental groups.

Exercises: 12

21 Nov 5 - 7

Homotopy equivalence of spaces: $X \sim Y$ if I have a pair of maps whose composition is homotopic to the identity.

Homotopy equivalence is an equivalence relation.

Examples:

1. $\mathbb{R}^n$ and point. Disc and point.
2. circle and cylinder and plane minus a point.
3. 8 and plane minus two points.

Deformation retraction: i, r such that $ir \sim 1, ri = 1$.

Homotopy equivalent spaces have isomorphic fundamental groups. This is a consequence of functoriality.

22 Nov 10 - 12

Free groups. Construction. Examples.

Universal property: any map from generators of F to G extends to a homomorphism $F \rightarrow G$.

Presentation: isomorphism between G and a quotient of F given by a surjective map.

Examples: cyclic groups. Dihedral groups.

Issue: given a presentation, it is typically difficult to understand the group. Even to understand if it is trivial!!!!

23 Nov 14 - 17

Seifert Van Kampen theorem.

Examples of how to compute fundamental groups.

1. sphere, projective planes, surfaces.
2. bouquet of circles.
3. surfaces minus points.
4. n-dimensional spheres.
5. union at a point of spaces.
6. necklace of spheres.

24 Nov 19

Review

25 Nov 21

Exam

26 Dec 1, 3

Things to do in the rest of the class could be: showing that the fund group of the circle is $\mathbb{Z}$, Brower fixed point theorem, fundamental theorem of algebra.

26.1 Applications of homotopy theory

1. The Brower Fixed Point Theorem
2. The Fundamental Theorem of Algebra
3. The Borsuk-Ulam Theorem (only if extra time)

27 Remaining of the semester(4 hours)

27.1 Proving that the fundamental group of the circle is the integers

Theorem: X a contractible space, G acting freely and properly discontinuously (all this means is you want the quotient map to be a covering) on X , then the fundamental group of X/G is G .

Have them prove this theorem as follows:

1. Give them a rough statement of the theorem: If X is a BLANK topological space, G acting on X NICELY, then ...
2. Get them to explore with experiments of group actions:
 - Circle, torus and projective plane.
 - Circle to circle, rotation action.
 - Disk $\mathbb{Z}_2$ action. Can do both reflection and $z \mapsto z^n$ action. Can show that there is not unique way to lift a homotopy.
3. ask them to look at what happens to loops. Can push loops from upstairs to downstairs, and can lift loops from downstairs to upstairs. See that cokernel of induced map on π_1 's is exactly G .
4. remember to analyze the case when there are fixed points. Show that a homotopy can be lifted in two different ways.