Notes

Renzo's math 676

1 Moduli Spaces: Categorical Approach.

1.1 Moduli Problems

In order to pose a **moduli problem** we need the following ingredients:

1. A class of (geometric) objects $\mathcal{P}$.
2. A notion of **family** of such objects.
3. The notion of equivalence of families. Note that families over one point are precisely the objects of $\mathcal{P}$, and hence we are in particular assigning the notion of equivalence of objects.

Intuition: A moduli space $\mathcal{M}_{\mathcal{P}}$ for the above problem consists of a space in the same category (or in a close enough one) as the objects parameterized, such that:

- points are in bijective correspondence with equivalence classes of objects in $\mathcal{P}$.
- there is a natural bijection between families over a given base B and functions $B \to \mathcal{M}_{\mathcal{P}}$.

In order to make the intuition more precise, we formally define what a family is.

Definition 1. *Let $\mathcal{P}$ define a class of objects in some category $\mathcal{C}$. Then for any $B \in \mathcal{C}$, a* **family** *of $\mathcal{P}$-objects over B is an object $X \in \mathcal{C}$ together with a surjective morphism*

$$\pi : X \to B$$

such that the fiber at each point ($X_b := \pi^{-1}(b)$ for any $b \in B$) is an object in $\mathcal{P}$.

A family immediately defines a set function $\varphi_\pi : B \to \mathcal{M}_{\mathcal{C}}$:

$$\varphi_\pi(b) := [X_b]$$

The image of a point b is the point in the moduli space is the (equivalence class of the) fiber of that point. In order to have a moduli space we like the most (a **fine** moduli space) we are making the following requirements:

1. That the set function is actually a morphism in the category $\mathcal{C}$.

2. That no two families give the same function.

3. That every function from B arises as the function associated to a family.

When the three conditions above are verified, we establish a fruitful dictionary between the geometry of the moduli space and the geometry of family of objects. Becoming fluent with using this dictionary and translating questions back and forth is one of the main goals of this class.

Intuition: If this whole dictionary with families seems a bit weird, let me try to convince you that in fact it is really natural. We would like the geometry of the moduli space to reflect the similarity of objects. For example, if our objects had (a complete set of) metric invariants and the moduli space had a metric, we want to say that objects with very close invariants correspond to points that are very near each other in the moduli space. Or, even better, that moving the invariants continuously would result in a continuous path in the moduli space.

Of course the problem is making such statments general and precise. The notion of a family achieves precisely that: it tells us that over a base B objetcs are varying in a "kosher" way if they fit together to form a larger object still in the right category.

Important Note: typically achieving this dictionary is far from simple, and a lot of times it is just not possible.

Exercise 1. *This exercise explores in extremely simple cases the failure of some of the above points to hold.*

1. *Dream of the moduli space of projective equivalence classes of reduced plane conics. Note that if such a space existed it would have two points. Construct a family of conics giving rise to a discontinuous function to such a space.*

2. *Wish for the moduli space of isomorphism classes of unit length plane segments up to rigid motion in the plane. Note that is such a space exists it can have only one point. Construct two families of segments over a circle that are not equivalent to each other. But they both must give the constant function: hence* 2 *fails.*

3. *Attempt to realize the moduli space of "(ordered!) pairs of opposite complex points" (i.e. pairs of points of the form $(z, -z)$). Convince yourself that if such a space existed it would have to be $\mathbb{C}$. Consider*

the identity function $\mathbb{C} \to \mathbb{C}$ and show that the corresponding family is not in the category of smoooth manifolds. This contradicts 3.

1.2 Moduli Functors

I would now like to rephrase all of the above in categorical language. This is nothing more than a reformulation, but it is useful to get acquainted with modern "slick" language, as that is what people use when they write papers so they look smart even when they say simple things.

Definition 2. *A* **moduli problem** *for a class $\mathcal{P}$ of objects in a category $\mathcal{C}$ is a contravariant functor:*

$$\mathcal{F}_{\mathcal{P}} : \mathcal{C} \to \mathcal{S}ets$$

that associates to each $B \in \mathcal{C}$ the set of isomorphism classes of families of $\mathcal{P}$-objects. To each morphism $f : B' \to B$ is associated the set map sending a family $X \to B$ to the pullback family $f^*(X) \to B'$.

Definition 3. *An object $\mathcal{M}_{\mathcal{P}} \in \mathcal{C}$ that represents the functor $\mathcal{F}_{\mathcal{P}}$ is called a* **fine moduli space** *for the moduli problem.*

Representing the functor means that the functor of points ($Hom_{\mathcal{C}}(-, \mathcal{M}_{\mathcal{P}})$) of $\mathcal{M}_{\mathcal{P}}$ is isomorphic (as a functor) to the moduli functor via a natural transformation $\mathbb{T}$.

Exercise 2. *I gave a couple minutes of thought whether this transformation (if it exists) should be unique (my intuition thinks so), then I got bored and lazy and didn't get to the end of it. Why don't you figure out and then tell me? Thanks.*

Exercise 3. *Meditate for a few minutes until you convince yourself that this is equivalent to the above discussion about families and the natural bijection with functions to the moduli space. In particular, be happy with the fact that evaluating the two functors on the terminal object in your category (which if you want to think geometrically, is just a point - according to your point of view, you might want to "dress" the point as $Spec(\mathbb{C})$ or $Spec(\mathbb{Z})$) recovers the invoked bijection between points of the moduli space and (equivalence classes of) objects you want to parameterize.*

Intuition: If you are funked out by this idea of understanding a space by thinking of its functor of points, think that this is a generalization of what we do when we talk about manifolds. We give up the idea of understanding the manifold globally, and rather focus on the local data. A differentiable atlas is the notion of understanding a family of (injective) functions that cover the set of points of the manifold (the charts), plus the transition functions on double overlaps. Here instead we want to understand ALL possible functions into $\mathcal{M}_{\mathcal{P}}$, and all possible pullbacks. This is of course not a proof (the formal proof is given by Yoneda's lemma - look it up if you want to dot the i's in this exposition), but it should make it fairly plausible that the knowledge of the functor of points should be enough to recover the (scheme, manifold, etc) structure of the moduli space.

1.3 Universal Families

When a moduli functor is representable by a fine moduli space, there is a very special object with a map to the moduli space, such that the fiber over each point $m \in \mathcal{M}_{\mathcal{P}}$ is precisely the (an) object (in the equivalence class) parameterized by the point m.

Definition 4. *Given a fine moduli space $\mathcal{M}_{\mathcal{P}}$, the* **Universal Family** $\mathcal{U}_{\mathcal{P}}$ *is the object in $\mathcal{C}$ corresponding to $\mathbb{T}(\mathbb{1}_{\mathcal{M}_{\mathcal{P}}})$. This is a family (whose projection function is called* **universal function***)*

$$\pi : \mathcal{U}_{\mathcal{P}} \to \mathcal{M}_{\mathcal{P}}$$

such that for any $m \in \mathcal{M}_{\mathcal{P}}$ corresponding to an (equivalence class of) object(s) $[X_m] \in \mathcal{P}$,

$$\pi^{-1}(m) = [X_m]$$

Exercise 4. *A very nice consequence of the existence of a universal family is that any other family of $\mathcal{P}$-objects is obtained from the universal family by pullback. Show that this is just a formal consequence of the categorical definitions we made*

Exercise 5. *Any scheme X is a fine moduli space for the functor "families of points of X", the universal family being X itself. This is all a big tautology, but make sure it makes perfect sense to you. It is a good exercise to unravel the definitions and keep them straight.*

Exercise 6. *Consider the moduli space of plane conics. Show that if we put no conditions (i.e. we allow reducible and non-reduced conics), then such a moduli space is a five dimensional projective space, with universal family given by the natural incidence relation.*

Exercise 7. *Show that a moduli space for squares (make sense of what this means!) cannot admit a universal family.*

Intuition: Typically moduli spaces that parameterize objects that have automorphisms tend to not admit universal families.

Here is an imprecise but intuitive argument to motivate why. Consider a circle as the identification space of the unit interval where you identify the endpoints. Consider a trivial family of objects over the interval and now define a family of objects over the circle by identifying points of the fibers over 0 and 1. In most cases, when you make this identification via the identity function or when you make via the action of a non-trivial automorphism of the fiber, you obtain two families that are isotrivial but not isomorphic. Isotrivial means that the object parameterized over each point of the base is the same (or alternatively that the natural map to the moduli space is a constant map).

A lot of geometric objects that we like to study tend to have automorphisms (e.g. symmetries). So for these kind of objects should we just give up the idea of forming moduli spaces? Well, yes and no. Certainly we cannot expect the best of life: a nice object in category $\mathcal{C}$, which parameterizes the objects we want to study in the sense of having this perfect dictionary between families of objects and maps to the moduli space. If we relax some of the above requirements, we obtain some ways to proceed:

stacks: we could relax the requirement of the moduli space being in $\mathcal{C}$, and ask for it to be in some enlarged category $\tilde{\mathcal{C}}$. In this case we still require the dictionary to work, and we make use of the dictionary to do geometry on the moduli space. This point of view is probably the most modern approach and it leads in the direction of stacks and orbifolds. We postpone this discussion to when we'll have sufficient muscles and examples for it.

rigidification: here the idea is to relax the idea that our space parameterizes precisely the objects we want. We instead parameterize "similar objects" by adding structure (marked points, sections of bundles, etc), that achieves the effect of killing nontrivial automorphisms (aka rigidifying the moduli problem). Then the new moduli problem becomes representable.

Intuition: Think of a circle. If you just have a circle, any rotation is a non-trivial map from the circle to itself. However imagine now to color one point of the circle, and require that admissible maps from the circle to the circle must send the colored point to itself. Now the only rotation that we allow is the identity. By talking of a circle with a marked point instead of just a circle, we have made trivial the automorphism group of the object we are interested in.

coarse moduli spaces: finally, we could relax our request of a perfect dictionary between families and functions to the moduli space, and just ask for the object in $\mathcal{C}$ that best approximates such dictionary. This leads to the notion of a coarse moduli space, wich we discuss next.

1.4 Coarse Moduli Spaces

Definition 5. *Let $\mathcal{F}$ be a moduli functor for a class of objects in some category $\mathcal{C}$. An object $M \in \mathcal{C}$ together with a natural transformation $T : \mathcal{F} \to Hom_{\mathcal{C}}(-, M_{\mathcal{P}})$ is a* **coarse moduli space** *for the moduli functor if:*

1. *The natural transformation evaluated on a point $T(pt.)$ is a set bijection.*
2. *For any other object $M' \in \mathcal{C}$ and natural transformation T' from the moduli functor to the functor of points of M', there is a unique morphism $\pi : M \to M'$ such that the associated natural transformation diagram commutes.*

Intuition: Let us parse this apparently mysterious definition to realize it is actually quite natural. First off, the natural transformation means that we can still use "half" of the dictionary: if we have a family of objects we get a corresponding map to the coarse moduli space. Point 1. is requiring that I have a bijection between the objects we want to parameterize and the points of M. Finally point 2. insures that we are putting the "simplest possible" (scheme) structure on M that makes things work. For example we don,'t want to put a reduced structure or nilpotents in our moduli space unless we really need them.

Exercise 8. *Show that if a coarse moduli space exists it is unique (up to canonical isomorphism). This is a consequence of point 2 in the definition.*

Exercise 9. *Show that the cuspidal cubic $y^2z = x^3 \subseteq \mathbb{P}^2$ has a natural bijection with $\mathbb{P}^1$ but it is not a coarse moduli space for the moduli problem of lines through the origin in the plane.*

Exercise 10. *Go back to the moduli problem of projective equivalence classes of reduced plane conics. Show that if we choose M to be just one point, one can (trivially) define a natural transformation T in such a way that 2. above is satisfied. Therefore the moduli problem does not admit a coarse moduli space.*

Intuition: The idea here is that a moduli space for projective equivalence classes should have two points (as we saw there are two such equivalence classes). However, since a reducible conic can be obtained as the limit of a pencil of smooth conics, the corresponding point $[C_{red}]$ in the moduli space should be infinitely close to the point $[C_{sm}]$, and any map from the wannabe moduli space to anywhere else sends the two points always in the same place. This allows a single point to satisfy point 2. and therefore be the only possible candidate for a coarse moduli space.

2 A few Hands on Examples

2.1 Projective Space

In this section we want to take an object we know and love (projective space) and observe it with a more critical eye, in order to observe some of the features we have discussed. For my psychological benefit we work over $\mathbb{C}$ (i.e. we can choose our category $\mathcal{C}$ to be whichever is most familiar to you between "Schemes over $Spec(\mathbb{C})$" and "Complex Manifolds") but I believe everything can be reproduced basically in the same way over an arbitrary base field.

2.1.1 The moduli Functor

For once (and probably the only time), let us spell out really carefully what moduli functor we wish $\mathbb{P}^n$ to represent.

For a given B, $\mathcal{F}_{\mathbb{P}^n}(B)$ is the set of equivalence classes of families of lines through the origin in $\mathbb{C}^{n+1}$. A family is a space

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "p"'] & B \times \mathbb{C}^{n+1} \arrow[d, "\pi_1"] & b \times L_b \arrow[l] \arrow[d] \\
& B & b \arrow[l]
\end{tikzcd}$$

where for every point $b \in B$, L_b is a line through the origin in $\mathbb{C}^{n+1}$. Two families X_1 and X_2 are equivalent if there is an isomorphism $F : X_1 \to X_2$ making the following diagram commutative:

$$\begin{tikzcd}
& & b \times L_b \arrow[dll] \arrow[dl] \arrow[dd] \\
X_1 \arrow[r, "F"] \arrow[dr, "p_1"] & X_2 \arrow[d, "p_2"] & \\
& B & b \arrow[l]
\end{tikzcd}$$

Exercise 11. *Note that in fact for two families X_1 and X_2 to be equivalent they need to be the same family. However the map F does not need to be trivial. This means that the objects that we are wishing to parameterize have no isomorphisms (there is only one family in each equivalence class), but do have automorphisms (there are non-trivial maps from a family to itself). Explicitly construct an example of an automorphism between two families of lines as above.*

Finally we need to define the functor on morphisms. The quick way to say is that it acts by pullback. Let us spell out what this means. Given a map $f : B_2 \to B_1$, we get a set function from B_1-families to B_2-families by associating to $p : X \to B_2$ the pullback family $f^*p : f^*X \to B_1$ defined by the following diagram:

$$\begin{tikzcd} B_2 \times \mathbb{C}^{n+1} \arrow[r, "f \times \mathbb{1}"] & B_1 \times \mathbb{C}^{n+1} \\ f^*X \arrow[u] \arrow[r] \arrow[d, "f^*p"] & X \arrow[u] \arrow[d, "p"] \\ B_2 \arrow[r, "f"] & B_1. \end{tikzcd}$$

Exercise 12. *Give a coordinate description of the pullback family.*

2.2 Representing the Functor: Local Coordinates

The most elementary (but maybe a bit laborious) approach to constructing a space that represents the above functor is to patch it together as a manifold using naturally defined charts.

Intuition: The main idea is that if we slice $\mathbb{C}^{n+1}$ with a hyperplane H NOT through the origin, we obtain a natural bijection between a subset of the objects we wish to parameterize (lines that are not parallel to H) and points in an n-dimensional affine space (the hyperplane itself). This bijection is obtained by simply assigning to a line not parallel to H its unique point of intersection with H. The topology of H seems to be a reasonable topology to have on our moduli space, in the sense that lines "close-by" correspond to points "close-by" and viceversa. Therefore we promote this bijection to becoming a local chart and what we have to check is that the induced transition functions are "good" (where this might mean bi-holomorphic, bi-regular, diffeomorphisms etc according to your point of view). An important point is that ANY plane not through the origin can be used as a local chart for $\mathbb{P}^n$. However, in order to construct $\mathbb{P}^n$ as a manifold, we only need a set of charts that cover all of the points of the set we want to turn into a manifold. We therefore have leeway in choosing appropriately a set of hyperplanes (translates of the coordinate hyperplanes) that makes our job as simple as possible.

Denote by $\mathbb{P}^n$ the set of points that we wish to become projective space, i.e. the set of families of lines over a single point: $\mathcal{F}_{\mathbb{P}^n}(pt.)$.

Let $H_i = \{x_i = 1\} \cong \mathbb{C}^n \subseteq \mathbb{C}^{n+1}$. There is a natural injective function

$$\varphi_i : H_i \to \mathbb{P}^n$$

sending $h \in H_i$ to the line between h and the origing in $\mathbb{C}^{n+1}$. Call $U_i \subset \mathbb{P}^n$ the image of φ_i. The collection $\{U_i : 0 \leq i \leq n\}$ covers the set $\mathbb{P}^n$. We want to show that $\mathbb{P}^n$ can be turned into a manifold by choosing $\{(H_i, \varphi_i) : 0 \leq i \leq n\}$ as an atlas for $\mathbb{P}^n$.

We must therefore check that the transition functions $\phi_{ij} := \varphi_i^{-1}\varphi_j : \varphi_j^{-1}(U_i \cap U_j) \to \varphi_i^{-1}(U_i \cap U_j)$ are "good".

Exercise 13. *Choose the natural isomorphism between H_i and $\mathbb{C}^n$ to be the natural projection forgetting the $i-th$ coordinate. Show that*

$$\phi_{ij}(x_0, \dots, \hat{x_j}, \dots, x_n) = \left(\frac{x_0}{x_i}, \dots, \hat{x_i}, \dots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

Check that these are indeed bi-regular (bi-holomorphic, diffeomorphic...) transitions functions.

Now we want to check that the manifold we constructed indeed represents the functor. We must define the natural transformation $\mathbb{T} : \mathcal{F} \to Hom(-, \mathbb{P}^n)$. This means we must find a functorial way to associate a map $B \to \mathbb{P}^n$ to a family of lines over B. Here I will just sketch the idea skipping all challenging details.

Given a family $p : X \to B$ I define a map $f_p : B \to \mathbb{P}^n$ by assigning $(n+1)$ compatible maps $f_{p,i}$ from open sets of B (that cover B) to each of the U_i's covering $\mathbb{P}^n$.

To define $f_{p,i}$ proceed as follows. Consider $X_i := X \cap B \times H_i \subset B \times \mathbb{C}^{n+1}$. Define $B_i := p(X_i)$. Check that B_i is an open set of B and that the various B_i's cover all of B. Use the inverse function theorem to get a function $\gamma_i : B_i \to X_i$. Then $f_{p,i}$ is defined as the composition $\varphi_i \pi_2 \gamma_i$, where π_2 is the projection to H_i.

One should check that the various $f'_{p,i}s$ are all compatible and indeed define a global function $f : B \to \mathbb{P}^n$. Finally, that this assignment is functorial and indeed defines the natural transformation $\mathbb{T}$ that we want and that this natural transformation is in fact an isomorphism of functors.

Exercise 14. *Describe the inverse natural transformation $\mathbb{T}^{-1}$.*

2.3 Representing the Functor: Global Approach

We now take a different point of view, and try to get projective space "all in one piece". In order to so we adopt the following strategy. Find a space X that we understand, and a group G acting on G in such a way that there is a natural bijection between G-orbits and lines in $\mathbb{C}^{n+1}$. Then we can realize projective space as the quotient space X/G.

Intuition: Of course I am being overly optimistic here...if G is a discrete group (e.g. finite), then it is not a big deal to assign some sort of topology to the orbit space. But if G is an algebraic group with an interesting algebraic structure then making sense of what algebraic structure to give the orbit space is much trickier. This is the content of Geometric Invariant Theory, and we will try to learn this later. For now we content ourselves of saying that in the case of projective space it can be done, and draw some interesting observations/consequences.

We can choose $X = \mathbb{C}^{n+1} \setminus \{\underline{0}\}$ and the group $G = \mathbb{C}^*$ acting by scaling all coordinates simultaneously:

$$t \cdot (x_0, x_1, \ldots, x_n) = (tx_0, tx_1, \ldots, tx_n).$$

Then, assuming that we know what it means (which we don't quite), we can define $\mathbb{P}^n := X/G$.

One way to try to describe what the above quotient means would be to describe what are the regular functions $f : X/G \to \mathbb{C}$. And we could take the standard point of view that functions on the quotient are regular functions on X that are invariant along the orbits.

Exercise 15. *Show that in this case the only regular functions on $\mathbb{P}^n$ are the constant functions.*

What we just did is correct, but sadly it really did not give us a lot of information about the structure of projective space... One way to proceed would be to remember that functions on a space form a sheaf, and that it is really the sheaf of functions that contains the information about the space: we can't just content ourselves with the global sections!!

This would basically lead us again towards a local approach. Notice for example that if we throw away the hyperplane $H_0 = \{x_0 = 0\}$ then the function x_0 becomes invertible. Now on this subset of projective space we have plenty of invariant functions, namely all polynomials in the variables $x_1/x_0, \ldots, x_n/x_0$. This is telling us that there is an open set isomorphic to $\mathbb{C}^n$ inside projective space. This is nothing else than the set U_0 described in the previous paragraph, only described in an algebraic fashion.

Another somewhat orthogonal approach would be to relax our requirement of our functions being invariant on orbits. We could allow to have an action of $\mathbb{C}^*$ on $\mathbb{C}$ (aka an irreducible representation ρ of $\mathbb{C}^*$) and ask for $\mathbb{C}^*$-equivariant functions:

$$F : \mathbb{C}^{n+1} \setminus \{\underline{0}\} \to \rho.$$

Irreducible representations of $\mathbb{C}^*$ are indexed by integers k ($\rho_k : t \cdot z = t^k z$). It is easy to note that the equivariant functions with respect to representations $\rho_k, k \geq 0$ are precisely homogeneous polynomials of degree k.

The direct sum (over all irreducible representations of $\mathbb{C}^*$) of equivariant functions is a graded ring

$$\mathbb{C}[\mathbb{P}^n] := \bigoplus_{k \geq 0} \{F_k \text{ homogeneous polynomial of degree } k\}$$

called the **homogeneous coordinate ring** of $\mathbb{P}^n$. This is (yet another) algebraic way to describe the scheme structure of proejctive space.

2.4 The Universal Family

Once we have $\mathbb{P}^n$ in our hands, it is not to hard to exhibit a universal family, since it can be described as the incidence set

$$\mathcal{U} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$$

of pairs $([l], x)$, with $x \in l$.

The universal family $\mathcal{U}$ is a line bundle on $\mathbb{P}^n$ that is denoted $\mathcal{O}_{\mathbb{P}^n}(-1)$. The name comes for example from analyzing the patching data of this line bundle, or from observing that it is naturally dual to the bundle whose sections are homogeneous polynomials of degree 1...but this is another story that we can't dwell on...

2.5 Grassmannians

Grassmannians are natural generalizations of projective space. The grassmannian $G(k, n)$ is a fine moduli space for families of k dimensional linear subspaces of a fixed vector space of dimension n. So in particular $\mathbb{P}^n = G(1, n)$. Grassmannians are compact manifolds, or smooth projective varieties (according to your point of view). We take a look at three different points of view to describe and understand these spaces.

Intuition: We proceed analogously to what we did for projective space. First we describe Grassmannian localli, by giving a natural set of local coordinate charts described by the geometry of the objects parameterized. Second, we think of Grassmannians as appropriate quotients of a quasi-projective variety by an algebraic group. The third point of view consists of constructing a natural inclusion from the Grassmannian to a subvariety of some large projective space, and then inducing a scheme structure on the Grassmannian using this inclusion. CAUTION! I'd like you to appreciate that there is something not automatic here: finding a set inclusion from the set of points of the moduli space you want to a scheme in such a way that the image is a subscheme is a standard strategy and starting point for constructing a moduli space. However it is not guaranteed that the (reduced induced) scheme structure of the image of such set map is necessarily the appropriate scheme structure to represent the moduli functor!! This problem luckily does not arise with Grassmannians, but it does arise for example when constructing the Hilbert Scheme. But for now, this is another story!

Once and for all fix an n-dimensional vector space $V \cong \mathbb{C}^n$ via the choice of a basis $e_1, \ldots, e_n$ and coordinates $(x_1, \ldots x_n)$.

2.5.1 Local coordinates

Intuition: For $\mathbb{P}^n$ our starting point to get a local chart was to observe that if we chose a hyperplane H NOT through the origin, then any line that intersected H did so in only one point. OK, so it seems natural now instead of choosing a hyperplane, to instead choose as a screen the translate away from the origin of a codimension k linear subspace H. A generic k-plane intersects H in one point (generic here means that we need to throw away the k-planes that are parallel to H or those that intersect it in more than one point - but these are two "closed" subsets). The problem is now that several k-planes intersect H in the same point (in fact a (k-1)-dimensional family...) so the function from k-planes to H is highly non injective...well then we can take more screens! In fact if we choose k screens (in general position, this meaning that together they span the full vector space V), then each k-plane that intersects all of them does so in one point, and this collection of k points identifies uniquely the k-plane. We have therefore given a local parameterization of the Grassmannian by the product of k linear spaces of dimension (n-k)! Just like in $\mathbb{P}^n$, while this procedure can be carried out with any general choice of screens, we will do ourselves a favor by choosing special screens that will make the exposition of the argument more tractable.

We construct one chart for the Grassmannian, that we call $U_{1\ldots k}$. By choosing any k indices from 1 to n one naturally gets an atlas (with $\binom{n}{k}$ charts) for the Grassmannian. After checking the transition functions are good (which we won't do!), one has proven that the Grassmannian is a smooth manifold of dimension $k(n-k)$.

Let H_1 be the $(n-k)$-dimensional plane in V defined by the equations $x_1 = 1, x_2 = x_3 = \ldots = x_k = 0$. Similarly, for $1 \leq i \leq k$, define H_i to be the plane where the i-th coordinate is 1 and all other of the first k coordinates are 0.

Exercise 16. *Convince yourself that any k linear subspace which is not parallel to any of the screens $H_1, \ldots, H_k$ intersects each of the H_i's in precisely one point, say P_i. Note that the origin and the collection of P_i's spans a unique k- plane.*

Therefore we have obtained an injective function from a subset of the Grassmannian to $\mathbb{C}^{k(n-k)}$, which we promote to becoming a local chart. It is tedious, but not particularly hard, to see that the transition functions are rational functions in the local coordinates that do not acquire singularities on the overlap of the two charts. However this is simpler to see by taking a more algebraic point of view, which is what we do next.

2.5.2 Quotient Space Approach

Our next point of view is to find a space with a group action in such a way that orbits correspond to k-subspaces of V.

A natural way to describe a k-subspace is as the span of k linearly independent vectors of V. Since we have picked a basis for V and we are doing things in coordinates, this is equivalent to giving a $k \times n$ matrix with maximal rank. Now if you act on the left with a $k \times k$ invertible matrix you do not change the image of the k-plane, but just change the set of vectors that span it. This is precisely the ambiguity we want to get rid of. Therefore we can present the Grassmannian as the quotient space:

$$G(k, n) := GL(k, \mathbb{C}) \backslash U,$$

where $U \subset M(k \times n, \mathbb{C})$ is the subset of matrices of rank k.

Exercise 17. *Prove that U is an open subset of $M(k \times n, \mathbb{C})$ (where the spaces of matrices is given the natural topology induced by its identification with $\mathbb{C}^{kn}$.*

We note that the dimension count checks. We are quotienting a manifold of dimension kn by the faithful action of a continuous Lie group of dimension k^2. The resulting quotient should have dimension $kn - k^2 = k(n-k)$.

Also note that the local coordinate presentation can be appropriately described in ths algebraic language. The chart $U_{1\ldots k}$ is then obtained by considering all matrices such that the leftmost $k \times k$ minor has non-zero determinant. Given such a matrix A, one can act on the left with an invertible $k \times k$ matrix in such a way that the above minor becomes the identity matrix. The remaining $k \times (n-k)$ block then gives the local coordinates presented in the previous section.

2.5.3 Plucker Embedding

We construct an injective function from the set of k-planes in $\mathbb{C}^n$ to some big projective space. We show that the image is a subvariety of projective space. We proceed to identify the Grassmannian with such subvariety.

Consider the $N = \binom{n}{k}$ dimensional projective space

$$\mathbb{P} := \mathbb{P}(\wedge^k \mathbb{C}^n).$$

Then we have a natural inclusion function:

$$pl : Gr(k,n) \to \mathbb{P}$$

given by sending a plane L to $[v_1 \wedge \ldots \wedge v_k]$ for any k-tuple of vectors spanning L. Note that if we choose a different set of k vectors $w_1, \ldots, w_k$ spanning L then the two wedge products differ by the determinant of the linear transformation sending v_i to w_i and are therefore projectively equivalent. This shows that pl is well defined.

Note that pl is injective more or less by definition. For example, if $v_1 \wedge \ldots \wedge v_k = a(w_1 \wedge \ldots \wedge w_k)$ for some non-zero constant a, one sees that each w_i is in the linear span of the v_j's by noting that

$$w_i \wedge v_1 \wedge \ldots \wedge v_k = a w_i \wedge (w_1 \wedge \ldots \wedge w_k) = 0$$

(because w_i appears twice in the wedge product).

The more interesting part of the task is now showing that the image of pl is a projective variety. I.e. finding a homogeneous ideal in the coordinates of $\mathbb{P}$ such that the associated vanishing locus is precisely the image of pl.

To do so we must go back to (multi)-linear algebra. First, recall that an element ω in $\wedge^k V$ is called **totally decomposible** if it is of the form $v_1 \wedge \ldots \wedge v_k$. Totally decomposible elements are characterized by the fact that there is a k dimensional subspace of $L \subset V$ (namely the span of the vectors that decompose ω) such that for any vector $v \in L$ we have $v \wedge \omega = 0$.

Another way to state this is that ω is totally decomposible if and only if the linear map:

$$\begin{array}{rccc} \varphi(\omega): & V & \to & \wedge^{k+1} V \\ & v & \mapsto & v \wedge \omega \end{array}$$

has rank $n-k$ (or equivalently $\leq n-k$). Consider the map:

$$\begin{array}{rccc} \varphi: & \wedge^k V & \to & Hom(V, \wedge^{k+1} V) \\ & \omega & \mapsto & \varphi(\omega) \end{array}$$

If we do things in coordinates, $\varphi(\omega)$ is represented by a $n \times \binom{n}{k+1}$ matrix with entries the homogeneous coordinates of $\mathbb{P}$, on which we are imposing the vanishing of all minors of size greater than $n-k$. We get a set of polynomial equations that cut out the image of pl , thus exhibiting the Grassmannian as a projective variety. However we note that these minors do not generate the ideal for the image of pl (i.e. the homogeneous ideal is the radical of the ideal generated by the above minors).

Exercise 18. *Follow this construction explicitly in the case of* $G(3,5)$.

In order to find generators for the homogeneous ideal of the image of pl we need to work a little harder. First we exhibit a natural identification

$$\star : \mathbb{P}(\wedge^k V) \to \mathbb{P}(\wedge^{n-k} V)$$

defined by sending the basis element corresponding to the wedge of k distinct e_i's to the wedge of the complementary e_i's. It is clear that for totally decomposible elements $\star$ sends a linear k space to its orthogonal complement.

Let us go back to our $\phi(\omega)$ and note that when ω is a totally decomposible element corresponding to a k plane W then the kernel of $\varphi(\omega)$ is precisely W. Now we consider the adjoint function to $\varphi(\omega)$ (corresponding to the transpose matrix). Adjointness means that for any $v \in V, \alpha \in \wedge^{k+1} V$:

$$\langle \varphi(\omega)(v), \alpha \rangle = \langle v, \varphi^t(\omega)(\alpha) \rangle,$$

implying that W is the orthogonal complement to the image of $\varphi^t(\omega)$.

Now we play the exact game with

$$\begin{array}{rccc} \psi(\omega): & V & \to & \wedge^{n-k+1} V \\ & v & \mapsto & v \wedge \star(\omega) \end{array}$$

Iff ω is totally decomposible (corresponding to W), then the kernel of $\psi(\omega)$ is the orthogonal complement $W^\perp$, and the image of the adjoint function $\psi^t(\omega)$ is the orthogonal complement to $W^\perp$, i.e. back to W.

At the end of the day ω is totally decomposible (and therefore is a point of the image of pl) if an only if the image of $\psi^t(\omega)$ is the orthogonal complement of the image of $\varphi^t(\omega)$. But such images are spanned by the columns of matrices whose entries are precisely the homogeneous coordinates of $\mathbb{P}$ (in the case of $\psi^t(\omega)$ after the identification given by $\star$).

For each pair of columns (one for each matrix of course) we get a quadratic homogeneous polynomial in the coordinates of $\mathbb{P}$ that the Grassmannian satisfies. The collection of all these relations are called **Plucker relations** and they generate the ideal of the image of pl.

Exercise 19. *Follow this construction explicitly in the case of* $G(3,5)$.

2.6 Elliptic Curves, Complex Tori and Plane Cubics.

An **elliptic curve** is a genus 1 smooth projective curve together with the choice of a point on it that we call 0. A complex torus is a complex analytic manifold of dimension one obtained by quotienting $\mathbb{C}$ by a non-degenerate lattice $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$. In this section we understand how there is a natural isomorphism between the two moduli functors. Since the two moduli spaces can be constructed independently in each of the two categories, then we have two very different ways to view the "same" moduli problem. We further make the connection with a third point of view, which is to think of these objects as projective equivalence classes of plane cubics.

We collect here some theorems from the theory of Riemann Surfaces:

genus-degree formula if X is a smooth projective plane curve of degree d then:

$$g_X = \binom{d-1}{2}$$

Bezout's Theorem if X, Y are projective plane curves of degrees d_X, d_Y, then

$$|X \cap Y| = d_X d_Y,$$

where the intersection points are counted with appropriate multiplicities.

Riemann-Hurwitz if $\pi : X \to Y$ is a map of Riemann surfaces of degree d, and ν is the degree of the ramification divisor, then

$$2g_X - 2 = d(2g_Y - 2) + \nu.$$

Riemann-Roch if D is a divisor on a smooth projective curve X, then

$$h^0(D) - h^1(D) = deg(D) + 1 - g_X.$$

Serre duality for any divisor D on a smooth projective curve X,

$$H^0(D) \cong H^1(K_X - D)^\vee$$

2.6.1 Take I: Complex Tori

A complex torus is obtained by taking the quotient space of $\mathbb{C}$ by a non-degenerate lattice Λ. We can think of Λ acting on the complex plane by translation. One therefore sees that the action is free and properly discontinuous, and since the complex plane is a complex manifold in the most obvious of ways, then the quotient space is also a complex manifold.

Remarks:

1. Topologically $\mathbb{C}/\Lambda$ is a torus (aka it has genus 1).

2. Different lattices can (and typically) induce different complex structures on the torus. Therefore we are interested on classifying these complex structures up to isomorphism.

Fact 1: Two complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic (eg biholomorphic) if there exists a biholomorphic function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(\Lambda_1) = \Lambda_2$$

Up to composing by a translation, we can assume that such maps preserve the origin. The only survising maps are then multiplication by nonzero scalars.

Intuition: Now we do something that you all have done thousands of times without paying attention to it, but is in fact an example of the philosophy of moduli spaces. At this point we want to classify lattices Λ up to the equivalence relation discussed in Fact 1. Since lattices are somewhat floppy and complicated to describe, we " add structure" and instead parameterize a lattice together with two vectors that generate it. Now it is quite easy to parameterize these, as all we have to do is choose two $\mathbb{R}$-linearly independent vectors in the plane. But we have the problem that there is not a unique pair of generators for a lattice, so we have to further quotient out by the action of a discrete group whose orbits correspond precisely to different choices of generators for the same lattice.

Fact 2: (v_1, v_2) and (w_1, w_2) are generators for the same lattice Λ if and only if there is a matrix $A = [a_{ij}] \in GL(2, \mathbb{Z})$ such that

$$w_i = \sum_j a_{ij} v_j$$

Putting together the two facts we obtain that the moduli space we are after is:

$$\frac{\mathbb{C} \times \mathbb{C} \smallsetminus \{\theta_1 = \theta_2\}}{\mathbb{C}^* \times GL(2, \mathbb{Z})}$$

Now, this orbit space is rightfully a bit complicated looking, so we try to do the quotienting a bit at a time.

step 1: First of all up to using the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we can assume that the angle between v_1 and v_2 is more than π.

step 2: now we can spend the $\mathbb{C}^\star$ rotation to assume $v_1 = \tau$, $v_2 = 1 \in \mathcal{H}$.

step 3: we now have a residual action of $SL(2,\mathbb{Z})$. Check that the action on τ is $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tau\right) = \frac{a\tau+b}{c\tau+d}$.

We finally expressed again our moduli space in terms of the much simpler quotient:

$$\frac{\mathcal{H}}{SL(2,\mathbb{Z})}$$

Exercise 20. *Find a fundamental domain for the above quotient.*

2.6.2 Take II: Plane Cubics

The degree formula tells us that a smooth projective plane cubic has genus 1. Any two curves related by a projectivity of the plane are certainly isomorphic. We therefore want to study the moduli problem of $s$mooth cubic plane curves up to projective equivalence.

There is a $\mathbb{P}^9$ worth of cubic curves. Smoothness is an open condition: the locus of non-smooth curves is a degree 12 hypersurface in $\mathbb{P}^9$ called the **discriminant variety**. It can be obtained by imposing the vanishing of two of the three partial derivatives of the universal cubic equation inside $\mathbb{P}^9 \times \mathbb{P}^2$ and then projecting down to $\mathbb{P}^9$. But the point is that we have an open set U of smooth cubics. We act on this set by projectivities, i.e. by projective classes of 3×3 invertible matrices. The parameter space we are after is then:

$$\mathcal{SC} := U/\mathbb{P}GL(3)$$

One quick observation we can make is that the dimension of such a space should be $1 = 9 - 8$. Of course just by looking at the above quotient we are kind of intimidated: we need to quotient a complicated open set by a complicated group. Rather than taking the quotient in one fell swoop, we proceed in steps to take a general equation and reduce it to a **standard form**. Let $C = \sum_{i+j+k=3} a_{ijk} X^i Y^j Z^k$ be a general smooth cubic.

step 1: find an inflection point P. Use a projectivity to send P to $(0:1:0)$ and the inflection tangent to the line $Z = 0$. This ensures that the cubic term of the affine equation in the (x, y) plane is just a multiple of x^3. We can even assume the coefficient to be 1.

step 2: use an appropriate transformation of the form $x \mapsto x, y \mapsto Ax + y$ to get rid of the xy term.

step 3: complete the square in x to get rid of the x^2 term. We can also arrange for the y^2 coefficient to be 1.At this point we have the equation in the form

$$y^2 = x^3 + \alpha x + \beta$$

step 4: it might seem like we are done, but a tranformation of the form $x \mapsto A^2 x, y \mapsto A^3 y$ still preserves the above form, giving the equation:

$$y^2 = x^3 + \frac{A^2\alpha}{A^6}x + \frac{\beta}{A^6}$$

By choosing $A = \alpha^{1/4}$, we can then get the x coefficient to be 1:

$$y^2 = x^3 + x + \frac{\beta}{\alpha^{3/2}}$$

step 5: we are now almost done, other than the fact that we can still act by $x \mapsto -x, y \mapsto iy$. Or in other words, that there are two square roots of α^3. Therefore the set in bijection with the objects we want is the set of pairs $\{\pm\frac{\beta}{\alpha^{3/2}}\}$. One way to get rid of this annoyance is to just take as a parameter the square of such quantity:

$$J := \frac{\beta^2}{\alpha^3}$$

After much huffing and puffing we have seen that the moduli space we are after is just a complex line parameterized by J. We will see later on that this is just a coarse moduli space. But for now let us move on to a different point of view.

2.6.3 Take III: Elliptic Curves

Let us now take an abstract approach to elliptic curves. First of all we study the automorphisms of an elliptic curve:

Lemma 1. *Let E be a smooth projective genus 1 curve. For any two points $P, Q \in E$ there is an automorphism $\sigma : E \to E$ such that:*

1. $\sigma^2 = \mathbb{1}$*;*
2. $\sigma(P) = Q$*;*
3. $R + \sigma(R) \sim P + Q$ *for any other point $R \in E$.*

Proof: Consider the divisor $D = P+Q$. By Riemann-Roch and Serre duality $h^0(D) = 2$. Therefore D defines a degree 2 map $\varphi_D : E \to \mathbb{P}^1$ such that P, Q are the preimages of the point $\infty \in \mathbb{P}^1$. This map embeds the field of rational functions $\mathbb{C}(t)$ into the field $\mathbb{C}(E)$ as a Galois extension with Galois group of order 2. We can take σ to be the nontrivial element of $Gal(\mathbb{C}(E)/\mathbb{C}(\mathbb{P}^1))$.

We see as a consequence that the group of automorphisms of a genus 1 curve is transitive, and in particular that any point can be choosen as our origin.

Intuition: We seem to be doing something silly by choosing the origin, but in fact this is an example of rigidifying the problem: the isomorphism classes of objects parameterized are the same, but by choosing an origin we have killed the infinite group of automorphisms that every object had. The content of the next lemma is that this degree 2 map to the projective line characterizes the genus 1 curve.

Lemma 2. *If f_1 and f_2 are two double covers $E \to \mathbb{P}^1$, then there is are automorphisms of E and of $\mathbb{P}^1$ that make the following square commute:*

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ f_1 \downarrow & & \downarrow f_2 \\ \mathbb{P}^1 & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

Proof: Choose P_1 and P_2 to be ramification points for f_1 and f_2. Let σ be the automorphism of E that takes P_1 to P_2. The maps f_1 and f_2 are determined by the linear systems $|2P_1|$ and $|2P_2|$, respectively. Therefore $f_2\sigma$ and f_1 differ by a choice of basis of $H^0(D)$ i.e. by an automorphism of $\mathbb{P}^1$.

We have discovered that any genus 1 curve admits a double cover to $\mathbb{P}^1$ and that once we mark a point to make the genus 1 curve into an elliptic curve, then the linear system giving rise to the double cover is unique.

So we want to parameterize elliptic curves by parameterizing double covers of $\mathbb{P}^1$ up to reparametrizations of $\mathbb{P}^1$. By the Riemann-Hurwitz formula a double cover of $\mathbb{P}_1$ has four branch points, which also uniquely determine the double cover.

Therefore:

$$\{Elliptic\, curves\} \longleftrightarrow \left\{4 \; points \; of \; \mathbb{P}^1 \; one \; of \; which \; is \; marked.\right\}/\mathbb{P}GL(2)$$

Since it is hairy to parameterize quartuple of points only one of which is special, we rigidify the problem and mark all four points. We then have to quotient out by an action of the symmetric group S_3 that permutes the three points that we do not want to consider special. So what we are after is:

$$\left\{Quartuples \; of \; marked \; points \; of \; \mathbb{P}^1\right\}/\mathbb{P}GL(2)/S_3$$

First we take care of the quotient by automorphisms of $\mathbb{P}^1$. We recall that there for any three points in $\mathbb{P}_1$ there is a unique mobius transformation that sends them to any other three points. We can therefore assume that the first three points are $0, 1, \infty$. Then the fourth point λ uniquely determines the isomorphism class of quartuples, and the quotient we are after is:

$$\frac{\mathbb{P}^1 \smallsetminus \{0,1,\infty\}}{S_3} = Spec\left(\mathbb{C}\left[t, \frac{1}{t}, \frac{1}{t-1}\right]\right)^{S_3}$$

Exercise 21. *Figure out how the symmetric group acts on the points of* $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$. *In particular, for a point* λ *show that the orbit is:*

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}$$

The invariant functions for the S_3 action are generated by the elementary symmetric functions in the above six rational functions.

Exercise 22. *Show that:*

$$\begin{aligned}
\sigma_0 = \sigma_6 &= 1 \\
\sigma_1 = \sigma_5 &= 3 \\
\sigma_2 = \sigma_4 = -&\frac{t^6 - 3t^5 + 5t^3 - 3t + 1}{t^2(t-1)^2} \\
\sigma_3 = &\frac{2t^6 - 6t^5 + 5t^4 + 5t^2 - 6t + 2}{t^2(t-1)^2}
\end{aligned}$$

and that σ_2 *and* σ_3 *satisfy the linear relation*

$$2\sigma_2 + \sigma_3 = 5$$

Therefore the moduli space we are looking after is:

$$Spec(\mathbb{C}[X,Y]/2X+Y-5) = \mathbb{C}$$

which is a fancy way to say it is the affine line. For historical reasons that I do not know the parameter for this line has been chosen to be neither X nor Y but

$$j = 2^8(X+Y+1) = 2^8\frac{\lambda^2-\lambda+1}{\lambda^2(1-\lambda)^2},$$

the classical j-invariant of an elliptic curve.

2.7 Equivalences

Here we whirlwind sketch why the above three problems are equivalent, without paying much attention to making any of the arguments either rigorous or complete. The idea is that this section should contain enough pointers and informations for any of you to be able to formulate precise arguments if you wish or need to do so.

First off, any smooth plane cubic is a genus one curve, just by the degree formula. The moment we choose an inflexion point and tangent and send that to the point/line at infinity, we are picking an origin, and therefore choosing an elliptic curve associated to the plane cubic.

Viceversa, if you start from an elliptic curve with origin P_0, the linear system $|3P_0|$ gives you a degree three embedding in $\mathbb{P}^2$. Another very concrete way of thinking about this is looking at the linear system $|6P_0|$ and note that there are 7 natural candidates to be in such a linear system (coming from powers and products of elements in $|3P_0|$. Since Riemann-Roch tells us that the linear system $|6P_0|$ must have dimension 6 there must be a linear relation among these 7 functions, giving precisely a cubic equation in the basis elements of $|3P_0|$.

From complex tori to plane cubics one must look at the theory of elliptic functions. For any given Λ Weierstrass gave us a Λ-periodic function:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

It is a fact that the field of elliptic functions is generated by $\wp$ and its derivative $\wp'$, and that these functions satifsy the algebraic relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

The map

$$T \to \mathbb{C}^2$$

sending $z \mapsto (\wp(z), \wp'(z))$ gives an embedding of the complex torus as an affine plane cubic, thus providing the connection with the algebraic world.

3 First Main Character: Rational Pointed Curves

3.1 Moduli of n Points on $\mathbb{P}^1$

Let us now consider the moduli space $M_{0,n}$ of all isomorphism classes of n ordered distinct marked points $p_i \in \mathbb{P}^1$. The subscript 0 is to denote the genus of our curve $\mathbb{P}^1$. Since the automorphism group $Aut(\mathbb{P}^1) = \mathbb{P}GL_2(\mathbb{C})$ allows us to move any three points on $\mathbb{P}^1$ to the ordered triple $(0, 1, \infty)$, the space $M_{0,n}$ reduces to a single point for $n \leq 3$.

Intuition: In fact there is a very delicate issue here. For $0, 1, 2$ marked points the objects we wish to parameterize still have many automorphisms. So the statement above refers to the coarse moduli space being just a point. The fine moduli space does not exist as a scheme. In fact they are negative dimensional Artin stacks, but this is another story...

Going one step up, $M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$: given a quadruple (p_1,p_2,p_3,p_4), we can always perform the unique automorphism of $\mathbb{P}^1$ sending (p_1,p_2,p_3) to $(0,1,\infty)$; the isomorphism class of the quadruple is then determined by the image of the fourth point.

The general case is similar. Any n-tuple $\underline{p} = (p_1,\dots,p_n)$ is equivalent to a n-tuple of the form $(0,1,\infty,\phi(p_4),\dots,\phi(p_n))$, where ϕ is the unique automorphism of $\mathbb{P}^1$ sending (p_1,p_2,p_3) to $(0,1,\infty)$. This shows

$$M_{0,n} = \overbrace{M_{0,4} \times ... \times M_{0,4}}^{n-3 \text{ times}} \setminus \{\text{all diagonals}\}.$$

If we define $U_n := M_{0,n} \times \mathbb{P}^1$, then the projection of U_n onto the first factor gives rise to a universal family

$$\begin{array}{c} U_n \\ \pi \downarrow\uparrow \sigma_i \\ M_{0,n} \end{array}$$

where the σ_i's are the universal sections:

- $\sigma_i(\underline{p}) = (\underline{p}, \phi(p_i)) \in U_n$.

This family is tautological since the fibre over a moduli point, which is the class of a marked curve, is the marked curve itself.
With U_n as its universal family, $M_{0,n}$ becomes a fine moduli space for isomorphism classes of n ordered distinct marked points on $\mathbb{P}^1$.

This is all fine except $M_{0,n}$ is not compact for $n \geq 4$. There are many reasons why compactness is an extremely desirable property for moduli spaces. As an extremely practical reason, proper (and if possible projective) varieties are much better behaved and understood than non compact ones. Also, a compact moduli space encodes information on how our objects can degenerate in families. For example, what happens when $p_1 \to p_2$ in $M_{0,4}$?

In general there are many ways to compactify a space. A "good" compactification $\overline{\mathcal{M}}$ of a moduli space $\mathcal{M}$ should have the following properties:

1. $\overline{\mathcal{M}}$ should be itself a moduli space, parametrizing some natural generalization of the objects of $\mathcal{M}$.

2. $\overline{\mathcal{M}}$ should not be a horribly singular space.

3. the boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ should be a normal crossing divisor.

4. it should be possible to describe boundary strata combinatorially in terms of simpler objects. This point may appear mysterious, but it will be clarified by the examples of stable curves and stable maps.

In the case of rational n-pointed curves there is a definite winner among compactifications.

3.2 Moduli of Rational Stable Curves

We will discuss the simple example of $M_{0,4}$; this hopefully will, without submerging us in combinatorial technicalities, provide intuition on the ideas and techniques used to compactify the moduli spaces of n-pointed rational curves.

A natural first attempt would be to just allow the points to come together, i.e. enlarge the collection of objects that we are considering from $\mathbb{P}^1$ with n ordered distinct marked points to $\mathbb{P}^1$ with n ordered, not necessarily distinct, marked points.

However, this will not quite work. For instance, consider the families

$$C_t = (0, 1, \infty, t) \quad \text{and} \quad D_t = (0, t^{-1}, \infty, 1).$$

For each $t \neq 0$, up to an automorphism of $\mathbb{P}^1$, $C_t = D_t$, thus corresponding to the same point in $M_{0,4}$. But for $t = 0$, C_0 has $p_1 = p_4$ whereas D_0 has $p_2 = p_3$. These configurations are certainly not equivalent up to an automorphism of $\mathbb{P}^1$ and so should be considered as distinct points in our compactification of $M_{0,4}$. Thus, we have a family with two distinct limit points (in technical terms we say that the space is **nonseparated**). This is not good.

Our failed attempt was not completely worthless though since it allowed us to understand that we want the condition $p_1 = p_4$ to coincide with $p_2 = p_3$, and likewise for the other two possible disjoint pairs. On the one hand this is very promising: 3 is the number of points needed to compactify $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to $\mathbb{P}^1$. On the other hand, it is now mysterious what modular interpretation to give to this compactification.

To do so, let us turn carefully to our universal family, illustrated in Figure 1. The natural first step is to fill in the three points on the base, to complete U_4 to $\mathbb{P}^1 \times \mathbb{P}^1$ and extend the sections by continuity.

We immediately notice a bothersome asymmetry in this picture: the point p_4 is the only one allowed to come together with all the other points: yet common sense, backed up by the explicit example just presented, suggests that there should be democracy among the four points. This fails where the diagonal section σ_4 intersects the three constant ones, i.e. at the three points $(0, 0)$, $(1, 1)$, (∞, ∞). Let us blow-up $\mathbb{P}^1 \times \mathbb{P}^1$ at these three points. This will make all the sections disjoint, and still preserve the smoothness and projectivity of our universal family.
The fibres over the three exceptional points are $\mathbb{P}^1 \cup E_i$: nodal rational curves. These are the new objects that we have to allow in order to obtain a good compactification of $M_{0,4}$.
Let us finally put everything together, and state things carefully.

Definition 6. *A* ***tree of projective lines*** *is a connected curve with the following properties:*

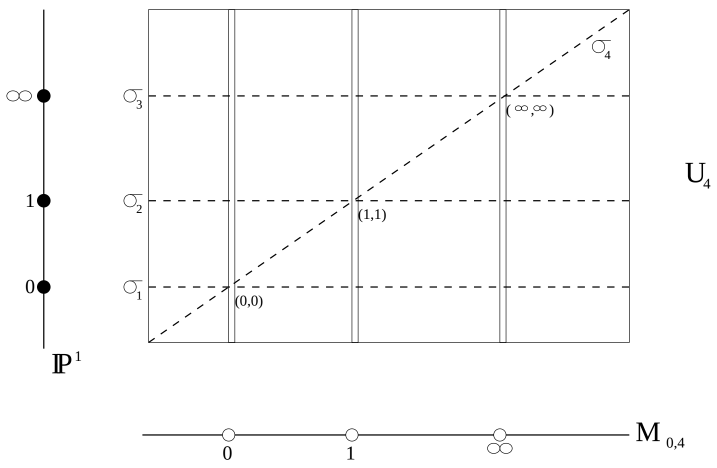

Figure 1: first attempt at compactifying U_4

1. *Each irreducible component is isomorphic to* $\mathbb{P}^1$.

2. *The points of intersection of the components are ordinary double points.*

3. *There are no closed circuits, i.e., if a node is removed then the curve becomes disconnected.*

These three properties are equivalent to saying that the curve has arithmetic genus zero. Each irreducible component will be called a twig. We will often draw a marked tree as in fig 2, where each line represents a twig.

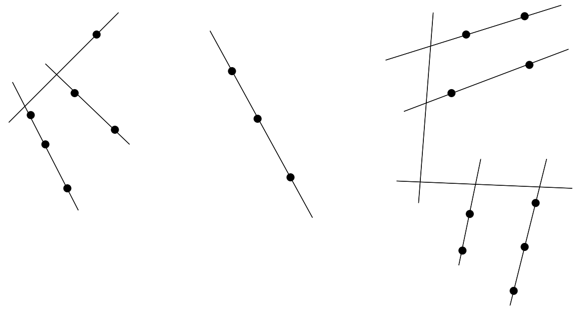

Figure 2: stable marked trees.

Definition 7. *A marked tree is* ***stable*** *if every twig has at least three special points (marks or nodes).*

This stability condition is equivalent to the existence of no nontrivial automorphisms of the tree that fix all of the marks.

Definition 8. $\overline{M}_{0,4} \cong \mathbb{P}^1$ *is the moduli space of isomorphism classes of four pointed stable trees. It is a fine moduli space, with universal family* $U_4 = Bl(\mathbb{P}^1 \times \mathbb{P}^1)$.

These results generalize to larger n.

Fact: The space $\overline{M}_{0,n}$ of n-pointed rational stable curves is a fine moduli space compactifying $M_{0,n}$. It is projective, and the universal family $\overline{U}_n$ is obtained from U_n via a finite sequence of blow-ups. (In particular all the diagonals need to be blown up in an appropriate order) . For further details see [KV07] or [?], [?].

One of the exciting features of this theory is that all these spaces are related to one another by natural morphisms. Consider the map

$$\pi_i : \overline{M}_{0,n+1} \to \overline{M}_{0,n},$$

defined by forgetting the ith mark. It is obviously defined if the ith mark does not belong to a twig with only three special points. If it does belong to such a twig, then our resulting tree will no longer be stable. In this case, we must perform what is called contraction.

Contraction: We need to consider two cases:

1. The remaining two special points are both nodes. We make the tree again stable by contracting this twig so that the two nodes are now one (see Figure 3).

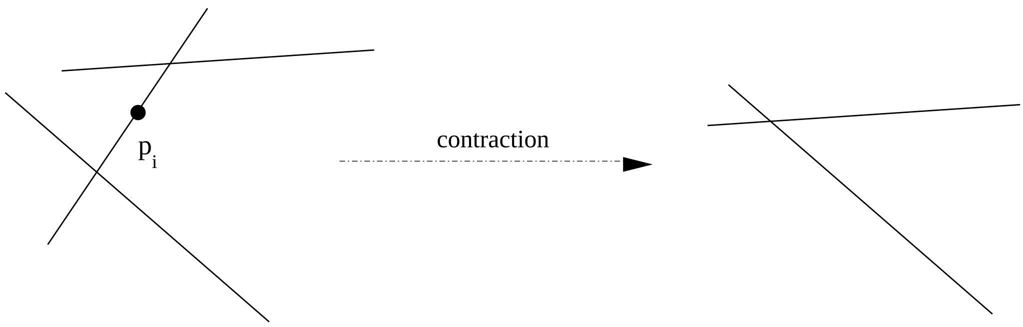

Figure 3: contracting a twig with only two nodes.

2. There is one other mark and one node on the twig in question. We make the tree stable by forgetting the twig and placing the mark where the node used to be (Figure4).

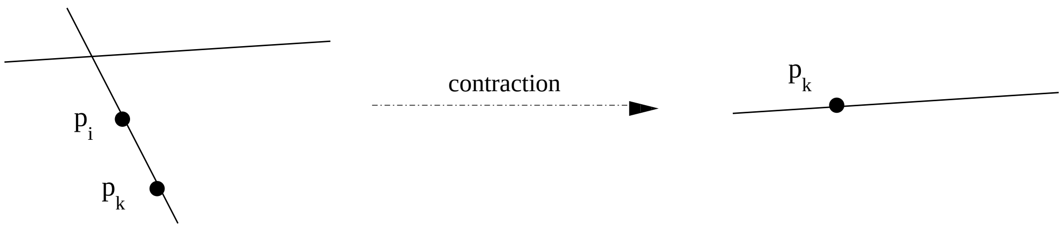

Figure 4: contracting a twig with one node and one mark.

We would like to construct a section σ_i of the family

$$\begin{array}{c} \overline{M}_{0,n+1} \\ \pi_k \downarrow\uparrow \sigma_i \\ \overline{M}_{0,n} \end{array}$$

by defining the kth mark to coincide with ith one. It should trouble you that in doing so we are not considering curves with distinct marked points, but we can get around this problem by "sprouting" a new twig so that the node is now where the ith mark was. The kth and the ith points now belong to this new twig.
This process of making stable a tree with two coinciding points is called **stabilization**.

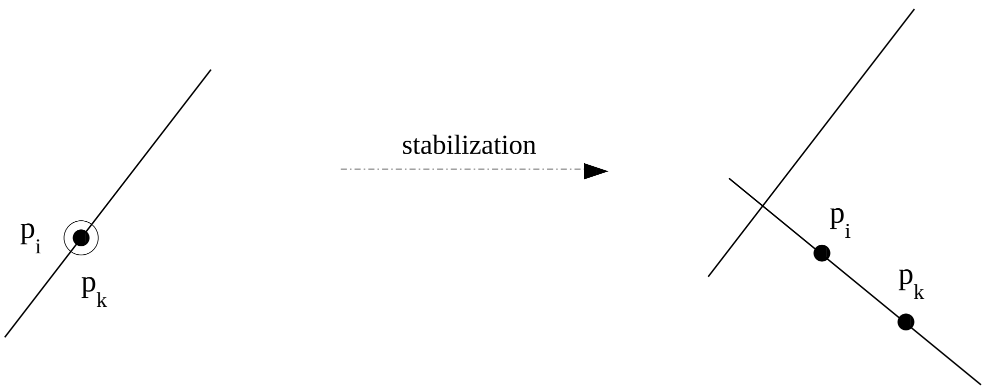

Figure 5: stabilization

Finally, we may now identify our universal family

$$\begin{array}{c} \overline{U}_n \\ \pi \downarrow \\ \overline{M}_{0,n} \end{array} \quad \text{with the family} \quad \begin{array}{c} \overline{M}_{0,n+1} \\ \pi_i \downarrow \\ \overline{M}_{0,n} \end{array}$$

as follows.
The fibre $\pi^{-1}([(C, p_1, \cdots, p_n)]) \subset \overline{U}_n$ is the marked curve itself. So any

point $p \in \overline{U}_n$ belonging to the fibre over C is actually a point on the stable n-pointed tree C, and may therefore be considered as an additional mark; stabilization may be necessary to ensure that our new $(n+1)$-marked tree is stable. Vice-versa, given an $(n+1)$ pointed curve C', we can think of the $(n+1)$st point as being a point on the n-marked curve obtained by forgetting the last marked point (eventually contracting, if needed); this way C' corresponds to a point on the universal family $\overline{U_n}$. These constructions realize an isomorphism between $\overline{U}_n$ and $\overline{M}_{0,n+1}$.

3.2.1 The boundary

We define the boundary to be the complement of $M_{0,n}$ in $\overline{M}_{0,n}$. It consists of all nodal stable curves.

Fact: the boundary is a union of irreducible components, corresponding to the different possible ways of arranging the marks on the various twigs; the codimension of a boundary component equals the number of nodes in the curves in that component. See [KV07] for more details.

The codimension 1 boundary strata of $\overline{M}_{0,n}$, called the **boundary divisors**, are in one-to-one correspondence with all ways of partitioning $[n] = A \cup B$ with the cardinality of both A and B strictly greater than 1.
A somewhat special class of boundary divisors consists of those with only two marked points on a twig. Together, these components are sometimes called the **soft boundary** and denoted by $D_{i,j}$. We can think of $D_{i,j}$ as the image of the ith section, σ_i, of the jth forgetful map, π_j (or vice-versa).

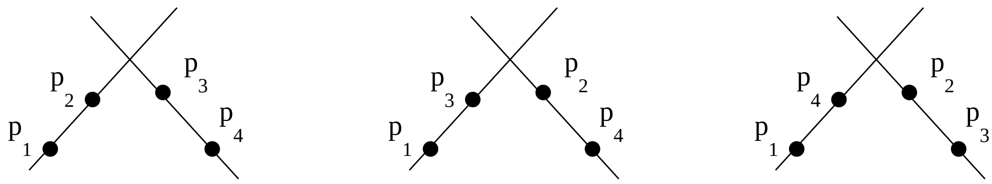

Figure 6: irreducible components of the boundary of $\overline{M}_{0,4}$

In Figures 6 and 7 we draw all boundary strata for $\overline{M}_{0,4}$ and $\overline{M}_{0,5}$.

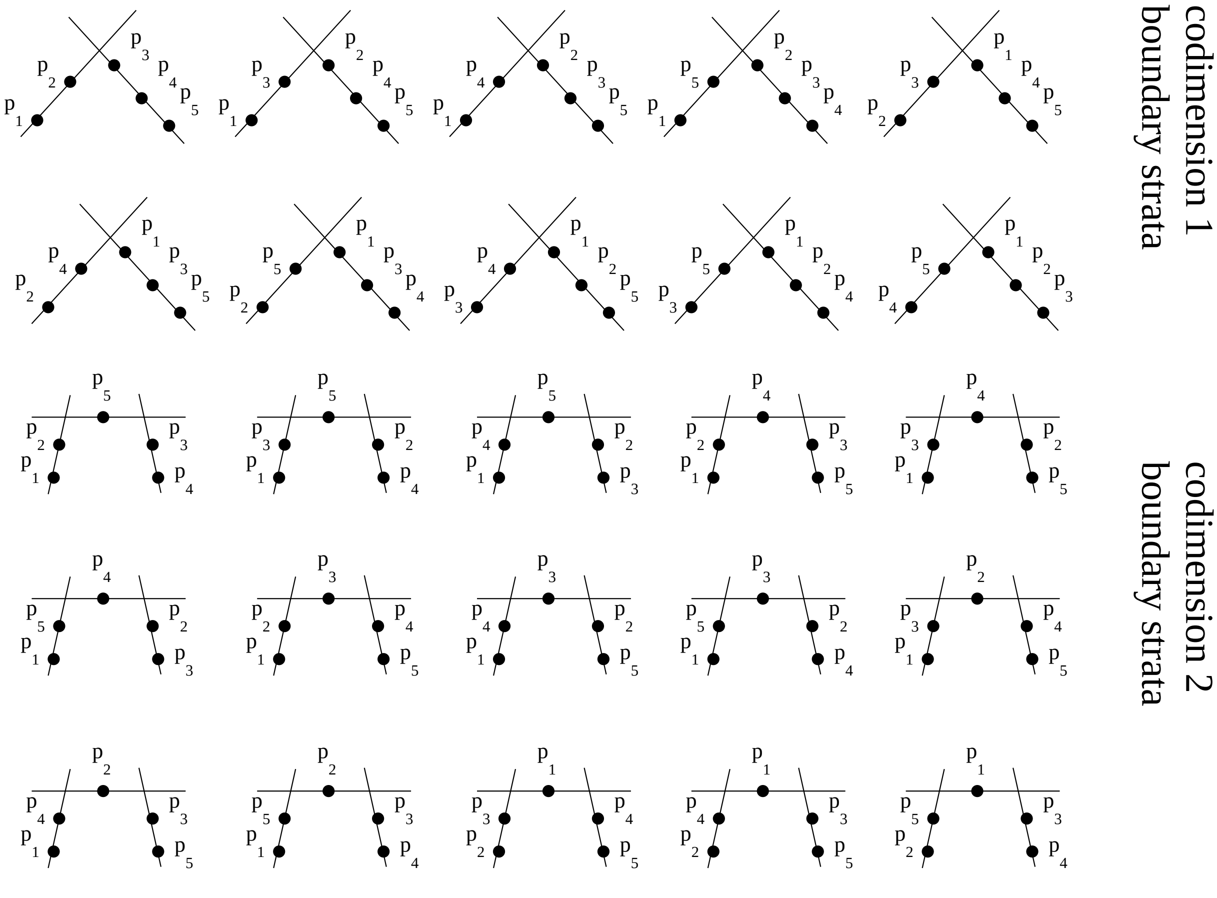

Figure 7: boundary cycles of $\overline{M}_{0,5}$

3.3 Constructions of $\overline{M}_{0,n}$

3.3.1 Knudsen's Construction

3.3.2 Keel's Construction

3.3.3 Kapranov's Construction

3.4 Intersection Theory on $\overline{M}_{0,n}$

3.4.1 A quick and dirty introduction to intersection theory

The main character here is the Chow ring, $A^*(X)$, of an algebraic variety X. The ring[1] $A^*(X)$ is, in some loose sense, the algebraic counterpart of the cohomology ring $H^*(X)$, and it allows us to make precise in the algebraic category the intuitive concepts of oriented intersection in topology.

We think of elements of the group $A^n(X)$ as formal finite sums of codimension n closed subvarieties (cycles), modulo an equivalence relation called rational equivalence. $A^*(X) = \oplus_0^{dim X} A^n(X)$ is a graded ring with product given by intersection.

Intersection is independent of the choice of representatives for the equivalence classes.

In topology, if we are interested in the cup product of two cohomology classes $\mathbf{a}$ and $\mathbf{b}$, we can choose representatives a and b that are transverse to each other. We can assume this since transversality is a generic condition: if a and b are not transverse then we can perturb them ever so slightly and make them transverse while not changing their classes. This being the case, then $a \cap b$ represents the cup product class $\mathbf{a} \cup \mathbf{b}$.

In algebraic geometry, even though this idea must remain the backbone of our intuition, things are a bit trickier. We will soon see examples of cycles that are rigid, in the sense that their representative is unique, and hence "unwigglable". Transversality then becomes an unattainable dream. Still, with the help of substantially sophisticated machinery (the interested reader can consult [Ful98]), we can define an algebraic version of intersection classes and a product that reduces to the "geometric" one when transversality can be achieved.

Throughout this paper, a bolded symbol will represent a class, the unbolded symbol a geometric representative. The intersection of two classes $\mathbf{a}$ and $\mathbf{b}$ will be denoted by $\mathbf{ab}$.

Example: the Chow Ring of Projective Space.

$$A^*(\mathbb{P}^n) = \frac{\mathbb{C}[\mathbf{H}]}{(\mathbf{H}^{n+1})},$$

[1] This is probably our greatest sloppiness. In order for $A^*(X)$ to be a ring we need X to be smooth. Since the spaces we will actually work with satisfy these hypotheses, we do not feel too guilty.

where $\mathbf{H} \in A^1(\mathbb{P}^n)$ is the class of a hyperplane H.

3.4.2 Characteristic Classes of Bundles

For every vector bundle there is a natural section $s_0 : B \to \mathcal{E}$ defined by

$$s_0(b) = (b, 0) \in \{b\} \times \mathbb{C}^n.$$

It is called the zero section, and it gives an embedding of B into $\mathcal{E}$.

A natural question to ask is if there exists another section $s : B \to \mathcal{E}$ which is disjoint form the zero section, i.e. $s(b) \neq s_0(b)$ for all $b \in B$. The **Euler class** of this vector bundle ($\mathbf{e}(\mathcal{E}) \in A^n(B)$) is defined to be the class of the self-intersection of the zero section: it measures obstructions for the above question to be answered affirmatively. This means that $\mathbf{e}(\mathcal{E}) = 0$ if and only if a never vanishing section exists. It easily follows from the Poincaré-Hopf theorem that for a manifold M, the following formula holds:

$$\mathbf{e}(TM) \cap [M] = \chi(M).$$

That is, the degree of the Euler class of the tangent bundle is the Euler characteristic.

The Euler class of a vector bundle is the first and most important example of a whole family of "special" cohomology classes associated to a bundle, called the **Chern classes** of $\mathcal{E}$. The k-th Chern class of $\mathcal{E}$, denoted $\mathbf{c}_k(\mathcal{E})$, lives in $A^k(B)$. In the literature you can find a wealth of definitions for Chern classes, some more geometric, dealing with obstructions to finding a certain number of linearly independent sections of the bundle, some purely algebraic. Such formal definitions, as important as they are (because they assure us that we are talking about something that actually exists!), are not particularly illuminating. In concrete terms, what you really need to know is that Chern classes are cohomology classes associated to a vector bundle that satisfy a series of really nice properties, which we are about to recall.

Let $\mathcal{E}$ be a vector bundle of rank n:

identity: by definition, $\mathbf{c}_0(\mathcal{E}) = 1$.

normalization: the n-th Chern class of $\mathcal{E}$ is the Euler class:

$$\mathbf{c}_n(\mathcal{E}) = \mathbf{e}(\mathcal{E}).$$

vanishing: for all $k > n$, $\mathbf{c}_k(\mathcal{E}) = 0$.

pull-back: Chern classes commute with pull-backs:

$$f^*\mathbf{c}_k(\mathcal{E}) = \mathbf{c}_k(f^*\mathcal{E}).$$

tensor products: if L_1 and L_2 are line bundles,

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Whitney formula: for every extension of bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

the k-th Chern class of $\mathcal{E}$ can be computed in terms of the Chern classes of $\mathcal{E}'$ and $\mathcal{E}''$, by the following formula:

$$\mathbf{c}_k(\mathcal{E}) = \sum_{i+j=k} \mathbf{c}_i(\mathcal{E}')\mathbf{c}_j(\mathcal{E}'').$$

Using the above properties it is immediate to see:

1. all the Chern classes of a trivial bundle vanish (except the 0-th, of course);
2. for a line bundle L, $\mathbf{c}_1(L^*) = -\mathbf{c}_1(L)$.

To show how to use these properties to work with Chern classes, we will now calculate the first Chern class of the tautological line bundle over $\mathbb{P}^1$. The tautological line bundle is

$$\begin{array}{c} \mathcal{S} \\ \pi \downarrow \\ \mathbb{P}^1, \end{array}$$

where $\mathcal{S} = \{(p, l) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid p \in l\}$. It is called tautological because the fiber over a point in $\mathbb{P}^1$ is the line that point represents.

Our tautological family fits into the short exact sequence of vector bundles over $\mathbb{P}^1$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{S} & \to & \mathbb{C}^2 \times \mathbb{P}^1 & \to & \mathcal{Q} & \to & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & \mathbb{P}^1 & & & & \end{array}$$

where $\mathcal{Q}$ is the bundle whose fibre over a line $l \in \mathbb{P}^1$ is the quotient vector space $\mathbb{C}^2/l$. Notice that $\mathcal{Q}$ is also a line bundle. From the above sequence, we have that

$$0 = \mathbf{c}_1(\mathbb{C}^2 \times \mathbb{P}^1) = \mathbf{c}_1(\mathcal{S}) + \mathbf{c}_1(\mathcal{Q}). \tag{1}$$

Since $\mathbb{P}^1$ is topologically a sphere, which has Euler characteristic 2, then

$$2 = \mathbf{c}_1(T\mathbb{P}^1) = \mathbf{c}_1(\mathcal{S}^*) + \mathbf{c}_1(\mathcal{Q}) = -\mathbf{c}_1(\mathcal{S}) + \mathbf{c}_1(\mathcal{Q}). \tag{2}$$

The second equality in 2 holds because $T\mathbb{P}^1$ is the line bundle $Hom(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$.

It now follows from (1) and (2) that $\mathbf{c}_1(\mathcal{S}) = -1$.

3.4.3 The Chow ring of $\overline{M}_{0,n}$

References

[Ful98] William Fulton. *Intersection Theory*. Springer, second edition, 1998.

[KV07] Joachim Kock and Israel Vainsencher. *An invitation to quantum cohomology*, volume 249 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2007. Kontsevich's formula for rational plane curves.