K-Theory **12:** 3–21, 1997.

Orbifolds, Sheaves and Groupoids

Dedicated to the memory of Bob Thomason

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(Received: December 1995)

Abstract. We characterize orbifolds in terms of their sheaves, and show that orbifolds correspond exactly to a specific class of smooth groupoids. As an application, we construct fibered products of orbifolds and prove a change-of-base formula for sheaf cohomology.

Mathematics Subject Classifications (1991): 22A22, 57S15, 55N30, 18B25.

Key words: orbifold, smooth groupoid, sheaf, equivariant cohomology.

The purpose of this paper is to give a characterization of orbifolds, or V-manifolds (Satake [15] and [16], Thurston [18]) in terms of their sheaves. A smooth manifold is completely determined by all its sheaves, with the smooth structure given by the particular sheaf of germs of smooth functions. We will show that an analogous result is true for orbifolds, and we characterize in various ways the categories of sheaves that so arise (Theorem 4.1 below). One such characterization is in terms of smooth groupoids (Connes [2, Def. II.5.2]). Each such groupoid $\mathcal{G}$ determines a category of $\mathcal{G}$-equivariant sheaves. We prove that this is the category of sheaves on some (uniquely determined) orbifold iff $\mathcal{G}$ is a groupoid of germs of diffeomorphisms with the property that the source and target maps of $\mathcal{G}$ jointly define a *proper* map $(s,t): G_1 \to G_0 \times G_0$. (Here G_0 and G_1 denote the spaces of objects and arrows of the groupoid.) Conversely, we construct for any orbifold such a groupoid $\mathcal{G}$ determining the same category of sheaves. This result shows that orbifolds are essentially the same as such ‘proper’ groupoids. (One can also easily reformulate this condition in terms of structures close to groupoids, such as pseudogroups [18, Def. 3.1.1] or S-atlases [3].) For a precise formulation and other such characterizations, we refer to Theorem 4.1 below.

These characterizations naturally lead to various applications. For example, using mappings between groupoids, we will show in Section 5 that one can construct fibered products of orbifolds in terms of ‘proper’ groupoids, and we prove a change-of-base formula for the sheaf cohomology of orbifolds, for fibered products along proper maps between orbifolds. Further applications will be described in a sequel [11] to this paper. Among other things, we will show that for an orbifold $\mathcal{M}$ and corresponding (i.e., having the same sheaves) groupoid $\mathcal{G}$, the sheaf cohomology of $\mathcal{M}$ with locally constant coefficients agrees with the ordinary (twisted) cohomology of the classifying space $B\mathcal{G}$. It will also be shown that the fundamental group of an orbifold $\mathcal{M}$ (as defined in [18, Def. 5.3.5]) is isomorphic to the Grothendieck fundamental group of locally constant sheaves on $\mathcal{M}$, and to the fundamental group of $\mathcal{G}$ as an S-atlas [3], and to the ordinary fundamental group of the classifying space $B\mathcal{G}$.

1. Orbifolds

In this section we briefly review the basic definitions concerning orbifolds, also called V-manifolds or Satake manifolds (Satake [15], [16], or Thurston [18]).

Let M be a space. Fix $n > 0$. An *orbifold chart* on M is given by a connected open subset $\tilde{U} \subseteq \mathbb{R}^n$, a finite group G of C^∞-automorphisms of $\tilde{U}$, and a map $\varphi\colon \tilde{U} \to M$, so that φ is G-invariant ($\varphi \circ g = \varphi$ for all $g \in G$) and induces a homeomorphism of $\tilde{U}/G$ onto an open subset $U = \varphi(\tilde{U})$ of M. An *embedding* $\lambda\colon (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ between two such charts is a smooth embedding $\lambda\colon \tilde{U} \to \tilde{V}$ with $\psi \circ \lambda = \varphi$. An *orbifold atlas* on M is a family $\mathcal{U} = \{(\tilde{U}, G, \varphi)\}$ of such charts, which cover M and are locally compatible in the following sense: given any two charts $(\tilde{U}, G, \varphi)$ for $U = \varphi(\tilde{U}) \subseteq M$ and $(\tilde{V}, H, \psi)$ for $V \subseteq M$, and a point $x \in U \cap V$, there exists an open neighbourhood $W \subseteq U \cap V$ of x and a chart $(\tilde{W}, K, \chi)$ for W such that there are embeddings $(\tilde{W}, K, \chi) \hookrightarrow (\tilde{U}, G, \varphi)$ and $(\tilde{W}, K, \chi) \hookrightarrow (\tilde{V}, H, \psi)$. Two such atlases are said to be equivalent if they have a common refinement (where an atlas $\mathcal{U}$ is said to refine $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$). An *orbifold* (of dimension n) is a paracompact Hausdorff space M equipped with an equivalence class of orbifold atlases.

Remarks. (1) Every orbifold atlas for M is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one. Therefore we shall often tacitly work with a maximal atlas.

(2) Note that for a chart $(\tilde{U}, G, \varphi)$, the set of non-fixed points of $g\colon \tilde{U} \to \tilde{U}$, $1 \neq g \in G$, is dense in $\tilde{U}$. (Indeed, a nontrivial automorphism of the connected set $\tilde{U}$ of finite order cannot fix a nonempty open set.)

(3) For two embeddings λ and $\mu\colon (\tilde{U}, G, \varphi) \rightrightarrows (\tilde{V}, H, \psi)$ there exists a unique $h \in H$ so that $\mu = h \circ \lambda$. In particular, since each $g \in G$ can be viewed as an embedding of $(\tilde{U}, G, \varphi)$ into itself, for the two embeddings λ and $\lambda \circ g$ there exists a unique $h \in H$ so that $\lambda \circ g = h \circ \lambda$. This h will be denoted $\lambda(g)$. In this way, the map $\lambda\colon \tilde{U} \to \tilde{V}$ induces an injective group homomorphism $\lambda\colon G \hookrightarrow H$. This is

proved in [16, Lem. 1] when the fixed point set is of codimension at least two. For the general case, see the Appendix.

(4) Let $\lambda: (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ be an embedding. If $h \in H$ is such that $\lambda(\tilde{U}) \cap h \cdot \lambda(\tilde{U}) \neq \emptyset$, then h belongs to the image of the group homomorphism $\lambda: G \hookrightarrow H$ just described (and hence $\lambda(\tilde{U}) = h \cdot \lambda(\tilde{U})$). Again this is proved in [16] when the fixed point set is of codimension at least two. It is easily seen to hold without this condition; see the Appendix.

(5) By the differentiable slice theorem for smooth group actions (see [9]), any orbifold has an atlas consisting of 'linear charts', i.e. charts of the form $(\mathbb{R}^n, G)$ where G is a finite group of linear transformations of $\mathbb{R}^n$.

(6) If $(\tilde{U}, G, \varphi)$ and $(\tilde{V}, H, \psi)$ are two charts for the same orbifold structure on M, and $\tilde{U}$ is simply connected, then there exists an embedding $(\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ whenever $\varphi(\tilde{U}) \subseteq \psi(\tilde{V})$ (cf. [16], footnote 2). In particular, our definition of an orbifold is equivalent to Satake's, except that (as in [18]) we do not require that the fixed point sets are of codimension at least two.

(7) If all the actions in the charts of an orbifold atlas on M are free, then M is a C^∞-manifold.

Finally, following Satake [15], we define smooth maps between orbifolds. Let $\mathcal{M} = (M, \mathcal{U})$ and $\mathcal{N} = (N, \mathcal{V})$ be orbifolds (given by orbifold atlases $\mathcal{U}$ and $\mathcal{V}$). A map $f: M \to N$ is said to be *smooth* if for any point $x \in M$ there are charts $(\tilde{U}, G, \varphi)$ around x and $(\tilde{V}, H, \psi)$ around $f(x)$, with the property that f maps $U = \varphi(\tilde{U})$ into $V = \psi(\tilde{V})$ and can be lifted to a C^∞-map $\tilde{f}: \tilde{U} \to \tilde{V}$ with $\psi \tilde{f} = f \varphi$. Note that smooth maps between orbifolds can be composed. In particular, two such orbifolds $\mathcal{M}$ and $\mathcal{N}$ are said to be *diffeomorphic* (or *equivalent*) if there are such smooth maps $f: M \to N$ and $g: N \to M$ with $f \circ g$ and $g \circ f$ the respective identity mappings.

In Section 5, we will consider a stricter notion of map between orbifolds, namely one which behaves well on sheaves.

2. Sheaves on Orbifolds

Let $\mathcal{M} = (M, \mathcal{U})$ be an orbifold (where $\mathcal{U}$ is a maximal atlas). A *sheaf* $\mathcal{S}$ on $\mathcal{M}$ is given by:

(i) for each chart $(\tilde{U}, G, \varphi)$ for $\mathcal{M}$ an (ordinary) sheaf $\mathcal{S}_{\tilde{U}}$ on the space $\tilde{U}$;

(ii) for each embedding $\lambda: (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ an isomorphism $\mathcal{S}(\lambda): \mathcal{S}_{\tilde{U}} \xrightarrow{\sim} \lambda^*(\mathcal{S}_{\tilde{V}})$; these isomorphisms are required to be functorial in λ.

A morphism between sheaves $\alpha: \mathcal{S} \to \mathcal{S}'$ is a system of ordinary sheaf maps $\alpha_{\tilde{U}}: \mathcal{S}_{\tilde{U}} \to \mathcal{S}'_{\tilde{U}}$ (one for each chart), compatible with the embeddings λ, in the sense that each diagram of the form

$$\begin{array}{ccc}
\mathcal{S}_{\tilde{U}} & \xrightarrow{\alpha_{\tilde{U}}} & \mathcal{S}'_{\tilde{U}} \\
{\scriptstyle \mathcal{S}(\lambda)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}'(\lambda)} \\
\lambda^*(\mathcal{S}_{\tilde{V}}) & \xrightarrow[\lambda^*(\alpha_{\tilde{V}})]{} & \lambda^*(\mathcal{S}'_{\tilde{V}})
\end{array}$$

commutes. This defines a category $\tilde{\mathcal{M}}$ of all sheaves (of sets) on $\mathcal{M}$. Sheaves of Abelian groups, rings, etc., are defined similarly.

EXAMPLES. (1) For each $\tilde{U}$ define $\mathcal{O}_{\tilde{U}}$ to be the sheaf of germs of C^∞-functions on $\tilde{U}$; each embedding $\lambda: \tilde{U} \hookrightarrow \tilde{V}$ induces a map $\mathcal{O}_{\tilde{U}} \xrightarrow{\sim} \lambda^*(\mathcal{O}_{\tilde{V}})$ by composition with λ^{-1}. This defines a sheaf $\mathcal{O}$ or $\mathcal{O}_{\mathcal{M}}$, called the *structure sheaf* of the orbifold.

(2) In a similar way, one can construct a sheaf Ω^q of differential q-forms.

Remarks. (1) To define a sheaf $\mathcal{S}$ (up to isomorphism), it is enough to specify the sheaves $\mathcal{S}_{\tilde{U}}$ and sheaf maps $\mathcal{S}(\lambda)$ for all the charts $\tilde{U}$ in some covering atlas for $\tilde{M}$ with the property that the images $U = \varphi(\tilde{U}) \subseteq M$ form a basis for the topology on M. (This is a consequence of the Comparison Lemma ([4, [Thm III 4.1]), and analogous to [17], Proposition 4.)

(2) Let $\mathcal{S}$ be a sheaf on $\mathcal{M}$, and let $(\tilde{U}, G, \varphi)$ be a chart in $\mathcal{U}$. Then each $g \in G$ defines an embedding $g: \tilde{U} \hookrightarrow \tilde{U}$, so there is a sheaf map $\mathcal{S}(g): \mathcal{S}_{\tilde{U}} \to g^*(\mathcal{S}_{\tilde{U}})$. If $s \in \mathcal{S}_{\tilde{U},x}$ is a point in the stalk over $x \in \tilde{U}$, write

$$g \cdot s = \mathcal{S}(g)(s) \in g^*(\mathcal{S}_{\tilde{U}})_x = \mathcal{S}_{\tilde{U}, g\cdot x}.$$

This defines an action by G on the sheaf $\mathcal{S}_{\tilde{U}}$. Thus we see that $\mathcal{S}_{\tilde{U}}$ is a *G-equivariant sheaf* on $\tilde{U}$.

(3) (Cf. Remark 7 in Sect. 1). If all actions in the charts are free, then a sheaf on $\mathcal{M}$ is the same thing as a sheaf on the underlying manifold M; $\mathcal{O}$ is then the usual structure sheaf of germs of C^∞-functions. (In the general case where the actions are not necessarily all free, a sheaf on M induces a sheaf on $\mathcal{M}$, but the category $\tilde{\mathcal{M}}$ is quite different from the category $\tilde{M}$ of sheaves on M.)

(4) The category $\tilde{\mathcal{M}}$ of sheaves on $\mathcal{M}$ is a topos. In fact $(\tilde{\mathcal{M}}, \mathcal{O})$ is a smooth étendue in the sense of Grothendieck–Verdier ([4, Exercise IV 9.8.2 j]), as will be discussed in the next section.

(5) For two orbifolds $\mathcal{M}$ and $\mathcal{N}$, any map $(\tilde{\mathcal{M}}, \mathcal{O}_{\mathcal{M}}) \to (\tilde{\mathcal{N}}, \mathcal{O}_{\mathcal{N}})$ of ringed topoi induces a smooth map between orbifolds $\mathcal{M} \to \mathcal{N}$. This is an easy consequence of the explicit description of such topos maps in terms of groupoids given in Section 3 below. Also, up to equivalence the orbifold $\mathcal{M}$ can be recovered from

the category of sheaves $\tilde{\mathcal{M}}$, as will be shown explicitly in (the proof of) our main theorem in Section 4.

Contrary to the last remark, a smooth map $f: \mathcal{M} \to \mathcal{N}$ between orbifolds (as defined in Section 1) does *not* in general induce a morphism of ringed topoi $(\tilde{\mathcal{M}}, \mathcal{O}_{\mathcal{M}}) \to (\tilde{\mathcal{N}}, \mathcal{O}_{\mathcal{N}})$. However, diffeomorphic orbifolds do have equivalent sheaf categories:

PROPOSITION 2.1. *Let $f: \mathcal{M} \xrightarrow{\sim} \mathcal{N}$ be a diffeomorphism between orbifolds. Then f induces an equivalence of sheaf categories $\tilde{\mathcal{M}} \cong \tilde{\mathcal{N}}$.* (*This equivalence is in fact one of ringed topoi*).

Proof. Let $f: M \to N$ be a diffeomorphism and write $e: N \to M$ for its two-sided smooth inverse. The main point will be to see that such an f induces a natural operation $f^*: \tilde{\mathcal{N}} \to \tilde{\mathcal{M}}$ on sheaves. It will be clear from the construction that e^* and f^* are mutually inverse up to natural isomorphism. In the proof we will use the following lemma.

LEMMA 2.2. *Let $(\tilde{U}, G, \varphi)$ and $(\tilde{W}, K, \chi)$ be charts for $\mathcal{M}$ over $U = \varphi(\tilde{U})$ and W respectively, and assume $U \subseteq W$. Also, suppose there is a point $x \in \tilde{U}$ so that $G = G_x$* (*e.g. this is the case if $\tilde{U}$ is a linear chart*). *Let $\alpha: \tilde{U} \to \tilde{W}$ be any smooth map so that $\psi\alpha = \varphi$. Then α is an embedding.*

Proof of Lemma 2.2 Note first that since $\varphi: \tilde{U} \to U$ is a proper map and $\alpha(\tilde{U}) \subseteq \tilde{W}$ is Hausdorff, $\alpha: \tilde{U} \to \alpha(\tilde{U})$ is a proper map onto its image. Thus if we can show that α is a local diffeomorphism, it must be a covering projection, and hence a diffeomorphism since $\alpha^{-1}\alpha(x) \subseteq \varphi^{-1}\varphi(x) = \{x\}$. Thus it remains to show that α is a local diffeomorphism. This being a local matter, we may assume that $\tilde{U}$ and $\tilde{W}$ are linear charts, and hence that there is an embedding $\lambda: \tilde{U} \hookrightarrow \tilde{W}$ (cf. Remarks 5 and 6 in Sect. 1). For $z \in \tilde{U}$ let $k_z = (d\alpha)_z \circ (d\lambda^{-1})_{\lambda(z)} \in \mathrm{Gl}(n, \mathbb{R})$. Let $D \subseteq \tilde{W}$ be the dense open set on which K acts freely and let $E = \lambda^{-1}(D)$. Then for $z \in E$ there is a unique $k \in K$ for which $\alpha(z) = k \cdot \lambda(z)$, hence $k = k_z$. This shows that $z \mapsto k_z$ maps the dense set $E \subseteq \tilde{U}$ into the finite subgroup $K \subseteq \mathrm{Gl}(n, \mathbb{R})$. By continuity, we have $k_z \in K$ for all $z \in \tilde{U}$. Since $\tilde{U}$ is connected, k_z is constant, with value k say. Thus $\alpha = k \circ \lambda$, and hence an embedding.

To continue the proof of Proposition 2.1, choose $x \in M$. Choose linear charts for which there are smooth liftings $\tilde{f}$ and $\tilde{e}$ as in the diagram below, where $x \in U$ and $f(x) \in V'$; this can be done in such a way that $V \subseteq V'$ and $\tilde{V}$ is simply connected, so that there is an embedding $\lambda: \tilde{V} \hookrightarrow \tilde{V}'$ between the charts for $\mathcal{N}$ (cf. Sect. 1, Remark 6), as indicated

$$\begin{array}{ccccccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{V} & \overset{\lambda}{\dashrightarrow} & \tilde{V}' & \xrightarrow{\tilde{e}} & \tilde{W} \\ \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\chi} \\ U & \xrightarrow[f]{} & V & \subseteq & V' & \xrightarrow[e]{} & W. \end{array}$$

By the lemma, $\tilde{e} \circ \lambda \circ \tilde{f}$ is a diffeomorphism onto its image. Hence $\tilde{e}$ and $\tilde{f}$ have maximal rank, so are C^∞-embeddings.

This proves that $\mathcal{M}$ has a basis of charts $(\tilde{U}_i, G_i, \varphi_i)$ with the property that for each i there is a chart $(\tilde{V}_i, H_i, \psi_i)$ for $\mathcal{N}$ and a lifting $\tilde{f}_i$ which is a C^∞-embedding

$$\begin{array}{ccccc} \tilde{U}_i & \xrightarrow[\tilde{f}_i]{\sim} & \tilde{f}_i(\tilde{U}_i) & \subseteq & \tilde{V}_i \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\psi_i} \\ U_i & \xrightarrow[f]{\sim} & f(U_i) & \subseteq & V_i \end{array}.$$

But then $\tilde{f}_i(\tilde{U}_i)$, with the evident action by G_i, is a chart over $f(U_i) \subseteq N$, compatible with the orbifold structure for $\mathcal{N}$, and hence belongs to the maximal atlas for $\mathcal{N}$. Thus if $\mathcal{S}$ is any sheaf on $\mathcal{N}$, we can define a sheaf $f^*(\mathcal{S})$ on $\mathcal{M}$ by specifying its values on this basis of charts $(\tilde{U}_i, G_i, \varphi_i)$, as

$$f^*(\mathcal{S})_{\tilde{U}_i} = \tilde{f}_i^*(\mathcal{S}_{\tilde{f}(\tilde{U}_i)}) = \tilde{f}_i^*(\mathcal{S}_{\tilde{V}_i}).$$

For any embedding $\lambda\colon \tilde{U}_i \to \tilde{U}_j$ between charts in this basis, $\mu = \tilde{f}_j \circ \lambda \circ \tilde{f}_i^{-1}$ is an embedding $\tilde{f}_i(\tilde{U}_i) \hookrightarrow \tilde{f}_j(\tilde{U}_j)$ between charts for $\mathcal{N}$, and we can define $f^*(\mathcal{S})(\lambda)$ to be the map

$$f^*(\mathcal{S})_{\tilde{U}_i} = \tilde{f}_i^*(\mathcal{S}_{\tilde{f}_i(\tilde{U}_i)}) \xrightarrow{\tilde{f}_i^*\mathcal{S}(\mu)} \tilde{f}_i^*\mu^*(\mathcal{S}_{\tilde{f}_j(\tilde{U}_j)}) \cong \lambda^*\tilde{f}_j^*(\mathcal{S}_{\tilde{f}_j(\tilde{U}_j)}) = f^*(\mathcal{S})_{\tilde{U}_j}.$$

This completes the description of the functor $f^*\colon \tilde{\mathcal{M}} \to \tilde{\mathcal{N}}$. The rest of the proof is straightforward.

3. Smooth Groupoids and Étendues

Recall that a (topological) groupoid $\mathcal{G}$ is given by a space G_0 ('of objects'), another space G_1 ('of arrows'), and continuous mappings $s, t\colon G_1 \rightrightarrows G_0$ for source and target, $u\colon G_0 \to G_1$ for units, $i\colon G_1 \to G_1$ for inverse and $m\colon G_1 \times_{G_0} G_1 =$

$\{(g,h)|g,h \in G_1, s(g) = t(h)\} \to G_1$ for multiplication (composition). We shall write $g \circ h = m(g,h)$ and $g^{-1} = i(g)$, and $g: x \to y$ to denote that $s(g) = x$ while $t(g) = y$. These mappings are required to satisfy the familiar identities

$$
\begin{aligned}
&s(g \circ h) = s(h), \quad t(g \circ h) = t(g), \quad su(x) = x = tu(x),\\
&s(g^{-1}) = t(g), \quad t(g^{-1}) = s(g),\\
&g \circ u(x) = g = u(y) \circ g \text{ (for } g: x \to y)\\
&g \circ g^{-1} = u(y), \quad g^{-1} \circ g = u(x),\\
&(g \circ h) \circ k = g \circ (h \circ k).
\end{aligned}
$$

As in [13] and [2], we shall call $\mathcal{G}$ a *smooth groupoid* if G_0 and G_1 are C^∞-manifolds, all structure maps are smooth, and moreover s and t are submersions (so the domain $G_1 \times_{G_0} G_1$ of the composition map m is again a manifold). If s and t are local diffeomorphisms we shall call $\mathcal{G}$ an *étale groupoid*. All groupoids in this paper will be (assumed) smooth. A *homomorphism* between smooth groupoids $\varphi: \mathcal{G} \to \mathcal{H}$ is given by two smooth maps $\varphi_0: G_0 \to H_0$ and $\varphi_1: G_1 \to H_1$ which commute with the structure maps of $\mathcal{G}$ and $\mathcal{H}$.

EXAMPLES. (1) For any manifold M, there is an étale groupoid $\Gamma(M)$ with M as space of objects. The space of arrows is the space of all germs of (local) diffeomorphisms, equipped with the sheaf topology.

(2) Let L be a Lie group acting smoothly on a manifold X. The 'translation groupoid' X_L of this action has X as space of objects and $L \times X$ as space of arrows. The source and target maps are defined

$$s(\lambda, x) = x, \qquad t(\lambda, x) = \lambda \cdot x,$$

while the multiplication m, the unit u and the inverse i of this groupoid X_L are defined from the corresponding structure group L, in the obvious way.

Let $\mathcal{G}$ be a (smooth) groupoid. A $\mathcal{G}$-equivariant sheaf (briefly, a $\mathcal{G}$-sheaf) is a sheaf E on G_0 with a right action by the space G_1 of arrows. Thus E is given by an étale space $p: E \to G_0$, together with an action map

$$\alpha: E \times_{G_0} G_1 \to E, \qquad \alpha(e,g) = e \cdot g,$$

defined for all pairs $e \in E, g \in G_1$ with $p(e) = t(g)$, satisfying the usual identities for an action ($p(e \cdot g) = s(g)$, $e \cdot (g \circ h) = (e \cdot g) \cdot h$ and $e \cdot u(x) = e$, whenever these make sense). A *map* of such $\mathcal{G}$-sheaves $f: E \to E'$ is a map of étale spaces over G_0 which respects the action. In this way we obtain a category

$$\mathrm{Sh}(\mathcal{G})$$

of all $\mathcal{G}$-sheaves.

EXAMPLES. (3) Let L act on X with translation groupoid X_L. An X_L-sheaf is exactly the same as an L-equivariant sheaf on X, i.e. an L-space E equipped with an L-equivariant étale map $E \to X$. The category of these L-equivariant sheaves is usually denoted by $\mathrm{Sh}_L(X)$ (rather than $\mathrm{Sh}(X_L)$).

(4) Suppose $\mathcal{G}$ is an étale groupoid. Let $\mathcal{O}_\mathcal{G}$ be the sheaf of germs of smooth functions on G_0. This sheaf carries a natural $\mathcal{G}$-action, defined as follows. Let $\sigma: U_y \to \mathbb{R}$ represent an element $\mathrm{germ}_y(\sigma)$ of $\mathcal{O}_\mathcal{G}$ at $y \in G_0$, and let $g: x \to y$ be an arrow of $\mathcal{G}$. Let $V_g \subseteq G_1$ be an open neighbourhood of g, so small that both s and t restrict to diffeomorphisms on V_g, and so that $t(V_g) \subseteq U_y$. Define $\mathrm{germ}_y(\sigma) \cdot g = \mathrm{germ}_x(\sigma \circ (s|V_g) \circ (t|V_g)^{-1})$. Then $\mathcal{O}_\mathcal{G}$ is naturally a $\mathcal{G}$-sheaf. It is in fact a sheaf of rings, called the *structure ring* of $\mathcal{G}$.

The category $\mathrm{Sh}(\mathcal{G})$ of all $\mathcal{G}$-sheaves is a topos (4, [Exercise IV 9.8.2 a]). The topoi that so arise from étale groupoids form a special class.

DEFINITION 3.1 [4]. A smooth *étendue* is a ringed topos $(\mathcal{T}, \mathcal{O}_\mathcal{T})$ for which there exists a (smooth) étale groupoid $\mathcal{G}$, and an equivalence of categories $\mathcal{T} \simeq \mathrm{Sh}(\mathcal{G})$, sending $\mathcal{O}_\mathcal{T}$ to the structure ring $\mathcal{O}_\mathcal{G}$ (up to isomorphism).

(The groupoid $\mathcal{G}$ is unique up to weak equivalence; see Remark 4 below.)

Remarks. (1) Any manifold M can be viewed as a smooth groupoid, with identity arrows only. The equivariant sheaves are exactly the ordinary sheaves on M. In this way, each manifold can be viewed as a smooth étendue.

(2) A homomorphism $\varphi: \mathcal{G} \to \mathcal{H}$ between groupoids induces an evident pullback functor $\varphi^*: \mathrm{Sh}(\mathcal{H}) \to \mathrm{Sh}(\mathcal{G})$. This defines a morphism of topoi (again denoted) $\varphi: \mathrm{Sh}(\mathcal{G}) \to \mathrm{Sh}(\mathcal{H})$.

(3) Say $\varphi: \mathcal{G} \to \mathcal{H}$ is a *weak equivalence* if the following two conditions hold: first, the map $s\pi_2: G_0 \times_{H_0} H_1 \to H_0$ is a surjective submersion (where $G_0 \times_{H_0} H_1 = \{(x, h) | x \in G_0, h \in H_1, \varphi(x) = t(h)\}$); secondly, the square

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & H_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\varphi \times \varphi} & H_0 \times H_0 \end{array}$$

is a fibered product. Say two groupoids $\mathcal{H}$ and $\mathcal{K}$ are *weakly* (or '*Morita*') *equivalent* if there exist such weak equivalences

$$\mathcal{H} \leftarrow \mathcal{G} \to \mathcal{K},$$

for some smooth groupoid $\mathcal{G}$. (Note that the notion of weak equivalence between groupoids is stable under pullback as described in Definition 5.1, so that 'weakly equivalent' defines an equivalence relation on smooth groupoids.) For such a weak equivalence $\varphi: \mathcal{G} \to \mathcal{H}$, the induced morphism $\mathrm{Sh}(\mathcal{G}) \to \mathrm{Sh}(\mathcal{H})$ is an equivalence of (ringed) topoi. *Thus weakly equivalent groupoids define essentially the same equivariant sheaves*.

(4) [10], [14] For a large class of groupoids, including those weakly equivalent to an étale groupoid, the morphisms of topoi of equivariant sheaves can be described in terms of the groupoids. Indeed, any map of ringed topoi $f: \mathrm{Sh}(\mathcal{G}) \to \mathrm{Sh}(\mathcal{H})$ is induced by a diagram of smooth groupoids

$$\mathcal{G} \xleftarrow[\varphi]{\sim} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H},$$

where φ is a weak equivalence (in the sense that $\varphi^* \circ f^* \cong \psi^*$). In other words, up to weak equivalence of groupoids, every topos map comes from a groupoid homomorphism. The intermediate groupoid $\mathcal{K}$ is unique up to weak equivalence, as in the formalism for categories of fractions ([6]). In particular, *if two ringed topoi* $\mathrm{Sh}(\mathcal{G})$ *and* $\mathrm{Sh}(\mathcal{H})$ *are equivalent, the groupoids* $\mathcal{G}$ *and* $\mathcal{H}$ *must be weakly equivalent*.

We shall need one more definition: let $\mathcal{G}$ be an étale groupoid, and let $\Gamma(G_0)$ be the étale groupoid of germs as described in Example 1 above. Each arrow $g: x \to y$ in $\mathcal{G}$ defines a germ $\gamma(g) = \mathrm{germ}_x((t|V_g) \circ (s|V_g)^{-1})$, as in the description of $\mathcal{O}_\mathcal{G}$ in Example 4 above. This defines a homomorphism

$$\gamma: \mathcal{G} \to \Gamma(G_0)$$

of groupoids (which is the identity on objects). We say that $\mathcal{G}$ is *effective* if γ is faithful, i.e. if $\gamma(g) = \gamma(h)$ implies $g = h$. (Such effective étale groupoids are essentially the same as S*-atlases* [3], and correspond closely to *pseudogroups* of diffeomorphisms [18, Def. 3.1.1].) For a weak equivalence $\mathcal{G} \to \mathcal{H}$ between étale groupoids, $\mathcal{G}$ is effective if and only if $\mathcal{H}$ is. In particular, it makes sense to define a smooth étendue $\mathcal{T}$ to be effective if $\mathcal{T} = \mathrm{Sh}(\mathcal{G})$ for some effective $\mathcal{G}$, because this $\mathcal{G}$ is unique up to weak equivalence.

4. Characterization of Orbifolds

We are now ready to state our main theorem.

THEOREM 4.1. *For any ringed topos* $\mathcal{T} = (\mathcal{T}, \mathcal{O}_T)$ *the following properties are equivalent:*

(1) $\mathcal{T} \cong \tilde{\mathcal{M}}$ *for some orbifold* $\mathcal{M}$ (*unique up to equivalence of orbifolds*).
(2) $\mathcal{T} \cong \mathrm{Sh}_L(X)$ (*the topos of equivariant sheaves*), *for some manifold* X *and a compact Lie group* L *acting smoothly on* X, *so that the action has finite isotropy groups and faithful slice representations.*

(3) *$\mathcal{T}$ is an effective smooth étendue such that the diagonal $\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is a proper map (between topoi).*

(4) *$\mathcal{T} \cong \mathrm{Sh}(\mathcal{G})$, for some effective étale groupoid $\mathcal{G}$ such that $(s,t) : G_1 \to G_0 \times G_0$ is a proper map (between manifolds).*

Remarks. (1) The symbols $\cong$ in the statement of the theorem are equivalences of ringed topoi. $\mathcal{T}$ has a structure ring $\mathcal{O}_T$ by assumption. $\tilde{\mathcal{M}}$ in Part 1 and $\mathrm{Sh}(\mathcal{G})$ in Part 4 have canonical structure rings $\mathcal{O}_M$ and $\mathcal{O}_{\mathcal{G}}$, as described in Section 2, Example 1 and Section 3, Example 4 above. The structure ring of the topos $\mathrm{Sh}_L(X)$ of equivariant sheaves, which makes the equivalence in Part 2 of the theorem one of ringed topoi, is *not* the obvious one, but the sheaf $\mathcal{S}$ of germs of functions on slices. For a point $x \in X$, the stalk $\mathcal{S}_x$ consists of germs of C^∞-functions $V_x \to \mathbb{R}$, where V_x is a slice at x, given by a 'slice representation' $L \times_{L_x} V_x \cong U_x$ for some L-invariant neighborhood U_x of x, see [9] or [1]. We will denote the equivalence class of a pair $(\alpha, v) \in L \times V_x$ by $\alpha \otimes v \in L \times_{L_x} V_x$; thus $\alpha\lambda \otimes v = \alpha \otimes \lambda v$ whenever $\lambda \in L_x$. There is a natural action of L on this sheaf $\mathcal{S}$: for $\lambda \in L$, a slice representation at x gives a similar one at $\lambda \cdot x$ by conjugation,

$$\tau\colon L \times_{L_{\lambda\cdot x}} V_x^\lambda \cong L \times_{L_x} V_x.$$

Here V_x^λ is the same vector space V_x, with action of $L_{\lambda\cdot x}$ induced by that of L_x via the conjugation homomorphism $L_{\lambda\cdot x} \to L_x$, $\alpha \mapsto \lambda^{-1}\alpha\lambda$; the map τ sends an equivalence class $\alpha \otimes v \in L \times_{L_{\lambda\cdot x}} V_x^\lambda$ to $\alpha\lambda \otimes v \in L \times_{L_x} V_x$. Thus λ acts on a germ $f_x\colon V_x \to \mathbb{R}$ of $\mathcal{S}_x$ by sending it to the germ of the same function on the slice V_x^λ at $\lambda \cdot x$.

(2) It is known that every orbifold $\mathcal{M}$ arises as the space of leaves of a foliation with compact leaves and finite holonomy groups. The étale groupoid $\mathcal{G}$ for which $\tilde{\mathcal{M}} \cong \mathrm{Sh}(\mathcal{G})$ (as in Part 4 of the statement) is then the holonomy groupoid (or 'graph', [7], [19]) of that foliation, constructed from a suitable complete transversal section.

(3) For a (smooth) groupoid $\mathcal{G}$, the property that the map $(s,t)\colon G_1 \to G_0 \times G_0$ is proper (as in Part 4 of the theorem) is easily seen to be invariant under weak equivalence (Section 3, Remark 3) of groupoids.

(4) From Section 3, Remark 4 it follows that two étale groupoids $\mathcal{G}$ and $\mathcal{G}'$ as in Part 4 of the theorem represent the same orbifold if and only if they are weakly equivalent. Similarly, two Lie group actions L on X and L' on X' as in Part 2 represent the same orbifold if and only if their translation groupoids X_L and $X'_{L'}$ are weakly equivalent.

Proof of Theorem 4.1. 1 $\Rightarrow$ 2. Since the underlying space M of the orbifold $\mathcal{M}$ is assumed to be paracompact and Hausdorff, there exists an invariant Riemannian metric on $\mathcal{M}$, and one can construct the bundle $X = \mathrm{OFr}(\mathcal{M}) \to \mathcal{M}$ of orthogonal frames. As pointed out in [16, Section 1.5], X is an orbifold for which all the local group actions are free, hence is an ordinary manifold (cf. Section 1, Remark 7). Thus its topos of sheaves $\tilde{X}$ is the topos of ordinary sheaves on the manifold (as noted in Section 2, Remark 3). On the other hand, X is a principal $\mathrm{O}(n)$-bundle over $\mathcal{M}$, so by descent theory, the category $\tilde{\mathcal{M}}$ of sheaves on $\mathcal{M}$ is equivalent to that of $\mathrm{O}(n)$-equivariant sheaves on X

$$\tilde{\mathcal{M}} \cong \mathrm{Sh}(X, \mathrm{O}(n)).$$

To complete the proof of the implication $1 \Rightarrow 2$ of the theorem, it thus suffices to show that the action of $\mathrm{O}(n)$ on X has finite isotropy and faithful slice representations. This being a local matter, it suffices to consider the case where the orbifold $\mathcal{M}$ is given by just one chart $\mathbb{R}^n$ acted upon by a finite group G of orthogonal transformations. In this case it is elementary to verify that for the induced action of $\mathrm{O}(n)$ on $X = \mathrm{OFr}(\mathbb{R}^n)/G$, the isotropy groups are conjugates (in $\mathrm{O}(n)$) of the isotropy groups of the action G on $\mathbb{R}^n$, while the slices are representations conjugate to the given representation of G in $\mathbb{R}^n$.

$2 \Rightarrow 4$: Let L act on X as stated in Part 2 of the theorem. As in Example 3 of Section 1, the topos $\mathrm{Sh}_L(X)$ of equivariant sheaves is the topos $\mathrm{Sh}(X_L)$ of sheaves for the translation groupoid X_L. For this groupoid, the map $(s,t)\colon X \times L \to X \times X$ is evidently proper. Since this propriety property is invariant under weak equivalence (Remark 3 above), it suffices to show that X_L is weakly equivalent to an effective étale groupoid. To this end, cover X by a collection $\{U_i\}$ of L-invariant open sets, for which there exist slice representations

$$\theta_i\colon L \times_{L_i} V_i \to U_i.$$

(Thus L_i is a finite subgroup of L, V_i is a linear L_i-representation, $L \times_{L_i} V_i$ is obtained from $L \times V_i$ by identifying $(\alpha, \lambda v)$ and $(\alpha\lambda, v)$ for $\lambda \in L_i$ (as in Remark 1 above), and θ_i is an L-equivariant diffeomorphism. Below, we will identify V_i with the subspace $\theta_i(1 \times V_i) \subseteq U_i$, and $1 \otimes 0$ with the point x.) Write $G_0 = \Sigma V_i$ for the disjoint sum of all the V_i, and

$$p\colon G_0 \to X$$

for the evident map $p(i, v) = \theta_i(1 \otimes v)$. The translation groupoid X_L induces a groupoid $\mathcal{G}$ with G_0 as space of objects and a homomorphism $p\colon \mathcal{G} \to X_L$, if we define the space of arrows G_1 by pullback

$$\begin{array}{ccc} G_1 & \longrightarrow & L \times X \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow[(p,p)]{} & X \times X. \end{array}$$

Note that this fibered product is transversal (precisely because the slices V_i are transversal to the L-orbits), so that G_1 is a smooth manifold. We claim that the homomorphism $p: \mathcal{G} \to X_L$ is a weak equivalence. For this, it suffices to observe that the map

$$L \times G_0 \to X,\ (\lambda, (i, v)) \mapsto \lambda \cdot p(i, v) = \theta_i(\lambda \otimes v)$$

is a surjective submersion. Indeed, its restriction to $L \times V_i$ is the covering projection $L \times V_i \to L \times_{L_i} V_i \cong U_i \subseteq X$. Next, we claim that the source and target maps of this groupoid $\mathcal{G}$ are étale, i.e. local diffeomorphisms. By symmetry (via the inverse $i: G_1 \to G_1$), one of them is étale if and only if the other is. Let us show that $t: G_1 \to G_0$ is étale. Consider the pullback

$$\begin{array}{ccc}
\mathrm{Hom}(V_i, V_j) & \longrightarrow & G_1 \\
\downarrow & & \downarrow{\scriptstyle (s,t)} \\
V_i \times V_j & \hookrightarrow & G_0 \times G_0.
\end{array}$$

Thus $\mathrm{Hom}(V_i, V_j) = \{(\lambda, x) | x \in V_i, \lambda \in L, \lambda \cdot x \in U_i \cap V_j\}$ is the set of arrows in $\mathcal{G}$ from points in V_i to points in V_j. Since $G_0 \times G_0$ is the disjoint sum of all these products $V_i \times V_j$, it is enough to show that

$$t: \mathrm{Hom}(V_i, V_j) \to V_j,\ t(\lambda, x) = \lambda \cdot x$$

is étale. But note that $\mathrm{Hom}(V_i, V_j)$ can also be constructed as the pullback

$$\begin{array}{ccccc}
\mathrm{Hom}(V_i, V_j) & \longrightarrow & L \times (V_i \cap U_j) & \hookrightarrow & L \times V_i \\
\Big\downarrow & & \downarrow & & \downarrow \\
 & & L \times_{L_i} (V_i \cap U_j) & \hookrightarrow & L \times_{L_i} V_i. \\
\Big\downarrow & & \Big\| \wr\theta_{ji} & & \\
V_j \cap U_i & \longrightarrow & L \times_{L_j} (V_j \cap U_i) & & \\
\cap\!\!\downarrow & & & & \\
V_j & & & &
\end{array}$$

Here $\theta_{ji} = \theta_j \circ \theta_i^{-1}: L \times_{L_i} (V_i \cap U_j) \xrightarrow{\sim} U_i \cap U_j \to L \times_{L_j} (V_j \cap U_i)$. This shows that $t: \mathrm{Hom}(V_i, V_j) \to V_j$ is the composition of a covering projection and an open inclusion, hence étale. Note that the restriction of this groupoid $\mathcal{G}$ to a summand V_i is exactly the translation groupoid $(V_i)_{L_i}$ of the slice representation $L \times_{L_i} V_i \cong U_i$. Thus $\mathcal{G}$ is an effective groupoid, precisely because the slice representations of the action by L on X are assumed to be faithful. This completes the proof of the implication 2 $\Rightarrow$ 4.

4 $\Rightarrow$ 1: Let $\mathcal{G}$ be an effective étale groupoid with the map $(s,t): G_1 \to G_0 \times G_0$ proper. Let M be the orbit space of $\mathcal{G}$ (i.e., the quotient space of G_0 obtained by identifying two points $x, y \in G_0$ if and only if there exists an arrow $g: x \to y$ in G_1). Write $\pi: G_0 \to M$ for the quotient map; this is an open map because s and t are open. Note that M is Hausdorff because (s,t) is proper, and paracompact (in fact M is a metric space as an open quotient of the manifold G_0).

We will describe an orbifold atlas for M, explicitly constructed from the groupoid $\mathcal{G}$. The local group actions and embeddings between charts for this atlas will correspond exactly to the arrows in $\mathcal{G}$, and it will be evident from the construction that sheaves on this orbifold $\mathcal{M}$ correspond to $\mathcal{G}$-sheaves.

Fix a point $x \in G_0$. Since $(s,t): G_1 \to G_0 \times G_0$ is proper and $s, t: G_1 \rightrightarrows G_0$ are both étale, $(s,t)^{-1}(x) = G_x$ is a finite group. For each $g \in G_x$ choose an open neighborhood W_g of g in G_1, so small that both $s|W_g$ and $t|W_g$ are diffeomorphisms into G_0, and so that these W_g are pairwise disjoint. We will now shrink these open sets W_g in a suitable way. First, let $U_x = \bigcap_{g \in G_x} s(W_g)$; this is an open neighborhood of x in G_0. Since (s,t) is proper, there exists an open neighborhood $V_x \subseteq U_x$ so that

$$(V_x \times V_x) \cap (s,t)\left(G_1 - \bigcup_g W_g\right) = \emptyset.$$

Thus for any $h \in G_1$

$$s(h), t(h) \in V_x \Rightarrow h \in W_g \text{ for some } g \in G_x. \tag{1}$$

Next, write

$$\tilde{g}: s(W_g) \xrightarrow{\sim} t(W_g)$$

for the diffeomorphism $t \circ (s|W_g)^{-1}$. Note that $V_x \subseteq s(W_g)$ for each $g \in G_x$, so each $\tilde{g}$ is defined on the open set V_x. Define a smaller neighborhood $N_x \subseteq V_x$ by

$$N_x = \{y \in V_x | \tilde{g}(y) \in V_x \text{ for all } g \in G_x\}.$$

Note that if $y \in N_x$ then for any $h \in G_x$, also $\tilde{h}(y) \in N_x$. (Indeed, pick any

$g \in G_x$. Then since $\tilde{h}(y) \in V$ and $y \in V$, both $\tilde{g}(\tilde{h}(y))$ and $(gh)^{\sim}(y)$ are defined, and equal. Hence $\tilde{g}(\tilde{h}(y)) \in V_x$.) Thus the group G_x acts on N_x, by

$$g \cdot x = \tilde{g}(x). \tag{2}$$

Now define, for each $g \in G_x$,

$$\begin{aligned} O_g &= W_g \cap s^{-1}(N_x) \\ &= W_g \cap (s,t)^{-1}(N_x \times N_x) \end{aligned}$$

(the last identity because $y \in N_x \Rightarrow \tilde{g}(y) \in N_x$ as just shown). Then by (1), for any $k \in G_1$,

$$s(k), t(k) \in N_x \Rightarrow k \in O_g \text{ for some } g \in G_x. \tag{3}$$

It follows that $G_1 \cap (s,t)^{-1}(N_x \times N_x)$ is the disjoint union of the open sets O_g. Thus the restriction of the groupoid $\mathcal{G}$ to N_x is exactly the translation groupoid of the action (2) of G_x on N_x. Furthermore, by (3), $N_x/G_x \hookrightarrow M$ is an open embedding. (Note that the neighborhoods N_x may be chosen arbitrary small.) This shows that G_0 has a basis of open sets N_x with group action G_x as above. To show that they form the atlas for an orbifold structure on M, it remains to construct suitable embeddings. To this end let (N_x, G_x) and (N_y, G_y) be two such charts, and suppose $z \in G_0$ is such that $\pi(z) \in \pi(N_x) \cap \pi(N_y)$. Let $g: z \to x' \in N_x$ and $h: z \to y' \in N_y$ be any arrows in G_1. Let W_g and W_h be neighborhoods for which s and t restrict to homeomorphisms, and let (N_z, G_z) be a chart as constructed above at z. We can choose W_g and W_h, and then N_z, so small that $s(W_g) = N_z = s(W_h)$, while $t(W_g) \subseteq N_x$ and $t(W_h) \subseteq N_y$. Then $\tilde{g} = t \circ (s|W_g)^{-1}: N_z \hookrightarrow N_x$, and similarly $\tilde{h}: N_z \hookrightarrow N_y$ are the required embeddings. This proves that these charts $(N_x, G_x, \pi: N_x \to N_x/G_x \subseteq M)$ form a well-defined orbifold structure $\mathcal{M}$ on M.

Finally, observe that for two small enough charts (N_x, G_x) and (N_y, G_y), any embedding $\lambda: N_x \to N_y$ must (locally) be of the form $\tilde{g}$ for an arrow $g: x \to y$ in G_1. Indeed, since $\pi \circ \lambda = \pi$ there is at least one arrow $h: x \to y$ in $\mathcal{G}$; and if N_x, N_y are chosen small enough then h defines an embedding $\tilde{h}: N_x \to N_y$. As in the proof of Lemma 2.2, it follows that $\lambda(z) = k \circ \tilde{h}(z)$ (for all $z \in N_x$), for a unique $k \in G_y$. thus $\lambda = \tilde{k} \circ \tilde{h} = \tilde{kh}$. It is now easy to see that the étale groupoid $\mathcal{G}$ and the constructed orbifold $\mathcal{M}$ have the same category of sheaves.

3 ⇔ 4: This equivalence is a formal property of proper maps between topoi (cf. [12] or Def. 5.3 below). Indeed, for any space S write $\tilde{S}$ for the associated topos of sheaves on S. Then $S \to S'$ is a proper map of spaces if and only if $\tilde{S} \to \tilde{S}'$ is a proper map of topoi. Moreover, for any smooth étendue $\mathcal{T}$ and any étale groupoid $\mathcal{G}$ representing $\mathcal{T}$ (as in 3.1 above), the square

$$\begin{array}{ccccc}
\tilde{G}_1 & \longrightarrow & \mathrm{Sh}(\mathcal{G}) & = & \mathcal{T}\\
\Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta}\\
\tilde{G}_0 \times \tilde{G}_0 & \xrightarrow[p]{} & \mathrm{Sh}(\mathcal{G}) \times \mathrm{Sh}(\mathcal{G}) & = & \mathcal{T} \times \mathcal{T}
\end{array}$$

is a pullback of topoi, where p^* is the functor which forgets the action by $\mathcal{G}$ on the sheaves in $\mathrm{Sh}(\mathcal{G})$. Since p is an open surjection, (s,t) is proper iff Δ is.

This, finally, completes the proof of the theorem.

5. Fibered Products of Orbifolds

By way of example, we shall outline in this section a construction of fibered products of orbifolds, and state a change-of-base formula for sheaf cohomology.

Starting from the equivalent characterizations of an orbifold in terms of its sheaves (Thm 4.1), it is natural to consider the category of orbifolds and (isomorphism classes of) smooth mappings between the associated smooth étendues of sheaves (here we call a map of smooth étendues *smooth* if it is a map of ringed topoi). We call such maps *strong maps*, in contrast to the weaker notion of smooth map described in Section 1. (Remark that by Proposition 2.1 a diffeomorphism as in Section 1 is a strong map.) By Remark 2 in Section 3, such strong maps correspond to smooth homomorphisms between groupoids. Explicitly, let $\mathcal{M}$ and $\mathcal{N}$ be orbifolds, and let $\mathcal{G}$ and $\mathcal{H}$ be smooth groupoids representing $\mathcal{M}$ and $\mathcal{N}$ as in Theorem 4.1; thus $\tilde{\mathcal{M}} \cong \mathrm{Sh}(\mathcal{G})$ and $\tilde{\mathcal{N}} \cong \mathrm{Sh}(\mathcal{H})$. For $\mathcal{M}$, one can take for $\mathcal{G}$ either an étale groupoid with proper map $G_1 \to G_0 \times G_0$ as in Part 4 of the Theorem, or the translation groupoid X_L of a group action as in Part 2; similarly for $\mathcal{N}$ and $\mathcal{H}$. A strong map $f\colon \mathcal{M} \to \mathcal{N}$ is represented by a diagram $\mathcal{G} \xleftarrow{\sim} \mathcal{K} \to \mathcal{H}$ of homomorphisms between smooth groupoids, where $\mathcal{K} \to \mathcal{G}$ is a weak equivalence. Thus, $\mathrm{Sh}(\mathcal{K}) \cong \mathrm{Sh}(\mathcal{G})$, and $\mathcal{K}$ still represents $\mathcal{M}$. This shows that, given a groupoid $\mathcal{H}$ representing $\mathcal{N}$, strong maps $f\colon \mathcal{M} \to \mathcal{N}$ are represented by smooth homomorphisms $\varphi\colon \mathcal{K} \to \mathcal{H}$, where $\mathcal{K}$ is a groupoid (depending on f) which represents $\mathcal{M}$.

For two (strong) maps $f\colon \mathcal{M} \to \mathcal{N}$ and $f'\colon \mathcal{M}' \to \mathcal{N}$, represented by homomorphisms between étale groupoids $\varphi\colon \mathcal{K} \to \mathcal{H}$ and $\varphi'\colon \mathcal{K}' \to \mathcal{H}$, it is a folklore fact (and an immediate consequence of the description of strong maps in terms of groupoids) that the pullback of topoi $\tilde{\mathcal{M}} \times_{\tilde{\mathcal{N}}} \tilde{\mathcal{M}}'$ is represented by the groupoid $\mathcal{P}$, defined as follows.

DEFINITION 5.1. For homomorphisms $\varphi\colon \mathcal{K} \to \mathcal{H}$ and $\varphi'\colon \mathcal{K}' \to \mathcal{H}$ the (pseudo) pullback of groupoids $\mathcal{P} = \mathcal{K} \times_{\mathcal{H}} \mathcal{K}'$, is described as follows. The space of objects P_0 of $\mathcal{P}$ is the fibered product $K_0 \times_{H_0} H_1 \times_{H_0} K_0'$, consisting of triples (x, g, x') with $x \in K_0$, $x' \in K_0'$ and $g\colon \varphi(x) \to \varphi'(x')$ an arrow in $\mathcal{H}$. An arrow $(x, g, x') \to$

(y, h, y') between such triples in $\mathcal{P}$ is a pair of arrows, $k\colon x \to y$ in $\mathcal{K}$ and $k'\colon x' \to y'$ in $\mathcal{K}'$, so that $\varphi'(k') \circ g = h \circ \varphi(k)$. Using this notation the space P_1 of arrows of $\mathcal{P}$ can be constructed as the fibered product $P_1 = K_1 \times_{H_0} H_1 \times_{H_0} K_1'$ of triples (k, g, k'), with source and target given by $s(k, g, k') = (s(k), g, s(k')) = (x, g, x')$, and $t(k, g, k') = (t(k), \varphi'(k') \circ g \circ \varphi(k)^{-1}, t(k')) = (y, h, y')$.

Say $f\colon \mathcal{M} \to \mathcal{N}$ and $f'\colon \mathcal{M}' \to \mathcal{N}$ are *transversal* if there are such representations $\varphi\colon \mathcal{K} \to \mathcal{H}$ and $\varphi'\colon \mathcal{K}' \to \mathcal{H}$ by étale groupoids, so that $\varphi_0\colon K_0 \to H_0$ and $\varphi_0'\colon K_0' \to H_0$ are transversal maps between manifolds in the usual sense. This definition does not depend on the choice of representing groupoids, and is equivalent to the obvious condition expressed in terms of the maps between tangent bundles $T(\mathcal{M}) \to T(\mathcal{N}) \leftarrow T(\mathcal{M}')$.

In the case that f and f' are transversal, the groupoid $\mathcal{P}$ just defined is again a smooth groupoid, and it is elementary to verify that $\mathcal{P}$ is an effective étale groupoid with $(s, t)\colon P_1 \to P_0 \times P_0$ proper. Thus, by Theorem 4.1 $\mathcal{P}$ represents a unique (up to diffeomorphism) orbifold, which we denote by $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$; so, by definition, $(\mathcal{M} \times_{\mathcal{N}} \mathcal{M}')^{\sim} \cong \mathrm{Sh}(\mathcal{P})$. This construction does not depend on the chosen representations $\varphi\colon \mathcal{K} \to \mathcal{H}$ and $\varphi'\colon \mathcal{K}' \to \mathcal{H}$: another choice, using weakly equivalent groupoids, results in a 'pullback' groupoid which is weakly equivalent to $\mathcal{P}$ and hence determines the same orbifold.

The construction of $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ can be summarized in the following proposition.

PROPOSITION 5.2. *Let $f\colon \mathcal{M} \to \mathcal{N}$ and $f'\colon \mathcal{M}' \to \mathcal{N}$ be strong maps between orbifolds, and assume that f is transversal to f'. Then there exists an orbifold $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$, unique up to diffeomorphism, for which there is a pullback square of topoi* (*of smooth étendues*)

$$\begin{array}{ccc} (\mathcal{M} \times_{\mathcal{N}} \mathcal{M}')^{\sim} & \longrightarrow & \tilde{M}' \\ \big\downarrow & & \big\downarrow{\scriptstyle f'} \\ \tilde{M} & \xrightarrow[f]{} & \tilde{N} \end{array} .$$

Moreover, $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ can be explicitly constructed from a groupoid $\mathcal{P}$, obtained from groupoid representations of f and f', as described above.

Remarks. (1) The underlying space of $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ is *not* in general the fibered product $M \times_N M'$ of the underlying spaces of $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}'$. (It is, of course, in case $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}'$ are manifolds.)

(2) The fibered product of covering spaces as described by Thurston in the proof of Proposition 5.3.3 of [18], forms a special case of the construction above.

We shall now define when a strong map $f\colon \mathcal{M} \to \mathcal{N}$ between orbifolds, i.e. a map $f\colon \tilde{\mathcal{M}} \to \tilde{\mathcal{N}}$ of smooth étendues, is *proper*. Let $\mathcal{H}$ be an étale groupoid so that $\mathrm{Sh}(\mathcal{H}) = \tilde{\mathcal{N}}$, as in Part 4 of Theorem 4.1. The manifold H_0 of objects can be viewed as an orbifold, and there is an evident groupoid homomorphism $H_0 \to \mathcal{H}$, giving a strong map of orbifolds $H_0 \to \mathcal{N}$. (Here H_0 also denotes the groupoid with identity arrows only, as in Remark 1 of Section 3.) Form the pullback orbifold $H_0 \times_{\mathcal{N}} \mathcal{M}$, and write Y for its underlying space. The projection $H_0 \times_{\mathcal{N}} \mathcal{M} \to H_0$ of orbifolds induces a map of underlying spaces $Y \to H_0$.

DEFINITION 5.3. A strong map $\mathcal{M} \to \mathcal{N}$ as above is called proper iff the map $Y \to H_0$, just described, is a proper map of topological spaces in the usual (Bourbaki) sense.

This condition does not depend on the choice of $\mathcal{H}$. Observe that by Part 2 of Theorem 4.1, there is a manifold X with an action by a compact Lie group L so that $(H_0 \times_{\mathcal{N}} \mathcal{M})^{\sim} \cong \mathrm{Sh}_L(X)$. Thus Y is the orbit space X/L. The map $H_0 \times_{\mathcal{N}} \mathcal{M} \to H_0$ is given by a smooth map $X \to H_0$ which is invariant under the action by L. Thus $X/L = Y \to H_0$ is proper iff $X \to H_0$ is. This definition is a special case of the more general notion of proper maps between topoi, defined in [8] and [12].

Write $\mathrm{Ab}(\tilde{\mathcal{M}})$ for the *Abelian* sheaves on $\mathcal{M}$. A strong map $f\colon \mathcal{M} \to \mathcal{N}$ induces functors on Abelian sheaves

$$f_*\colon \mathrm{Ab}(\tilde{\mathcal{M}}) \rightleftarrows \mathrm{Ab}(\tilde{\mathcal{N}}) \;:\; f^*$$

in the usual way, where f^* is left adjoint to f_* and f^* is exact. Writing $R^n f_*$ for the right derived functor, we can state the change-of-base formula, analogous to the one for proper maps between schemes (Artin, [5]).

THEOREM 5.4. *For any fibered product square of orbifolds* (*and strong maps*)

$$\begin{array}{ccc} \mathcal{M} \times_{\mathcal{N}} \mathcal{M}' & \xrightarrow{p'} & \mathcal{M}' \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle f'} \\ \mathcal{M} & \xrightarrow[f]{} & \mathcal{N} \end{array},$$

if f' is proper then for any Abelian sheaf A on $\mathcal{M}'$, the canonical map

$$f^* R^n f'_*(A) \xrightarrow{\sim} R^n p_*(p'^* A)$$

is an isomorphism.

Using the various equivalent definitions of proper maps described above the proof is fairly straightforward, and uses étale descent and an equivariant version of ordinary proper base change for manifolds. Details are given in Pronk's PhD-thesis (unpublished).

Appendix

In [16], certain properties of embeddings of charts in an orbifold atlas were proved using that the fixed point set of the action of a group element had codimension at least two. In this Appendix we give a different proof of these statements, which works without this codimension requirement.

PROPOSITION A.1. *Let λ, μ be two embeddings $(\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$. Then there exists a unique $h \in H$ such that $\mu = h \circ \lambda$.*

Proof. Choose $\lambda x \in \lambda(\tilde{U}) \subseteq \tilde{V}$ to be a nonfixed point. Since $\psi(\lambda x) = \psi(\mu x)$ and λx is a nonfixed point, there is a unique $h \in H$ sending λx to μx. By continuity this element h is the same for all points λx in the same connected component of the set of non-fixed points in $\lambda\tilde{U} \subseteq \tilde{V}$. We have to show that this also holds for λx and $\lambda x'$ in different connected components of the nonfixed point set of $\tilde{V}$. Different connected components of the nonfixed-point set are being separated by one or more fixed point sets of codimension 1. However for λx and $\lambda x'$ in different connected components there is always a path connecting λx and $\lambda x'$ that crosses only one fixed point set of codimension 1 at a time. So in order to prove that the same element $h \in H$ also sends $\lambda x'$ to $\mu x'$ for x' in a different connected component of the nonfixed point set, it is sufficient to prove this for λx and $\lambda x'$ in a connected open neighborhood $W \subseteq \lambda\tilde{U} \subseteq \tilde{V}$ such that $W\cap$ fixed point set $\subseteq L$, and $W \cap L$ has codimension 1.

Denote the connected component of ($W\cap$ nonfixed-point set) containing λx (resp. $\lambda x'$), by W_x (resp. $W_{x'}$). So assume $h \cdot \lambda(x) = \mu(x)$ and let $h_{x'}$ be the (unique) element of H such that $h_{x'} \cdot \lambda(x') = \mu(x')$. Suppose that $h_{x'} \neq h$. By continuity we have that $h \cdot z = \mu(z) = h_{x'} \cdot z$ for all $z \in L \cap \overline{W}_x \cap \overline{W}_{x'}$. So h and $h_{x'}$ agree on $(L \cap \overline{W}_x \cap \overline{W}_{x'}) \subset \tilde{V}$, which has a nonempty interior in L. Pick $z_0 \in L \cap \overline{W}_x \cap \overline{W}_{x'}$ and let $(\tilde{V}', H', \psi')$ be a small linear chart around z_0 with an embedding ν: $\tilde{V}' \hookrightarrow \tilde{V}$, such that $z_0 \in \nu(\tilde{V}') \subseteq W$. Note that $h^{-1} \circ h_{x'} \in H'$, since it keeps z_0 fixed, and we can consider it as an element of $\mathrm{O}(n, \mathbb{R})$. Since $h^{-1} \circ h_{x'} \neq$ id and it keeps the hyperplane L fixed, it has to be the reflection in this plane. So $h^{-1} \circ h_{x'}(W_{x'}) \cap W_x \neq \emptyset$. Pick x_0 in this intersection and write $W_x^* = \lambda^{-1}(W_x)$ and $W_{x'}^* = \lambda^{-1}(W_{x'})$. Then $x_0 \in \lambda W_x^*$ and $h_{x'}^{-1} \circ h(x_0) \in \lambda W_{x'}^*$. It follows that $h(x_0) \in \mu(W_x^*)$ and $h(x_0) = h_{x'} \circ h_{x'}^{-1} \circ h(x_0) \in \mu(W_{x'}^*)$. So $\mu(W_x^*) \cap \mu(W_{x'}^*) \neq \emptyset$, whereas $W_x^* \cap W_{x'}^* = \emptyset$. This contradicts the fact that μ is an embedding. So $h_x = h_{x'}$ and this finishes the proof of the proposition.

LEMMA A.2. *Let λ be an embedding $(\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ of the atlas $\mathcal{U}$. If*

$h(\lambda(\tilde{U})) \cap \lambda(\tilde{U}) \neq \emptyset$ for some $h \in H$, then $h(\lambda(\tilde{U})) = \lambda(\tilde{U})$ and h belongs to the image of the isomorphism of G onto a subgroup of H as defined in the remark above.

Proof. Choose a nonfixed point $\lambda x \in h(\lambda(\tilde{U})) \cap \lambda(\tilde{U})$. Suppose $\lambda x = h\lambda y$. Then $\varphi x = \varphi y$, so there exists a unique $g \in G$ with $gy = x$. Consider $\lambda(g) \in H$, this diffeomorphism is equal to h on an open neighborhood of λy, so they are equal everywhere (cf. Sect. 1, Remark 2).

Acknowledgements

The authors would like to thank MSRI and Aarhus University for their hospitality, and NWO as well as the Danish Natural Science Research Council for support.

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