CAMBRIDGE TRACTS IN MATHEMATICS

171

ORBIFOLDS AND STRINGY TOPOLOGY

ALEJANDRO ADEM, JOHANN LEIDA
& YONGBIN RUAN

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE www.cambridge.org/9780521870047

This page intentionally left blank

CAMBRIDGE TRACTS IN MATHEMATICS

171 Orbifolds and Stringy Topology

CAMBRIDGE TRACTS IN MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit
http://www.cambridge.org/series/sSeries.asp?code=CTM

145 Isoperimetric Inequalities. By I. CHAVEL
146 Restricted Orbit Equivalence for Actions of Discrete Amenable Groups. By J. KAMMEYER and D. RUDOLPH
147 Floer Homology Groups in Yang–Mills Theory. By S. K. DONALDSON
148 Graph Directed Markov Systems. By D. MAULDIN and M. URBANSKI
149 Cohomology of Vector Bundles and Syzygies. By J. WEYMAN
150 Harmonic Maps, Conservation Laws and Moving Frames. By F. HÉLEIN
151 Frobenius Manifolds and Moduli Spaces for Singularities. By C. HERTLING
152 Permutation Group Algorithms. By A. SERESS
153 Abelian Varieties, Theta Functions and the Fourier Transform. By A. POLISHCHUK
154 Finite Packing and Covering, K. BÖRÖCZKY, JR
155 The Direct Method in Soliton Theory. By R. HIROTA. Edited and translated by A. NAGAI, J. NIMMO, and C. GILSON
156 Harmonic Mappings in the Plane. By P. DUREN
157 Affine Hecke Algebras and Orthogonal Polynomials. By I. G. MACDONALD
158 Quasi-Frobenius Rings. By W. K. NICHOLSON and M. F. YOUSIF
159 The Geometry of Total Curvature. By K. SHIOHAMA, T. SHIOYA, and M. TANAKA
160 Approximation by Algebraic Numbers. By Y. BUGEAD
161 Equivalence and Duality for Module Categories. By R. R. COLBY, and K. R. FULLER
162 Lévy Processes in Lie Groups. By MING LIAO
163 Linear and Projective Representations of Symmetric Groups. By A. KLESHCHEV
164 The Covering Property Axiom, CPA. K. CIESIELSKI and J. PAWLIKOWSKI
165 Projective Differential Geometry Old and New. By V. OVSIENKO and S. TABACHNIKOV
166 The Lévy Laplacian. By M. N. FELLER
167 Poincaré Duality Algebras, Macaulay's Dual Systems, and Steenrod Operations. By D. M. MEYER and L. SMITH
168 The Cube: A Window to Convex and Discrete Geometry. By C. ZONG
169 Quantum Stochastic Processes and Noncommutative Geometry. By K. B. SINHA and D. GOSWAMI

Orbifolds and Stringy Topology

ALEJANDRO ADEM
University of British Columbia

JOHANN LEIDA
University of Wisconsin

YONGBIN RUAN
University of Michigan

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521870047

First published in print format 2007

ISBN-13 978-0-511-28528-8 eBook (Adobe Reader)
ISBN-10 0-511-28288-5 eBook (Adobe Reader)

ISBN-13 978-0-521-87004-7 hardback
ISBN-10 0-521-87004-6 hardback

Contents

Introduction

Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra, and string theory, among others. What is more, although the word "orbifold" was coined relatively recently,[1] orbifolds actually have a much longer history. In algebraic geometry, for instance, their study goes back at least to the Italian school under the guise of *varieties with quotient singularities*. Indeed, surface quotient singularities have been studied in algebraic geometry for more than a hundred years, and remain an interesting topic today. As with any other singular variety, an algebraic geometer aims to remove the singularities from an orbifold by either *deformation* or *resolution*. A deformation changes the defining equation of the singularities, whereas a resolution removes a singularity by blowing it up. Using combinations of these two techniques, one can associate many smooth varieties to a given singular one. In complex dimension two, there is a natural notion of a *minimal resolution*, but in general it is more difficult to understand the relationships between all the different desingularizations.

Orbifolds made an appearance in more recent advances towards Mori's birational geometric program in the 1980s. For Gorenstein singularities, the higher-dimensional analog of the minimal condition is the famous *crepant* resolution, which is minimal with respect to the canonical classes. A whole zoo of problems surrounds the relationship between crepant resolutions and Gorenstein orbifolds: this is often referred to as *McKay correspondence*. The McKay correspondence is an important motivation for this book; in complex dimension two it was solved by McKay himself. The higher-dimensional version has attracted increasing attention among algebraic geometers, and the existence of crepant resolutions in the dimension three case was eventually solved by an

[1] According to Thurston [148], it was the result of a democratic process in his seminar.

array of authors. Unfortunately, though, a Gorenstein orbifold of dimension four or more does not possess a crepant resolution in general. Perhaps the best-known example of a higher-dimensional crepant resolution is the Hilbert scheme of points of an algebraic surface, which forms a crepant resolution of its symmetric product. Understanding the cohomology of the Hilbert scheme of points has been an interesting problem in algebraic geometry for a considerable length of time.

Besides resolution, deformation also plays an important role in the classification of algebraic varieties. For instance, a famous conjecture of Reid [129] known as *Reid's fantasy* asserts that any two Calabi–Yau 3-folds are connected to each other by a sequence of resolutions or deformations. However, deformations are harder to study than resolutions. In fact, the relationship between the topology of a deformation of an orbifold and that of the orbifold itself is one of the major unresolved questions in orbifold theory.

The roots of orbifolds in algebraic geometry must also include the theory of *stacks*, which aims to deal with singular spaces by enlarging the concept of "space" rather than finding smooth desingularizations. The idea of an algebraic stack goes back to Deligne and Mumford [40] and Artin [7]. These early papers already show the need for the stack technology in fully understanding moduli problems, particularly the moduli stack of curves. Orbifolds are special cases of topological stacks, corresponding to "differentiable Deligne and Mumford stacks" in the terminology of [109].

Many of the orbifold cohomology theories we will study in this book have roots in and connections to cohomology theories for stacks. The book [90] of Laumon and Moret-Bailly is a good general reference for the latter. Orbifold Chen–Ruan cohomology, on the other hand, is closely connected to quantum cohomology – it is the *classical limit* of an orbifold quantum cohomology also due to Chen–Ruan. Of course, stacks also play an important role in the quantum cohomology of smooth spaces, since moduli stacks of maps from curves are of central importance in defining these invariants. For more on quantum cohomology, we refer the reader to McDuff and Salamon [107]; the original works of Kontsevich and Manin [87, 88], further developed in an algebraic context by Behrend [19] with Manin [21] and Fantechi [20], have also been very influential.

Stacks have begun to be studied in earnest by topologists and others outside of algebraic geometry, both in relation to orbifolds and in other areas. For instance, topological modular forms (tmf), a hot topic in homotopy theory, have a great deal to do with the moduli stack of elliptic curves [58].

Outside of algebraic geometry, orbifolds were first introduced into topology and differential geometry in the 1950s by Satake [138, 139], who called

them *V-manifolds*. Satake described orbifolds as topological spaces generalizing smooth manifolds. In the same work, many concepts in smooth manifold theory such as de Rham cohomology, characteristic classes, and the Gauss–Bonnet theorem were generalized to V-manifolds. Although they are a useful concept for such problems as finite transformation groups, V-manifolds form a straightforward generalization of smooth manifolds, and can hardly be treated as a subject in their own right. This was reflected in the first twenty years of their existence. Perhaps the first inkling in the topological literature of additional features worthy of independent interest arose in Kawasaki's *V-manifold index theorem* [84, 85] where the index is expressed as a summation over the contribution of fixed point sets, instead of via a single integral as in the smooth case. This was the first appearance of the twisted sectors, about which we will have much more to say later.

In the late 1970s, V-manifolds were used seriously by Thurston in his geometrization program for 3-manifolds. In particular, Thurston invented the notion of an *orbifold fundamental group*, which was the first true invariant of an orbifold structure in the topological literature.[2] As noted above, it was during this period that the name V-manifold was replaced by the word orbifold. Important foundational work by Haefliger [64–68] and others inspired by foliation theory led to a reformulation of orbifolds using the language of groupoids. Of course, groupoids had also long played a central role in the development of the theory of stacks outlined above. Hence the rich techniques of groupoids can also be brought to bear on orbifold theory; in particular the work of Moerdijk [111–113] has been highly influential in developing this point of view. As a consequence of this, fundamental algebraic topological invariants such as classifying spaces, cohomology, bundles, and so forth have been developed for orbifolds.

Although orbifolds were already clearly important objects in mathematics, interest in them was dramatically increased by their role in string theory. In 1985, Dixon, Harvey, Vafa, and Witten built a conformal field theory model on singular spaces such as $\mathbb{T}^6/G$, the quotient of the six-dimensional torus by a smooth action of a finite group. In conformal field theory, one associates a Hilbert space and its operators to a manifold. For orbifolds, they made a surprising discovery: the Hilbert space constructed in the traditional fashion is not consistent, in the sense that its partition function is not modular. To recover modularity, they introduced additional Hilbert space factors to build a

[2] Of course, in algebraic geometry, invariants of orbifold structures (in the guise of stacks) appeared much earlier. For instance, Mumford's calculation of the Picard group of the moduli stack of elliptic curves [117] was published in 1965.

stringy Hilbert space. They called these factors *twisted sectors*, which intuitively represent the contribution of singularities. In this way, they were able to build a *smooth* stringy theory out of a singular space. Orbifold conformal field theory is very important in mathematics and is an impressive subject in its own right. In this book, however, our emphasis will rather be on topological and geometric information.

The main topological invariant obtained from orbifold conformal field theory is the *orbifold Euler number*. If an orbifold admits a crepant resolution, the string theory of the crepant resolution and the orbifold's string theory are thought to lie in the same family of string theories. Therefore, the orbifold Euler number should be the same as the ordinary Euler number of a crepant resolution. A successful effort to prove this statement was launched by Roan [131, 132], Batyrev and Dais [17], Reid [130] and others. In the process, the orbifold Euler number was extended to an orbifold Hodge number. Using intuition from physics, Zaslow [164] essentially discovered the correct stringy cohomology group for a global quotient using ad hoc methods. There was a very effective motivic integration program by Denef and Loeser [41, 42] and Batyrev [14, 16] (following ideas of Kontsevich [86]) that systematically established the equality of these numbers for crepant resolutions. On the other hand, motivic integration was not successful in dealing with finer structures, such as cohomology and its ring structure.

In this book we will focus on explaining how this problem was dealt with in the joint work of one of the authors (Ruan) with Chen [38]. Instead of guessing the correct formulation for the cohomology of a crepant resolution from orbifold data, Chen and Ruan approached the problem from the sigma-model quantum cohomology point of view, where the starting point is the space of maps from a Riemann surface to an orbifold. The heart of this approach is a correct theory of *orbifold morphisms*, together with a classification of those having domain an orbifold Riemann surface. The most surprising development is the appearance of a new object – the *inertia orbifold* – arising naturally as the target of an evaluation map, where for smooth manifolds one would simply recover the manifold itself. The key conceptual observation is that the components of the inertia orbifold should be considered the geometric realization of the conformal theoretic twisted sectors. This realization led to the successful construction of an orbifold quantum cohomology theory [37], and its classical limit leads to a new cohomology theory for orbifolds. The result has been a new wave of activity in the study of orbifolds. One of the main goals of this book is to give an account of *Chen–Ruan cohomology* which is accessible to students. In particular, a detailed treatment of orbifold morphisms is one of our basic themes.

Besides appearing in Chen–Ruan cohomology, the inertia orbifold has led to interesting developments in other orbifold theories. For instance, as first discussed in [5], the twisted sectors play a big part in orbifold K-theory and twisted orbifold K-theory. Twisted K-theory is a rapidly advancing field; there are now many types of twisting to consider, as well as interesting connections to physics [8, 54, 56].

We have formulated a basic framework that will allow a graduate student to grasp those essential aspects of the theory which play a role in the work described above. We have also made an effort to develop the background from a variety of viewpoints. In Chapter 1, we describe orbifolds very explicitly, using their manifold-like properties, their incarnations as groupoids, and, last but not least, their aspect as singular spaces in algebraic geometry. In Chapter 2, we develop the classical notions of cohomology, bundles, and morphisms for orbifolds using the techniques of Lie groupoid theory. In Chapter 3, we describe an approach to orbibundles and (twisted) K-theory using methods from equivariant algebraic topology. In Chapter 4, the heart of this book, we develop the Chen–Ruan cohomology theory using the technical background developed in the previous chapters. Finally, in Chapter 5 we describe some significant calculations for this cohomology theory.

As the theory of orbifolds involves mathematics from such diverse areas, we have made a selection of topics and viewpoints from a large and rather opaque menu of options. As a consequence, we have doubtless left out important work by many authors, for which we must blame our ignorance. Likewise, some technical points have been slightly tweaked to make the text more readable. We urge the reader to consult the original references.

It is a pleasure for us to thank the Department of Mathematics at the University of Wisconsin-Madison for its hospitality and wonderful working conditions over many years. All three of us have mixed feelings about saying farewell to such a marvelous place, but we must move on. We also thank the National Science Foundation for its support over the years. Last but not least, all three authors want to thank their wives for their patient support during the preparation of this manuscript. This text is dedicated to them.

1
Foundations

1.1 Classical effective orbifolds

Orbifolds are traditionally viewed as singular spaces that are locally modeled on a quotient of a smooth manifold by the action of a finite group. In algebraic geometry, they are often referred to as varieties with quotient singularities. This second point of view treats an orbifold singularity as an intrinsic structure of the space. For example, a codimension one orbifold singularity can be treated as smooth, since we can remove it by an analytic change of coordinates. This point of view is still important when we consider resolutions or deformations of orbifolds. However, when working in the topological realm, it is often more useful to treat the singularities as an additional structure – an *orbifold structure* – on an underlying space in the same way that we think of a smooth structure as an additional structure on a topological manifold. In particular, a topological space is allowed to have several different orbifold structures. Our introduction to orbifolds will reflect this latter viewpoint; the reader may also wish to consult the excellent introductions given by Moerdijk [112, 113].

The original definition of an orbifold was due to Satake [139], who called them *V-manifolds*. To start with, we will provide a definition of *effective* orbifolds equivalent to Satake's original one. Afterwards, we will provide a refinement which will encompass the more modern view of these objects. Namely, we will also seek to explain their definition using the language of groupoids, which, although it has the drawback of abstractness, does have important technical advantages. For one thing, it allows us to easily deal with ineffective orbifolds, which are generically singular. Such orbifolds are unavoidable in nature. For instance, the moduli stack of elliptic curves [117] (see Example 1.17) has a $\mathbb{Z}/2\mathbb{Z}$ singularity at a generic point. The definition below appears in [113].

Definition 1.1 Let X be a topological space, and fix $n \geq 0$.

- An n-dimensional *orbifold chart* on X is given by a connected open subset $\widetilde{U} \subseteq \mathbb{R}^n$, a finite group G of smooth automorphisms of $\widetilde{U}$, and a map $\phi : \widetilde{U} \to X$ so that ϕ is G-invariant and induces a homeomorphism of $\widetilde{U}/G$ onto an open subset $U \subseteq X$.
- An *embedding* $\lambda : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ between two such charts is a smooth embedding $\lambda : \widetilde{U} \hookrightarrow \widetilde{V}$ with $\psi\lambda = \phi$.
- An *orbifold atlas* on X is a family $\mathcal{U} = \{(\widetilde{U}, G, \phi)\}$ of such charts, which cover X and are locally compatible: given any two charts $(\widetilde{U}, G, \phi)$ for $U = \phi(\widetilde{U}) \subseteq X$ and $(\widetilde{V}, H, \psi)$ for $V \subseteq X$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ of x and a chart $(\widetilde{W}, K, \mu)$ for W such that there are embeddings $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \phi)$ and $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \psi)$.
- An atlas $\mathcal{U}$ is said to *refine* another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are said to be *equivalent* if they have a common refinement.

We are now ready to provide a definition equivalent to the classical definition of an effective orbifold.

Definition 1.2 An *effective orbifold* $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases.

There are some important points to consider about this definition, which we now list. Throughout this section we will always assume that our orbifolds are effective.

1. We are assuming that for each chart $(\widetilde{U}, G, \phi)$, the group G is acting smoothly and effectively[1] on $\widetilde{U}$. In particular G will act freely on a dense open subset of $\widetilde{U}$.
2. Note that since smooth actions are locally smooth (see [31, p. 308]), any orbifold has an atlas consisting of linear charts, by which we mean charts of the form $(\mathbb{R}^n, G, \phi)$, where G acts on $\mathbb{R}^n$ via an orthogonal representation $G \subset O(n)$.
3. The following is an important technical result for the study of orbifolds (the proof appears in [113]): given two embeddings of orbifold charts $\lambda, \mu : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$, there exists a unique $h \in H$ such that $\mu = h \cdot \lambda$.

[1] Recall that a group action is *effective* if $gx = x$ for all x implies that g is the identity. For basic results on topological and Lie group actions, we refer the reader to Bredon [31] and tom Dieck [152].

4. As a consequence of the above, an embedding of orbifold charts $\lambda : (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ induces an injective group homomorphism, also denoted by $\lambda : G \hookrightarrow H$. Indeed, any $g \in G$ can be regarded as an embedding from $(\widetilde{U}, G, \phi)$ into itself. Hence for the two embeddings λ and $\lambda \cdot g$, there exists a unique $h \in H$ such that $\lambda \cdot g = h \cdot \lambda$. We denote this element $h = \lambda(g)$; clearly this correspondence defines the desired monomorphism.
5. Another key technical point is the following: given an embedding as above, if $h \in H$ is such that $\lambda(\widetilde{U}) \cap h \cdot \lambda(\widetilde{U}) \neq \emptyset$, then $h \in \operatorname{im} \lambda$, and so $\lambda(\widetilde{U}) = h \cdot \lambda(\widetilde{U})$.
6. If $(\widetilde{U}, G, \phi)$ and $(\widetilde{V}, H, \psi)$ are two charts for the same orbifold structure on X, and if $\widetilde{U}$ is simply connected, then there exists an embedding $(\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ whenever $\phi(\widetilde{U}) \subset \psi(\widetilde{V})$.
7. Every orbifold atlas for X is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one. As with manifolds, we tend to work with a maximal atlas.
8. If the finite group actions on all the charts are free, then X is locally Euclidean, hence a manifold.

Next we define the notion of smooth maps between orbifolds.

Definition 1.3 Let $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ be orbifolds. A map $f : X \to Y$ is said to be *smooth* if for any point $x \in X$ there are charts $(\widetilde{U}, G, \phi)$ around x and $(\widetilde{V}, H, \psi)$ around $f(x)$, with the property that f maps $U = \phi(\widetilde{U})$ into $V = \psi(\widetilde{V})$ and can be lifted to a smooth map $\tilde{f} : \widetilde{U} \to \widetilde{V}$ with $\psi \tilde{f} = f \phi$.

Using this we can define the notion of *diffeomorphism* of orbifolds.

Definition 1.4 Two orbifolds $\mathcal{X}$ and $\mathcal{Y}$ are *diffeomorphic* if there are smooth maps of orbifolds $f : X \to Y$ and $g : Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$.

A more stringent definition for maps between orbifolds is required if we wish to preserve fiber bundles (as well as sheaf-theoretic constructions) on orbifolds. The notion of an *orbifold morphism* will be introduced when we discuss orbibundles; for now we just wish to mention that a diffeomorphism of orbifolds is in fact an orbifold morphism, a fact that ensures that orbifold equivalence behaves as expected.

Let X denote the underlying space of an orbifold $\mathcal{X}$, and let $x \in X$. If $(\widetilde{U}, G, \phi)$ is a chart such that $x = \phi(y) \in \phi(\widetilde{U})$, let $G_y \subseteq G$ denote the *isotropy subgroup* for the point y. We claim that up to conjugation, this group does not depend on the choice of chart. Indeed, if we used a different chart, say $(\widetilde{V}, H, \psi)$, then by our definition we can find a third chart $(\widetilde{W}, K, \mu)$ around x together with

embeddings $\lambda_1 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \phi)$ and $\lambda_2 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \psi)$. As we have seen, these inclusions are equivariant with respect to the induced injective group homomorphisms; hence the embeddings induce inclusions $K_y \hookrightarrow G_y$ and $K_y \hookrightarrow H_y$. Now applying property 5 discussed above, we see that these maps must also be onto, hence we have an isomorphism $H_y \cong G_y$. Note that if we chose a different preimage y', then G_y is conjugate to $G_{y'}$. Based on this, we can introduce the notion of a *local group* at a point $x \in X$.

Definition 1.5 Let $x \in X$, where $\mathcal{X} = (X, \mathcal{U})$ is an orbifold. If $(\widetilde{U}, G, \psi)$ is any local chart around $x = \psi(y)$, we define the *local group* at x as

$$G_x = \{g \in G \mid gy = y\}.$$

This group is uniquely determined up to conjugacy in G.

We now use the notion of local group to define the singular set of the orbifold.

Definition 1.6 For an orbifold $\mathcal{X} = (X, \mathcal{U})$, we define its *singular set* as

$$\Sigma(\mathcal{X}) = \{x \in X \mid G_x \neq 1\}.$$

This subspace will play an important role in what follows.

Before discussing any further general facts about orbifolds, it seems useful to discuss the most natural source of examples for orbifolds, namely, compact transformation groups. Let G denote a compact Lie group acting smoothly, effectively and *almost freely* (i.e., with finite stabilizers) on a smooth manifold M. Again using the fact that smooth actions on manifolds are locally smooth, we see that given $x \in M$ with isotropy subgroup G_x, there exists a chart $U \cong \mathbb{R}^n$ containing x that is G_x-invariant. The orbifold charts are then simply (U, G_x, π), where $\pi : U \to U/G_x$ is the projection map. Note that the quotient space $X = M/G$ is automatically paracompact and Hausdorff. We give this important situation a name.

Definition 1.7 An *effective quotient orbifold* $\mathcal{X} = (X, \mathcal{U})$ is an orbifold given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M; here $X = M/G$ and $\mathcal{U}$ is constructed from a manifold atlas using the locally smooth structure.

An especially attractive situation arises when the group G is finite; following established tradition, we single out this state of affairs.

Definition 1.8 If a finite group G acts smoothly and effectively on a smooth manifold M, the associated orbifold $\mathcal{X} = (M/G, \mathcal{U})$ is called an *effective global quotient*.

More generally, if we have a compact Lie group acting smoothly and almost freely on a manifold M, then there is a group extension

$$1 \to G_0 \to G \to G_{\text{eff}} \to 1,$$

where $G_0 \subset G$ is a finite group and G_{eff} acts *effectively* on M. Although the orbit spaces M/G and M/G_{eff} are identical, the reader should note that the structure on $X = M/G$ associated to the full G action will not be a classical orbifold, as the constant kernel G_0 will appear in all the charts. However, the main properties associated to orbifolds easily apply to this situation, an indication that perhaps a more flexible notion of orbifold is required – we will return to this question in Section 1.4. For a concrete example of this phenomenon, see Example 1.17.

1.2 Examples

Orbifolds are of interest from several different points of view, including representation theory, algebraic geometry, physics, and topology. One reason for this is the existence of many interesting examples constructed from different fields of mathematics. Many new phenomena (and subsequent new theorems) were first observed in these key examples, and they are at the heart of this subject.

Given a finite group G acting smoothly on a compact manifold M, the quotient M/G is perhaps the most natural example of an orbifold. We will list a number of examples which are significant in the literature, all of which arise as global quotients of an n-torus. To put them in context, we first describe a general procedure for constructing group actions on $\mathbb{T}^n = (\mathbb{S}^1)^n$. The group $GL_n(\mathbb{Z})$ acts by matrix multiplication on $\mathbb{R}^n$, taking the lattice $\mathbb{Z}^n$ to itself. This then induces an action on $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$. In fact, one can easily show that the map induced by looking at the action in homology, $\Phi : \text{Aut}(\mathbb{T}^n) \to GL_n(\mathbb{Z})$, is a split surjection. In particular, if $G \subset GL_n(\mathbb{Z})$ is a finite subgroup, then this defines an effective G-action on $\mathbb{T}^n$. Note that by construction the G-action lifts to a proper action of a discrete group Γ on $\mathbb{R}^n$; this is an example of a *crystallographic group*, and it is easy to see that it fits into a group extension of the form $1 \to (\mathbb{Z})^n \to \Gamma \to G \to 1$. The first three examples are all special cases of this construction, but are worthy of special attention due to their role in geometry and physics (we refer the reader to [4] for a detailed discussion of this class of examples).

Example 1.9 Let $\mathcal{X} = \mathbb{T}^4/(\mathbb{Z}/2\mathbb{Z})$, where the action is generated by the involution τ defined by

$$\tau(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{-it_1}, e^{-it_2}, e^{-it_3}, e^{-it_4}).$$

Note that under the construction above, τ corresponds to the matrix $-I$. This orbifold is called the *Kummer surface*, and it has sixteen isolated singular points.

Example 1.10 Let $\mathbb{T}^6 = \mathbb{C}^3/\Gamma$, where Γ is the lattice of integral points. Consider $(\mathbb{Z}/2\mathbb{Z})^2$ acting on $\mathbb{T}^6$ via a lifted action on $\mathbb{C}^3$, where the generators σ_1 and σ_2 act as follows:

$$\begin{aligned}\sigma_1(z_1, z_2, z_3) &= (-z_1, -z_2, z_3),\\ \sigma_2(z_1, z_2, z_3) &= (-z_1, z_2, -z_3),\\ \sigma_1\sigma_2(z_1, z_2, z_3) &= (z_1, -z_2, -z_3).\end{aligned}$$

Our example is $\mathcal{X} = \mathbb{T}^6/(\mathbb{Z}/2\mathbb{Z})^2$. This example was considered by Vafa and Witten [155].

Example 1.11 Let $\mathcal{X} = \mathbb{T}^6/(\mathbb{Z}/4\mathbb{Z})$. Here, the generator κ of $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{T}^6$ by

$$\kappa(z_1, z_2, z_3) = (-z_1, iz_2, iz_3).$$

This example has been studied by Joyce in [75], where he constructed five different desingularizations of this singular space. The importance of this accomplishment lies in its relation to a conjecture of Vafa and Witten, which we discuss in Chapter 4.

Algebraic geometry is another important source of examples of orbifolds. Our first example of this type is the celebrated *mirror quintic*.

Example 1.12 Suppose that Y is a degree five hypersurface of $\mathbb{C}P^4$ given by a homogeneous equation

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \phi z_0 z_1 z_2 z_3 z_4 = 0, \tag{1.1}$$

where ϕ is a generic constant. Then Y admits an action of $(\mathbb{Z}/5\mathbb{Z})^3$. Indeed, let λ be a primitive fifth root of unity, and let the generators e_1, e_2, and e_3 of $(\mathbb{Z}/5\mathbb{Z})^3$ act as follows:

$$\begin{aligned}e_1(z_0, z_1, z_2, z_3, z_4) &= (\lambda z_0, z_1, z_2, z_3, \lambda^{-1} z_4),\\ e_2(z_0, z_1, z_2, z_3, z_4) &= (z_0, \lambda z_1, z_2, z_3, \lambda^{-1} z_4),\\ e_3(z_0, z_1, z_2, z_3, z_4) &= (z_0, z_1, \lambda z_2, z_3, \lambda^{-1} z_4).\end{aligned}$$

The quotient $\mathcal{X} = Y/(\mathbb{Z}/5\mathbb{Z})^3$ is called the *mirror quintic*.

Example 1.13 Suppose that M is a smooth manifold. One can form the *symmetric product* $X_n = M^n/S_n$, where the symmetric group S_n acts on M^n by

permuting coordinates. Tuples of points have isotropy according to how many repetitions they contain, with the diagonal being the fixed point set. This set of examples has attracted a lot of attention, especially in algebraic geometry. For the special case when M is an algebraic surface, X_n admits a beautiful resolution, namely the Hilbert scheme of points of length n, denoted $X^{[n]}$. We will revisit this example later, particularly in Chapter 5.

Example 1.14 Let G be a finite subgroup of $GL_n(\mathbb{C})$ and let $\mathcal{X} = \mathbb{C}^n/G$; this is a singular complex manifold called a *quotient singularity*. $\mathcal{X}$ has the structure of an algebraic variety, arising from the algebra of G-invariant polynomials on $\mathbb{C}^n$. These examples occupy an important place in algebraic geometry related to McKay correspondence. In later applications, it will often be important to assume that $G \subset SL_n(\mathbb{C})$, in which case $\mathbb{C}^n/G$ is said to be *Gorenstein*. We note in passing that the Gorenstein condition is essentially the local version of the definition of SL-*orbifolds* given on page 15.

Example 1.15 Consider

$$\mathbb{S}^{2n+1} = \left\{ (z_0, \ldots, z_n) \mid \sum_i |z_i|^2 = 1 \right\} \subseteq \mathbb{C}^{n+1},$$

then we can let $\lambda \in \mathbb{S}^1$ act on it by

$$\lambda(z_0, \ldots, z_n) = (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n),$$

where the a_i are coprime integers. The quotient

$$\mathrm{W}\mathbb{P}(a_0, \ldots, a_n) = \mathbb{S}^{2n+1}/\mathbb{S}^1$$

is called a *weighted projective space*, and it plays the role of the usual projective space in orbifold theory. $\mathrm{W}\mathbb{P}(1, a)$, is the famous *teardrop*, which is the easiest example of a non-global quotient orbifold. We will use the orbifold fundamental group to establish this later.

Example 1.16 Generalizing from the teardrop to other two-dimensional orbifolds leads us to consider *orbifold Riemann surfaces*, a fundamental class of examples that are not hard to describe. We need only specify the (isolated) singular points and the order of the local group at each one. If x_i is a singular point with order m_i, it is understood that the local chart at x_i is $D/\mathbb{Z}_{m_i}$ where D is a small disk around zero and the action is $e \circ z = \lambda z$ for e the generator of $\mathbb{Z}_{m_i}$ and $\lambda^{m_i} = 1$.

Suppose that an orbifold Riemann surface Σ has genus g and k singular points. Thurston [149] has shown that it is a global quotient if and only if $g + 2k \geq 3$ or $g = 0$ and $k = 2$ with $m_1 = m_2$. In any case, an orbifold Riemann

surface is always a quotient orbifold, as it can be expressed as $X^3/\mathbb{S}^1$, where X^3 is a 3-manifold called a *Seifert fiber manifold* (see [140] for more on Seifert manifolds).

Example 1.17 Besides considering orbifold structures on a single surface, we can also consider various moduli spaces – or rather, moduli *stacks* – of (non-orbifold) curves. As we noted in the introduction to this chapter, these were among the first orbifolds in which the importance of the additional structure (such as isotropy groups) became evident [7]. For simplicity, we describe the orbifold structure on the moduli space of elliptic curves.

For our purposes, elliptic curves may be defined to be those tori $\mathbb{C}/L$ obtained as the quotient of the complex numbers $\mathbb{C}$ by a lattice of the form $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}^*$, where $\tau \in \mathbb{C}^*$ satisfies $\operatorname{im} \tau > 0$. Suppose we have two elliptic curves $\mathbb{C}/L$ and $\mathbb{C}/L'$, specified by elements τ and τ' in the Poincaré upper half plane $H = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$. Then $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic if there is a matrix in $SL_2(\mathbb{Z})$ that takes τ to τ', where the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

The moduli stack or orbifold of elliptic curves is then the quotient $H/SL_2(\mathbb{Z})$. This is a two-dimensional orbifold, although since the matrix $-\operatorname{Id}$ fixes every point of H, the action is not effective. We could, however, replace $G = SL_2(\mathbb{Z})$ by $G_{\text{eff}} = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(\pm \operatorname{Id})$ to obtain an associated effective orbifold. The only points with additional isotropy are the two points corresponding to $\tau = i$ and $\tau = e^{2\pi i/3}$ (which correspond to the square and hexagonal lattices, respectively). The first is fixed by a cyclic subgroup of $SL_2(\mathbb{Z})$ having order 4, while the second is fixed by one of order 6.

In Chapter 4, we will see that understanding certain moduli stacks involving orbifold Riemann surfaces is central to Chen–Ruan cohomology.

Example 1.18 Suppose that (Z, ω) is a symplectic manifold admitting a *Hamiltonian* action of the torus $\mathbb{T}^k$. This means that the torus is acting effectively by symplectomorphisms, and that there is a *moment map* $\mu : Z \to \mathfrak{t}^*$, where $\mathfrak{t}^* \cong \mathbb{R}^k$ is the dual of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}^k$. Any $v \in \mathfrak{t}$ generates a one-parameter subgroup. Differentiating the action of this one-parameter subgroup, one obtains a vector field X_v on Z. The moment map is then related to the action by requiring the equation

$$\omega(X_v, X) = d\mu(X)(v)$$

to hold for each $X \in TZ$.

One would like to study $Z/\mathbb{T}^k$ as a symplectic space, but of course even if the quotient space is smooth, it will often fail to be symplectic: for instance, it could have odd dimension. To remedy this, take a regular value $c \in \mathbb{R}^k$ of μ. Then $\mu^{-1}(c)$ is a smooth submanifold of Z, and one can show that $\mathbb{T}^k$ acts on it. The quotient $\mu^{-1}(c)/\mathbb{T}^k$ will always have a symplectic structure, although it is usually only an orbifold and not a manifold. This orbifold is called the *symplectic reduction* or *symplectic quotient* of Z, and is denoted by $Z//\mathbb{T}^k$.

The symplectic quotient depends on the choice of the regular value c. If we vary c, there is a chamber structure for $Z//\mathbb{T}^k$ in the following sense. Namely, we can divide $\mathbb{R}^k$ into subsets called *chambers* so that inside each chamber, $Z//\mathbb{T}^k$ remains the same. When we cross a wall separating two chambers, $Z//\mathbb{T}^k$ will undergo a surgery operation similar to a flip in algebraic geometry. The relation between the topology of Z and that of $Z//\mathbb{T}^k$ and the relation between symplectic quotients in different chambers have long been interesting problems in symplectic geometry – see [62] for more information.

The construction of the symplectic quotient has an analog in algebraic geometry called the *geometric invariant theory (GIT) quotient*. Instead of $\mathbb{T}^k$, one has the complex torus $(\mathbb{C}^*)^k$. The existence of an action by $(\mathbb{C}^*)^k$ is equivalent to the condition that the induced action of $\mathbb{T}^k$ be Hamiltonian. The choice of c corresponds to the choice of an ample line bundle L such that the action of $(\mathbb{C}^*)^k$ lifts to the total space of L. Taking the level set $\mu^{-1}(c)$ corresponds to the choice of semi-stable orbits.

Example 1.19 The above construction can be used to construct explicit examples. A convenient class of examples are *toric varieties*, where $Z = \mathbb{C}^r$. The combinatorial datum used to define a Hamiltonian toric action is called a *fan*. Most explicit examples arising in algebraic geometry are complete intersections of toric varieties.

Example 1.20 Let G denote a Lie group with only finitely many components. Then G has a maximal compact subgroup K, unique up to conjugacy, and the homogeneous space $X = G/K$ is diffeomorphic to $\mathbb{R}^d$, where $d = \dim G - \dim K$. Now let $\Gamma \subset G$ denote a discrete subgroup. Γ has a natural left action on this homogeneous space; moreover, it is easy to check that this is a proper action, due to the compactness of K. Consequently, all the stabilizers $\Gamma_x \subseteq \Gamma$ are finite, and each $x \in X$ has a neighborhood U such that $\gamma U \cap U = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma_x$. Clearly, this defines an orbifold structure on the quotient space X/Γ. We will call this type of example an *arithmetic orbifold*; they are of fundamental interest in many areas of mathematics, including topology and number

theory. Perhaps the favorite example is the orbifold associated to $SL_n(\mathbb{Z})$, where the associated symmetric space on which it acts is $SL_n(\mathbb{R})/SO_n \cong \mathbb{R}^d$, with $d = \frac{1}{2}n(n-1)$.

1.3 Comparing orbifolds to manifolds

One of the reasons for the interest in orbifolds is that they have geometric properties akin to those of manifolds. A central topic in orbifold theory has been to elucidate the appropriate adaptations of results from manifold theory to situations involving finite group quotient singularities.

Given an orbifold $\mathcal{X} = (X, \mathcal{U})$ let us first consider how the charts are glued together to yield the space X. Given $(\widetilde{U}, G, \phi)$ and $(\widetilde{V}, H, \psi)$ with $x \in U \cap V$, there is by definition a third chart $(\widetilde{W}, K, \mu)$ and embeddings λ_1, λ_2 from this chart into the other two. Here W is an open set with $x \in W \subset U \cap V$. These embeddings give rise to diffeomorphisms $\lambda_1^{-1} : \lambda_1(\widetilde{W}) \to \widetilde{W}$ and $\lambda_2 : \widetilde{W} \to \lambda_2(\widetilde{W})$, which can be composed to provide an equivariant diffeomorphism $\lambda_2\lambda_1^{-1} : \lambda_1(\widetilde{W}) \to \lambda_2(\widetilde{W})$ between an open set in $\widetilde{U}$ and an open set in $\widetilde{V}$. The word "equivariant" needs some explanation: we are using the fact that an embedding is an equivariant map with respect to its associated injective group homomorphism, and that the local group K associated to $\widetilde{W}$ is isomorphic to the local groups associated to its images. Hence we can regard $\lambda_2\lambda_1^{-1}$ as an equivariant diffeomorphism of K-spaces. We can then proceed to glue $\widetilde{U}/G$ and $\widetilde{V}/H$ according to the induced homeomorphism of subsets, i.e., identify $\phi(\tilde{u}) \sim \psi(\tilde{v})$ if $\lambda_2\lambda_1^{-1}(\tilde{u}) = \tilde{v}$. Now let

$$Y = \bigsqcup_{\widetilde{U} \in \mathcal{U}} (\widetilde{U}/G)/\sim$$

be the space obtained by performing these identifications on the orbifold atlas. The maps $\phi : \widetilde{U} \to X$ induce a homeomorphism $\Phi : Y \to X$.

This procedure is, of course, an analog of what takes place for manifolds, except that our gluing maps are slightly more subtle. It is worth noting that we can think of $\lambda_2\lambda_1^{-1}$ as a *transition function*. Given another λ_1' and λ_2', we have seen that there must exist unique $g \in G$ and $h \in H$ such that $\lambda_1' = g\lambda_1$ and $\lambda_2' = h\lambda_2$. Hence the resulting transition function is $h\lambda_2\lambda_1^{-1}g^{-1}$. This can be restated as follows: there is a transitive $G \times H$ action on the set of all of these transition functions.

We now use this explicit approach to construct a *tangent bundle* for an orbifold $\mathcal{X}$. Given a chart $(\widetilde{U}, G, \phi)$, we can consider the tangent bundle $T\widetilde{U}$; note that by assumption G acts smoothly on $\widetilde{U}$, hence it will also act smoothly

on $T\widetilde{U}$. Indeed, if $(\tilde{u}, v)$ is a typical element there, then $g(\tilde{u}, v) = (g\tilde{u}, Dg_{\tilde{u}}(v))$. Moreover, the projection map $T\widetilde{U} \to \widetilde{U}$ is equivariant, from which we obtain a natural projection $p : T\widetilde{U}/G \to U$ by using the map ϕ. Next we describe the fibers of this map. If $x = \phi(\tilde{x}) \in U$, then

$$p^{-1}(x) = \{G(z, v) \mid z = \tilde{x}\} \subset T\widetilde{U}/G.$$

We claim that this fiber is homeomorphic to $T_{\tilde{x}}\widetilde{U}/G_x$, where as before G_x denotes the local group at x, i.e., the isotropy subgroup of the G-action at $\tilde{x}$. Define $f : p^{-1}(x) \to T_{\tilde{x}}\widetilde{U}/G_x$ by $f(G(\tilde{x}, v)) = G_x v$. Then $G(\tilde{x}, v) = G(\tilde{x}, w)$ if and only if there exists a $g \in G$ such that $g(\tilde{x}, v) = (\tilde{x}, w)$, and this happens if and only if $g \in G_x$ and $D_{\tilde{x}}g(v) = w$. This is equivalent to the assertion that $G_x v = G_x w$. So f is both well defined and injective. Continuity and surjectivity are clear, establishing our claim. What this shows is that we have constructed (locally) a bundle-like object where the fiber is no longer a vector space, but rather a quotient of the form $\mathbb{R}^n/G_0$, where $G_0 \subset GL_n(\mathbb{R})$ is a finite group.

It should now be clear how to construct the tangent bundle on an orbifold $\mathcal{X} = (X, \mathcal{U})$: we simply need to glue together the bundles defined over the charts. Our resulting space will be an orbifold, with an atlas $T\mathcal{U}$ comprising local charts $(T\widetilde{U}, G, \pi)$ over $TU = T\widetilde{U}/G$ for each $(\widetilde{U}, G, \phi) \in \mathcal{U}$. We observe that the gluing maps $\lambda_{12} = \lambda_2\lambda_1^{-1}$ we discussed earlier are smooth, so we can use the transition functions $D\lambda_{12} : T\lambda_1(\widetilde{W}) \to T\lambda_2(\widetilde{W})$ to glue $T\widetilde{U}/G \to U$ to $T\widetilde{V}/H \to V$. In other words, we define the space TX as an identification space $\bigsqcup_{\widetilde{U} \in \mathcal{U}} (T\widetilde{U}/G)/\sim$, where we give it the minimal topology that will make the natural maps $T\widetilde{U}/G \to TX$ homeomorphisms onto open subsets of TX. We summarize this in the next proposition.

Proposition 1.21 *The tangent bundle of an n-dimensional orbifold $\mathcal{X}$, denoted by $T\mathcal{X} = (TX, T\mathcal{U})$, has the structure of a $2n$-dimensional orbifold. Moreover, the natural projection $p : TX \to X$ defines a smooth map of orbifolds, with fibers $p^{-1}(x) \cong T_{\tilde{x}}\widetilde{U}/G_x$.*

In bundle theory, one of the classical constructions arising from a vector bundle is the associated principal $GL_n(\mathbb{R})$ bundle. In the case of a paracompact Hausdorff base space, we can reduce the structural group to $O(n)$ by introducing a fiberwise inner product. This construction applied to a manifold M gives rise to a principal $O(n)$-bundle, known as the frame bundle of M; its total space $\mathrm{Fr}(M)$ is a manifold endowed with a free, smooth $O(n)$-action such that $\mathrm{Fr}(M)/O(n) \cong M$. We now proceed to adapt this construction to orbifolds using the basic method of constructing a principal bundle from a vector bundle, namely, by replacing the fibers with their automorphism groups as explained by Steenrod in [146].

In this case, given a local chart $(\widetilde{U}, G, \phi)$ we choose a G-invariant inner product on $T\widetilde{U}$. We can then construct the manifold

$$\mathrm{Fr}(\widetilde{U}) = \{(\tilde{x}, B) \mid B \in O(T_{\tilde{x}}\widetilde{U})\}$$

and consider the induced left G-action on it:

$$g(\tilde{x}, B) = (g\tilde{x}, Dg_{\tilde{x}}B).$$

Since we have assumed that the G-action on $\widetilde{U}$ is effective, the G-action on frames is free, and so the quotient $\mathrm{Fr}(\widetilde{U})/G$ is a smooth manifold. It has a right $O(n)$ action inherited from the natural translation action on $\mathrm{Fr}(\widetilde{U})$, given by $[\tilde{x}, B]A = [\tilde{x}, BA]$. Note that this action is transitive on fibers; indeed, $[\tilde{x}, A] = [\tilde{x}, I]A$. The isotropy subgroup for this orbit consists of those orthogonal matrices A such that $(\tilde{x}, A) = (g\tilde{x}, Dg_{\tilde{x}})$ for some $g \in G$. This simply means that $g \in G_x$ and $A = Dg_{\tilde{x}}$; the differential establishes an injection $G_x \to O(T_{\tilde{x}}\widetilde{U})$. We conclude that G_x is precisely the isotropy subgroup of this action, and that the fiber is simply the associated homogeneous space $O(n)/G_x$. If we take the quotient by this action in $\mathrm{Fr}(\widetilde{U})/G$, we obtain (up to isomorphism) the natural projection $\mathrm{Fr}(\widetilde{U})/G \to U$.

Now we proceed as before, and glue these local charts using the appropriate transition functions.

Definition 1.22 The *frame bundle* of an orbifold $\mathcal{X} = (X, \mathcal{U})$ is the space $\mathrm{Fr}(\mathcal{X})$ obtained by gluing the local charts $\mathrm{Fr}(\widetilde{U})/G \to U$ using the $O(n)$-transition functions obtained from the tangent bundle of X.

This object has some useful properties, which we now summarize.

Theorem 1.23 *For a given orbifold $\mathcal{X}$, its frame bundle* $\mathrm{Fr}(\mathcal{X})$ *is a smooth manifold with a smooth, effective, and almost free $O(n)$-action. The original orbifold $\mathcal{X}$ is naturally isomorphic to the resulting quotient orbifold* $\mathrm{Fr}(\mathcal{X})/O(n)$.

Proof We have already remarked that $\mathrm{Fr}(\mathcal{X})$ is locally Euclidean. By gluing the local frame bundles as indicated, we obtain a compatible $O(n)$-action on the whole space. We know that the isotropy is finite, and acts non-trivially on the tangent space to $\mathrm{Fr}(\mathcal{X})$ due to the effectiveness hypothesis on the original orbifold. The local charts obtained for the quotient space $\mathrm{Fr}(\mathcal{X})/O(n)$ are of course equivalent to those for X; indeed, locally this quotient is of the form $V \times_G O(n) \to V/G$, where $G \subset O(n)$ via the differential. $\square$

The following is a very important consequence of this theorem.

Corollary 1.24 *Every classical n-orbifold $\mathcal{X}$ is diffeomorphic to a quotient orbifold for a smooth, effective, and almost free $O(n)$-action on a smooth manifold M.*

What we see from this is that classical orbifolds can all be studied using methods developed for almost free actions of compact Lie groups. Note that an orbifold can be expressed as a quotient in different ways, which will be illustrated in the following result.

Proposition 1.25 *Let M be a compact manifold with a smooth, almost free and effective action of G, a compact Lie group. Then the frame bundle* $\mathrm{Fr}(M)$ *of M has a smooth, almost free $G \times O(n)$ action such that the following diagram of quotient orbifolds commutes:*

$$\begin{tikzcd} \mathrm{Fr}(M) \arrow[r, "/O(n)"] \arrow[d, "/G"] & M \arrow[d, "/G"] \\ \mathrm{Fr}(M/G) \arrow[r, "/O(n)"] & M/G \end{tikzcd},$$

In particular, we have a natural isomorphism $\mathrm{Fr}(M)/G \cong \mathrm{Fr}(M/G)$.

Proof The action of $G \times O(n)$ is defined just as we defined the action on the local frame bundle $\mathrm{Fr}(U)$. Namely if $(g, A) \in G \times O(n)$, and $(m, B) \in \mathrm{Fr}(M)$, then we let $(g, A)(m, B) = (gm, ABDg_m^{-1})$. If we divide by the G action (as before), we obtain $\mathrm{Fr}(M/G)$, and the remaining $O(n)$ action is the one on the frames. If we take the quotient by the $O(n)$ action first, then we obtain M by definition, and obviously the remaining G action is the original one on M. □

Note here that the quotient orbifold M/G is also the quotient orbifold $\mathrm{Fr}(M/G)/O(n)$. We shall say that these are two distinct *orbifold presentations* for $X = M/G$.

It is clear that we can define the notion of orientability for an orbifold in terms of its charts and transition functions. Moreover, if an orbifold $\mathcal{X}$ is orientable, then we can consider oriented frames, and so we obtain the oriented frame bundle $\mathrm{Fr}^+(\mathcal{X})$ with an action of $SO(n)$ analogous to the $O(n)$ action previously discussed.

Example 1.26 Let Σ denote a compact orientable Riemann surface of genus $g \geq 2$, and let G denote a group of automorphisms of Σ. Such a group must necessarily be finite and preserve orientation. Moreover, the isotropy subgroups are all cyclic. Let us consider the global quotient orbifold $\mathcal{X} = \Sigma/G$, which is orientable. The oriented frame bundle $\mathrm{Fr}^+(\Sigma)$ is a compact 3-manifold with

an action of $G \times SO(2)$. The G-action is free, and the quotient $\mathrm{Fr}^+(\Sigma/G)$ is again a 3-manifold, now with an $SO(2)$-action, whose quotient is $\mathcal{X}$.

The so-called tangent bundle $T\mathcal{X}$ is, of course, not a vector bundle, unless $\mathcal{X}$ is a manifold. It is an example of what we will call an *orbibundle*. More generally, given any continuous functor F from vector spaces to vector spaces (see [110, p. 31]), we can use the same method to extend F to orbibundles, obtaining an orbibundle $F(T\mathcal{X}) \to X$ with fibers $F(T_{\tilde{x}}\widetilde{U})/G_x$. In particular, this allows us to construct the cotangent bundle $T^*\mathcal{X}$ and tensor products $S^k(T\mathcal{X})$, as well as the exterior powers $\bigwedge T^*\mathcal{X}$ used in differential geometry and topology.

We will also need to define what we mean by a section of an orbibundle. Consider for example the tangent bundle $T\mathcal{X} \to \mathcal{X}$. A *section s* consists of a collection of sections $s : \widetilde{U} \to T\widetilde{U}$ for the local charts which are (1) equivariant with respect to the action of the local group G and (2) compatible with respect to transition maps and the associated gluing. Alternatively, we could study orbibundles via the frame bundle: there is an $O(n)$ action on the tangent bundle $T\,\mathrm{Fr}(\mathcal{X}) \to \mathrm{Fr}(\mathcal{X})$ of $\mathrm{Fr}(\mathcal{X})$, and the tangent bundle for $\mathcal{X}$ can be identified with the resulting quotient $T\,\mathrm{Fr}(\mathcal{X})/O(n) \to \mathrm{Fr}(\mathcal{X})/O(n)$. In this way we can identify the sections of $T\mathcal{X} \to \mathcal{X}$ with the $O(n)$-equivariant sections of the tangent bundle of $\mathrm{Fr}(\mathcal{X})$. This point of view can, of course, be applied to any quotient orbifold.

From this we obtain a whole slew of classical invariants for orbifolds that are completely analogous to the situation for manifolds. Below we will list orbifold versions of some useful constructions that we will require later. Given that our goal is to develop stringy invariants of orbifolds, we will not dwell on these fundamental but well-understood aspects of orbifold theory; rather, we will concentrate on aspects relevant to current topics such as orbifold cohomology, orbifold K-theory, and related topics.

Definition 1.27 Let $\mathcal{X}$ denote an orbifold with tangent bundle $T\mathcal{X}$.

1. We call a non-degenerate symmetric 2-tensor of $S^2(T\mathcal{X})$ a *Riemannian metric* on $\mathcal{X}$.
2. An *almost complex structure* on $\mathcal{X}$ is an endomorphism $J : T\mathcal{X} \to T\mathcal{X}$ such that $J^2 = -\,\mathrm{Id}$.
3. We define a *differential k-form* as a section of $\bigwedge^k T^*\mathcal{X}$; the exterior derivative is defined as for manifolds in the usual way. Hence we can define the de Rham cohomology $H^*(\mathcal{X})$.
4. A *symplectic structure* on $\mathcal{X}$ is a non-degenerate closed 2-form.
5. We call $\mathcal{X}$ a *complex* orbifold if all the defining data are holomorphic. For complex orbifolds, we can define Dolbeault cohomology in the usual way.

For an almost complex orbifold $\mathcal{X}$ with underlying space X, we define its *canonical bundle* as $K_{\mathcal{X}} = \bigwedge_{\mathbb{C}}^{m} T^*\mathcal{X}$, where m is the dimension of $\mathcal{X}$ and we are providing the cotangent bundle with a complex structure in the usual way. Note that $K_{\mathcal{X}}$ is a complex orbibundle over $\mathcal{X}$, and that the fiber at any given point $x \in X$ is of the form $\mathbb{C}/G_x$. The action of G_x on the fiber $\mathbb{C}$ can be thought of as follows: G_x acts on the fiber of the tangent bundle, which may be identified with $\mathbb{C}^m$ using the complex structure. The induced action on the fiber $\mathbb{C}$ is via the determinant associated to this representation. Hence if $G_x \subset SL_m(\mathbb{C})$ for all $x \in X$, then the canonical bundle will be an honest line bundle. In that case, we will say that $\mathcal{X}$ is an *SL-orbifold*. $\mathcal{X}$ is *Calabi–Yau* if $K_{\mathcal{X}}$ is a trivial line bundle. Note that if $\mathcal{X}$ is compact, then there always exists an integer $N > 0$ such that $K_{\mathcal{X}}^N$ is an honest line bundle. For instance, take N to be the least common multiple of the exponents of the isotropy groups of $\mathcal{X}$.

As in the manifold case, it turns out that de Rham cohomology of an orbifold $\mathcal{X}$ is isomorphic to the singular cohomology of the underlying space with real coefficients, and so it is independent of the orbifold structure. We can also define de Rham cohomology with compact supports, and it will again agree with the compactly supported singular version. Nevertheless, we will study both of these theories in more detail and generality in the next chapter so that we can extend them to Chen–Ruan cohomology in Chapter 4.

Using the frame bundle of an orbifold, we see that techniques applicable to quotient spaces of almost free smooth actions of Lie groups will yield results about orbifolds. For example, we have (see [6]):

Proposition 1.28 *If a compact, connected Lie group G acts smoothly and almost freely on an orientable, connected, compact manifold M, then $H^*(M/G;\mathbb{Q})$ is a Poincaré duality algebra. Hence, if $\mathcal{X}$ is a compact, connected, orientable orbifold, then $H^*(X;\mathbb{Q})$ will satisfy Poincaré duality.*

In this section we have only briefly touched on the many manifold-like properties of orbifolds. In later sections we will build on these facts to develop the newer, "stringy" invariants which tend to emphasize *differences* instead of similarities between them.

1.4 Groupoids

In this section we will reformulate the notion of an orbifold using the language of groupoids. This will allow us to define a more general version of an orbifold, relaxing our effectiveness condition from the previous sections. As we have noted already, ineffective orbifolds occur in nature, and it turns out that many

natural and useful constructions, such as taking the twisted sectors of an orbifold, force one outside the effective category. Maybe even more importantly, the groupoid language seems to be best suited to a discussion of *orbifold morphisms* and the *classifying spaces* associated to orbifold theory. The price one pays is that of a somewhat misleading abstraction, which can detract from the geometric problems and examples which are the actual objects of our interest. We will keep a reasonable balance between these points of view in the hope of convincing the reader that both are worthwhile and are valuable perspectives on the subject. This section is based on the excellent exposition due to Moerdijk [112]; the reader should consult his paper for a full account.

Recall that a *groupoid* is a (small) category in which every morphism is an isomorphism. One can think of groupoids as simultaneous generalizations of groups and equivalence relations, for a groupoid with one object is essentially the same thing as the automorphism group of that object, and a groupoid with only trivial automorphisms determines and is determined by an equivalence relation on the set of objects. Now, just as one studies group objects in the topological and smooth categories to obtain topological and Lie groups, one can also study groupoids endowed with topologies.

Definition 1.29 A *topological groupoid* $\mathcal{G}$ is a groupoid object in the category of topological spaces. That is, $\mathcal{G}$ consists of a space G_0 of *objects* and a space G_1 of *arrows*, together with five continuous structure maps, listed below.

1. The *source map* $s : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its *source* $s(g)$.
2. The *target map* $t : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its *target* $t(g)$. For two objects $x, y \in G_0$, one writes $g : x \to y$ or $x \xrightarrow{g} y$ to indicate that $g \in G_1$ is an arrow with $s(g) = x$ and $t(g) = y$.
3. The composition map $m : G_1 \, {}_s\!\times_t G_1 \to G_0$. If g and h are arrows with $s(h) = t(g)$, one can form their *composition* hg, with $s(hg) = s(g)$ and $t(hg) = t(h)$. If $g : x \to y$ and $h : y \to z$, then hg is defined and $hg : x \to z$. The *composition map*, defined by $m(h, g) = hg$, is thus defined on the fibered product
$$G_1 \, {}_s\!\times_t G_1 = \{(h, g) \in G_1 \times G_1 \mid s(h) = t(g)\},$$
and is required to be associative.
4. The *unit* (or *identity*) map $u : G_0 \to G_1$, which is a two-sided unit for the composition. This means that $su(x) = x = tu(x)$, and that $gu(x) = g = u(y)g$ for all $x, y \in G_0$ and $g : x \to y$.
5. An *inverse map* $i : G_1 \to G_1$, written $i(g) = g^{-1}$. Here, if $g : x \to y$, then $g^{-1} : y \to x$ is a two-sided inverse for the composition, which means that $g^{-1}g = u(x)$ and $gg^{-1} = u(y)$.

Definition 1.30 A *Lie groupoid* is a topological groupoid $\mathcal{G}$ where G_0 and G_1 are smooth manifolds, and such that the structure maps, s, t, m, u and i, are smooth. Furthermore, s and $t : G_1 \to G_0$ are required to be submersions (so that the domain $G_1 \, {}_s\!\times_t \, G_1$ of m is a smooth manifold). We always assume that G_0 and G_1 are Hausdorff.

Our first examples are well known.

Example 1.31 Let M be a smooth manifold and let $G_0 = G_1 = M$. This gives rise to a Lie groupoid whose arrows are all units – all five structure maps are the identity $M \to M$. Thus, this construction is often referred to as the *unit groupoid* on M.

Example 1.32 Suppose a Lie group K acts smoothly on a manifold M from the left. One defines a Lie groupoid $K \ltimes M$ by setting $(K \ltimes M)_0 = M$ and $(K \ltimes M)_1 = K \times M$, with $s : K \times M \to M$ the projection and $t : K \times M \to M$ the action. Composition is defined from the multiplication in the group K, in an obvious way. This groupoid is called the *action groupoid* or *translation groupoid* associated to the group action. The unit groupoid is the action groupoid for the action of the trivial group. On the other hand, by taking M to be a point we can view any Lie group K as a Lie groupoid having a single object.

Some authors write $[M/G]$ for the translation groupoid, although more often that notation indicates the *quotient stack*. For more on the stack perspective, see [50, 109].

Example 1.33 Let $(X, \mathcal{U})$ be a space with an manifold atlas $\mathcal{U}$. Then we can associate to it a groupoid $\mathcal{G}_{\mathcal{U}}$ in the following way: the space of objects is the disjoint union

$$\bigsqcup_{\alpha} U_\alpha$$

of all the charts, and the arrows are the fibered products

$$\bigsqcup_{\alpha,\beta} U_\alpha \times_X U_\beta,$$

where (x_1, x_2) in $U_\alpha \times_X U_\beta$ is an arrow from x_1 to x_2, so that $|\mathcal{G}_{\mathcal{U}}| \cong X$.

Example 1.34 Let M denote a connected manifold. Then the *fundamental groupoid* $\Pi(M)$ of M is the groupoid with $\Pi(M)_0 = M$ as its space of objects, and an arrow $x \to y$ for each homotopy class of paths from x to y.

Definition 1.35 Let $\mathcal{G}$ be a Lie groupoid. For a point $x \in G_0$, the set of all arrows from x to itself is a Lie group, denoted by G_x and called the *isotropy* or

local group at x. The set $ts^{-1}(x)$ of targets of arrows out of x is called the *orbit* of x. The *orbit space* $|\mathcal{G}|$ of $\mathcal{G}$ is the quotient space of G_0 under the equivalence relation $x \sim y$ if and only if x and y are in the same orbit.[2] Conversely, we call $\mathcal{G}$ a *groupoid presentation* of $|\mathcal{G}|$.

At this stage, we impose additional restrictions on the groupoids we consider, as we shall see that the groupoids associated to orbifolds are rather special. The following definitions are essential in characterizing such groupoids.

Definition 1.36 Let $\mathcal{G}$ be a Lie groupoid.

- $\mathcal{G}$ is *proper* if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map. Note that in a proper Lie groupoid $\mathcal{G}$, every isotropy group is compact.
- $\mathcal{G}$ is called a *foliation groupoid* if each isotropy group G_x is discrete.
- $\mathcal{G}$ is *étale* if s and t are local diffeomorphisms. If $\mathcal{G}$ is an étale groupoid, we define its *dimension* $\dim \mathcal{G} = \dim G_1 = \dim G_0$. Note that every étale groupoid is a foliation groupoid.

Let us try to understand the effects that these conditions have on a groupoid.

Proposition 1.37 *If $\mathcal{G}$ is a Lie groupoid, then for any $x \in G_0$ the isotropy group G_x is a Lie group. If $\mathcal{G}$ is proper, then every isotropy group is a compact Lie group. In particular, if $\mathcal{G}$ is a proper foliation groupoid, then all of its isotropy groups are finite.*

Proof Recall that given $x \in G_0$, we have defined its isotropy group as

$$G_x = \{g \in G_1 \mid (s, t)(g) = (x, x)\} = (s, t)^{-1}(x, x) = s^{-1}(x) \cap t^{-1}(x) \subset G_1.$$

Given that s and t are submersions, we see that G_x is a closed, smooth submanifold of G_1, with a smooth group structure, so G_x is a Lie group. Therefore, for a proper Lie groupoid $\mathcal{G}$ all the G_x are compact Lie groups. Now if $\mathcal{G}$ is also a foliation groupoid, each G_x is a compact discrete Lie group, and hence is finite. □

In particular, when we regard a Lie group G as a groupoid having a single object, the result is a proper étale groupoid if and only if G is finite. We call such groupoids *point orbifolds*, and denote them by $\bullet^G$. As we shall see, even this seemingly trivial example can exhibit interesting behavior.

[2] The reader should take care not to confuse the quotient functor $|\mathcal{G}|$ with the geometric realization functor, which some authors write similarly. In this book, $|\mathcal{G}|$ will always mean the quotient unless specifically stated otherwise.

Consider the case of a general proper étale groupoid $\mathcal{G}$. Given $x \in G_0$, there exists a sufficiently small neighborhood U_x of x such that G_x acts on U_x in the following sense. Given $g \in G_x$, let $\phi : U_x \to V_g$ be a local inverse to s; assume furthermore that t maps V_g diffeomorphically onto U_x. Now define $\tilde{g} : U_x \to U_x$ as the diffeomorphism $\tilde{g} = t\phi$. This defines a group homomorphism $G_x \to \mathrm{Diff}(U_x)$. At this point the reader should be starting to see an orbifold structure emerging from these groupoids – we will revisit this construction and make the connection explicit shortly. For now, note that the construction above actually produces a well-defined germ of a diffeomorphism.

Definition 1.38 We define an *orbifold groupoid* to be a proper étale Lie groupoid. An orbifold groupoid $\mathcal{G}$ is *effective* if for every $x \in G_0$ there exists an open neighborhood U_x of x in G_0 such that the associated homomorphism $G_x \to \mathrm{Diff}(U_x)$ is injective.

Other authors sometimes use the term orbifold groupoid for proper foliation Lie groupoids. As we shall see, up to "Morita equivalence" this amounts to the same thing. Next, we discuss morphisms between groupoids and their natural transformations.

Definition 1.39 Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A *homomorphism* $\phi : \mathcal{H} \to \mathcal{G}$ consists of two smooth maps, $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$, that together commute with all the structure maps for the two groupoids $\mathcal{G}$ and $\mathcal{H}$. Often, one omits the subscripts when the context makes it clear whether we are talking about objects or arrows.

Definition 1.40 Let $\phi, \psi : \mathcal{H} \to \mathcal{G}$ be two homomorphisms. A *natural transformation* α from ϕ to ψ (notation: $\alpha : \phi \to \psi$) is a smooth map $\alpha : H_0 \to G_1$ giving for each $x \in H_0$ an arrow $\alpha(x) : \phi(x) \to \psi(x)$ in G_1, natural in x in the sense that for any $h : x \to x'$ in H_1 the identity $\psi(h)\alpha(x) = \alpha(x')\phi(h)$ holds.

Definition 1.41 Let $\phi : \mathcal{H} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{G}$ be homomorphisms of Lie groupoids. The *fibered product* $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is the Lie groupoid whose objects are triples (y, g, z), where $y \in H_0, z \in K_0$ and $g : \phi(y) \to \psi(z)$ in G_1. Arrows $(y, g, z) \to (y', g', z')$ in $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ are pairs (h, k) of arrows, $h : y \to y'$ in H_1 and $k : z \to z'$ in K_1, with the property that $g'\phi(h) = \psi(k)g$. We represent this in the following diagram:

$$
\begin{tikzcd}
y \arrow[d, "h"'] & \phi(y) \arrow[r, "g"] \arrow[d, "\phi(h)"'] & \psi(z) \arrow[d, "\psi(k)"] & z \arrow[d, "k"] \\
y' & \phi(y') \arrow[r, "g'"] & \psi(z') & z'
\end{tikzcd}
$$

Composition in $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is defined in an obvious way.

The fibered product of two Lie groupoids is a Lie groupoid as soon as the space $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 = H_0 \times_{G_0} G_1 \times_{G_0} K_0$ is a manifold. For instance, this is certainly the case when the map $t\pi_2 : H_0 \times_{G_0} G_1 \to G_0$ is a submersion. The fibered product sits in a square of homomorphisms

$$\begin{tikzcd} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{H} \arrow[r, "\phi"] & \mathcal{G} \end{tikzcd}, \tag{1.2}$$

which commutes up to a natural transformation, and it is universal with this property.

Definition 1.42 A homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ between Lie groupoids is called an *equivalence* if

(i) the map

$$t\pi_1 : G_1 \,{}_s\!\times_\phi H_0 \to G_0$$

defined on the fibered product of manifolds $\{(g, y) \mid g \in G_1, y \in H_0, s(g) = \phi(y)\}$ is a surjective submersion;

(ii) the square

$$\begin{tikzcd} H_1 \arrow[r, "\phi"] \arrow[d, "{(s,t)}"'] & G_1 \arrow[d, "{(s,t)}"] \\ H_0 \times H_0 \arrow[r, "\phi\times\phi"] & G_0 \times G_0 \end{tikzcd}$$

is a fibered product of manifolds.

The first condition implies that every object $x \in G_0$ can be connected by an arrow $g : \phi(y) \to x$ to an object in the image of ϕ, i.e., ϕ is *essentially surjective* as a functor. The second condition implies that ϕ induces a diffeomorphism

$$H_1(y, z) \to G_1(\phi(y), \phi(z))$$

from the space of all arrows $y \to z$ in H_1 to the space of all arrows $\phi(y) \to \phi(z)$ in G_1. In particular, then, ϕ is *full* and *faithful* as a functor. Taken together, these conditions are thus quite similar to the usual notion of equivalence of categories. If instead of Definition 1.42 we require that the map $\phi : H_0 \to G_0$ already be a surjective submersion, then we say that ϕ is a *strong equivalence*.

It is clear that a homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ induces a continuous map $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ between quotient spaces; moreover, if ϕ is an equivalence, $|\phi|$ is a homeomorphism.

A more subtle but extremely useful notion is that of *Morita equivalence* of groupoids.

Definition 1.43 Two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ are said to be *Morita equivalent* if there exists a third groupoid $\mathcal{H}$ and two equivalences

$$\mathcal{G} \xleftarrow{\phi} \mathcal{H} \xrightarrow{\phi'} \mathcal{G}'.$$

Using the fibered product of groupoids, it can be shown that this defines an equivalence relation.

It turns out that given an equivalence between groupoids $\phi : \mathcal{H} \to \mathcal{G}$, this implies that there are strong equivalences $f_1 : \mathcal{K} \to \mathcal{H}$ and $f_2 : \mathcal{K} \to \mathcal{G}$. In particular, $\mathcal{H}$ is Morita equivalent to $\mathcal{G}$ via strong equivalences. Hence the notion of Morita equivalence can be defined with either kind of equivalence and they produce exactly the same result. Sometimes (for technical purposes) we will prefer to use strong equivalences in our Morita equivalence relation.

A number of properties are invariant under Morita equivalence; for example if $\phi : \mathcal{H} \to \mathcal{G}$ is a Morita equivalence, $\mathcal{H}$ is proper if and only if $\mathcal{G}$ is proper. Similarly, $\mathcal{H}$ is a foliation groupoid if and only if $\mathcal{G}$ is one. However, being étale is *not* invariant under Morita equivalence. In fact, a result of Crainic and Moerdijk [39] shows that a Lie groupoid is a foliation groupoid if and only if it is Morita equivalent to an étale groupoid. On the other hand, one can show that given two Morita equivalent étale groupoids one of them is effective if and only if the other one is too.

We now spell out the relationship between the classical orbifolds defined at the beginning of this chapter and orbifold groupoids. Let $\mathcal{G}$ be an orbifold groupoid, and consider the topological space $|\mathcal{G}|$, the orbit space of the groupoid.

Proposition 1.44 *Let $\mathcal{G}$ be a proper, effective, étale groupoid. Then its orbit space $X = |\mathcal{G}|$ can be given the structure of an effective orbifold, explicitly constructed from the groupoid $\mathcal{G}$.*

Proof We follow the exposition in [113]. Let $\pi : G_0 \to X$ denote the quotient map, where we identify two points $x, y \in G_0$ if and only if there exists an arrow $g : x \to y$ in G_1. As s and t are both open, so is π; also, X is Hausdorff (because $(s, t) : G_1 \to G_0 \times G_0$ is proper) and paracompact (actually, a metric space). Fix a point $x \in G_0$. We have seen that G_x is a finite group. For each $g \in G_x$, choose an open neighborhood W_g of g in G_1, sufficiently small so that both s and t restrict to diffeomorphisms into G_0, and such that these W_g are pairwise disjoint. Next, we further shrink these open sets: let $U_x = \cap_{g \in G_x} s(W_g)$. Using properness of (s, t) again, we get an open neighborhood $V_x \subset U_x$ so

that

$$(V_x \times V_x) \cap (s,t)(G_1 - \cup_g W_g) = \emptyset.$$

So for any $h \in G_1$, if $s(h)$ and $t(h)$ are in V_x, then $h \in W_g$ for some $g \in G_x$.

Now consider the diffeomorphism $t \circ (s|_{W_g})^{-1} = \tilde{g} : s(W_g) \to t(W_g)$. As $V_x \subset s(W_g)$ for all $g \in G_x$, each $\tilde{g}$ is defined on the open set V_x. Define a still smaller neighborhood $N_x \subset V_x$ by

$$N_x = \{y \in V_x \mid \tilde{g}(y) \in V_x \quad \forall g \in G_x\}.$$

Then if $y \in N_x$, for any $g \in G_x$ we will have $\tilde{g}(y) \in N_x$. Thus the group G_x acts on N_x via $g \cdot x = \tilde{g}(x)$. Note that our assumption that $\mathcal{G}$ is an effective groupoid ensures that this action of G_x is effective. For each $g \in G_x$ we can define $O_g = W_g \cap s^{-1}(N_x) = W_g \cap (s,t)^{-1}(N_x \times N_x)$. For each $k \in G_1$, if $s(k), t(k) \in N_x$, then $k \in O_g$ for some $g \in G_x$. From this we see that $G_1 \cap (s,t)^{-1}(N_x \times N_x)$ is the disjoint union of the open sets O_g.

We conclude from this that the restriction of the groupoid $\mathcal{G}$ over N_x is isomorphic to the translation groupoid $G_x \ltimes N_x$, and $N_x/G_x \subset X$ is an open embedding. We conclude that G_0 has a basis of open sets N_x, each with G_x-action as described before. To verify that they form an atlas for an orbifold structure on X, we just need to construct suitable embeddings between them. Let (N_x, G_x) and (N_y, G_y) denote two such charts, and let $z \in G_0$ be such that $\pi(z) \in \pi(N_x) \cap \pi(N_y)$. Let $g : z \to x' \in N_x$ and $h : x \to y' \in N_y$ be any arrows in G_1. Let W_g and W_h be neighborhoods for which s and t restrict to diffeomorphisms, and let (N_z, G_z) be a chart at z. Choose W_g, W_h, and N_z sufficiently small so that $s(W_g) = N_z = s(W_h)$, while $t(W_g) \subset N_x$ and $t(W_h) \subset N_y$). Then $\tilde{g} = t \circ (s|_{W_g})^{-1} : N_z \hookrightarrow N_x$, together with $\tilde{h} : N_z \hookrightarrow N_y$ are the required embeddings. To summarize: we have shown that the charts $(N_x, G_x, \pi : N_x \to N_x/G_x \subseteq X)$ form a well-defined orbifold structure for X. $\square$

The following basic theorem appears in [113].

Theorem 1.45 *Two effective orbifold groupoids $\mathcal{G}$ and $\mathcal{G}'$ represent the same effective orbifold up to isomorphism if and only if they are Morita equivalent.*

Conversely, if we are given an effective orbifold $\mathcal{X}$, we have seen that it is equivalent to the quotient orbifold arising from the $O(n)$ action on its frame bundle $\mathrm{Fr}(\mathcal{X})$. Let $\mathcal{G}_{\mathcal{X}} = O(n) \ltimes \mathrm{Fr}(\mathcal{X})$ denote the associated action groupoid; then it is clear that $|\mathcal{G}_{\mathcal{X}}| \cong \mathcal{X}$ as orbifolds. One can also show (using slices) that $O(n) \ltimes \mathrm{Fr}(\mathcal{X})$ is Morita equivalent to an effective orbifold groupoid.

Remark 1.46 In general, the question of whether or not every ineffective orbifold has a quotient presentation M/G for some compact Lie group G remains open. Some partial results, and a reduction of the problem to one involving equivariant gerbes, appear in [69].

We now pause to consider what we have learned. Given an orbifold $\mathcal{X}$, with underlying space X, its structure is completely described by the Morita equivalence class of an associated effective orbifold groupoid $\mathcal{G}$ such that $|\mathcal{G}| \cong X$. Based on this, we now give the general definition of an orbifold, dropping the classical effective condition.

Definition 1.47 An *orbifold structure* on a paracompact Hausdorff space X consists of an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. If $\phi : \mathcal{H} \to \mathcal{G}$ is an equivalence, then $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ is a homeomorphism, and we say the composition $f \circ |\phi| : |\mathcal{H}| \to X$ defines an *equivalent* orbifold structure on X.

If $\mathcal{G}$ represents an orbifold structure for X, and if $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent, then from the above the two define an equivalent orbifold structure on X.

Definition 1.48 An *orbifold* $\mathcal{X}$ is a space X equipped with an equivalence class of orbifold structures. A specific such structure, given by $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$, is called a *presentation* of the orbifold $\mathcal{X}$.

Example 1.49 If we allow the weights to have a common factor, the weighted projective space $W\mathbb{P}(a_0, \ldots, a_n) = \mathbb{S}^{2n+1}/\mathbb{S}^1$ will fail to be effective. However, it is still an orbifold under our extended definition. The same is true for the moduli stack of elliptic curves $SL_2(\mathbb{Z}) \ltimes H$ in Example 1.17.

We can now use the groupoid perspective to introduce a suitable notion of a map between orbifolds. Given an orbifold atlas, we want to be allowed to take a refinement before defining our map. In the groupoid terminology, this corresponds to allowing maps from $\mathcal{H}$ to $\mathcal{G}$ which factor through a Morita equivalence. Hence, we need to consider pairs

$$\mathcal{H} \xleftarrow{\epsilon} \mathcal{H}' \xrightarrow{\phi} \mathcal{G}, \tag{1.3}$$

where ϵ is an equivalence and ϕ is a homomorphism of groupoids. We call the pair (ϵ, ϕ) an *orbifold morphism* or *generalized map* between groupoids. We define a map $\mathcal{Y} \to \mathcal{X}$ between two orbifolds presented by groupoids $\mathcal{G}_{\mathcal{Y}}$ and $\mathcal{G}_{\mathcal{X}}$ to consist of a continuous map of underlying spaces $|\mathcal{G}_{\mathcal{Y}}| \to |\mathcal{G}_{\mathcal{X}}|$, together with a generalized map of orbifold groupoids for which the following diagram

commutes:

$$\begin{tikzcd} \mathcal{G}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{G}_{\mathcal{X}} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}.$$

We will not dwell here on the notion of a map between orbifolds, as full precision actually requires that we first construct a quotient category by identifying homomorphisms for which there exists a natural transformation between them, and then "invert" all arrows represented by equivalences. This is called a category of fractions, in the sense of Gabriel and Zisman (see [112, p. 209]). Roughly speaking, what we have described is a definition of orbifolds as a full subcategory of the category of Lie groupoids and generalized maps. We remark that these generalized maps are often referred to as *good* or *strong* maps in the literature. Their main use is in pulling back bundle data, as we shall see when we revisit them in Section 2.4.

Given a Lie groupoid $\mathcal{G}$, we can associate an important topological construction to it, namely its classifying space $B\mathcal{G}$. Moreover, this construction is well behaved under Morita equivalence, so the resulting space will depend largely on the orbifold the groupoid represents. In particular, the classifying space allows us to study the "homotopy type" of an orbifold $\mathcal{X}$, and define many other invariants besides.

We recall the basic construction, which is due to Segal (see [141], [143]). Let $\mathcal{G}$ be a Lie groupoid, and for $n \geq 1$, let G_n be the iterated fibered product

$$G_n = \{(g_1, \ldots, g_n) \mid g_i \in G_1, s(g_i) = t(g_{i+1}), i = 1, \ldots, n-1\}. \tag{1.4}$$

Together with the objects G_0, these G_n have the structure of a simplicial manifold, called the *nerve* of $\mathcal{G}$. Here we are really just thinking of $\mathcal{G}$ as a category. Following the usual convention, we define *face operators* $d_i : G_n \to G_{n-1}$ for $i = 0, \ldots, n$, given by

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0, \\ (g_1, \ldots, g_{n-1}) & i = n, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{otherwise,} \end{cases}$$

when $n > 1$. Similarly, we define $d_0(g) = s(g)$ and $d_1(g) = t(g)$ when $n = 1$.

For such a simplicial space, we can glue the disjoint union of the spaces $G_n \times \Delta^n$ as follows, where Δ^n is the topological n-simplex. Let

$$\delta_i : \Delta^{n-1} \to \Delta^n$$

be the linear embedding of Δ^{n-1} into Δ^n as the ith face. We define the *classifying space* of $\mathcal{G}$ (the geometric realization of its nerve) as the identification space

$$B\mathcal{G} = \bigsqcup_n (G_n \times \Delta^n)/(d_i(g), x) \sim (g, \delta_i(x)). \tag{1.5}$$

This is usually called the *fat realization* of the nerve, meaning that we have chosen to leave out identifications involving degeneracies. The two definitions (fat and thin) will produce homotopy equivalent spaces provided that the topological category has sufficiently good properties (see [143, p. 309]). Another good property of the fat realization is that if every G_n has the homotopy type of a CW-complex, then the realization will also have the homotopy type of a CW-complex ([143]). For the familiar groupoids that we will encounter in the theory of orbifolds – e.g., an action groupoid for a compact Lie group acting on a manifold – these technical subtleties do not really matter.

A homomorphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$ induces a continuous map $B\phi : B\mathcal{H} \to B\mathcal{G}$. In particular, an important basic property is that a strong equivalence of groupoids induces a weak homotopy equivalence between classifying spaces: $B\mathcal{H} \simeq B\mathcal{G}$. Intuitively, this stems from the fact that a strong equivalence induces an equivalence of (non-topological) categories between $\mathcal{H}$ and $\mathcal{G}$; for a full proof, see Moerdijk [111]. In fact, the same is true if ϕ is just a (weak) equivalence, and so Morita equivalent groupoids will have weakly homotopy equivalent classifying spaces. Therefore, for any point $y \in H_0$, an equivalence $\phi : \mathcal{H} \to \mathcal{G}$ induces an isomorphism of all the homotopy groups $\pi_n(B\mathcal{H}, y) \to \pi_n(B\mathcal{G}, \phi(y))$. From this we see that the weak homotopy type of an orbifold $\mathcal{X}$ can be defined as that of $B\mathcal{G}$, where $\mathcal{G}$ is any orbifold groupoid representing $\mathcal{X}$. So we discover that we can obtain orbifold invariants by applying (weak) homotopy functors to the classifying space.

Definition 1.50 Let $\mathcal{X}$ be an orbifold, and let $\mathcal{G}$ be any groupoid representing its orbifold structure via a given homeomorphism $f : |\mathcal{G}| \to X$. We define the *nth orbifold homotopy group* of $\mathcal{X}$ based at $x \in X$ to be

$$\pi_n^{\mathrm{orb}}(\mathcal{X}, x) = \pi_n(B\mathcal{G}, \tilde{x}), \tag{1.6}$$

where $\tilde{x} \in G_0$ maps to x under the map $G_0 \to X$, which is the composition of the canonical quotient map $G_0 \to |\mathcal{G}|$ with the homeomorphism f.

Note that, as abstract groups, this definition is independent of the choice of representing groupoid, and of the choice of lifting. We remark that the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{X}, x)$ can also be described in terms of an

appropriate version of covering spaces, as in Thurston's original definition – we will describe this in Section 2.2.

When the groupoid happens to be a topological group G, we obtain the more familiar classifying space BG, which (up to homotopy) can be expressed as a quotient EG/G. Here, EG is a contractible free G space, called the universal G-space for principal G-bundles. Similarly, given any G-space M, we can construct its *Borel construction*. This is defined as $EG \times_G M = (EG \times M)/G$, where G acts diagonally on the product $EG \times M$. Looking at the identifications in this situation, one sees that the situation for BG extends to more general action groupoids, and we have a basic and important description of the classifying space.

Proposition 1.51 *Let $\mathcal{G} = G \ltimes M$ be the action groupoid associated to a compact Lie group G acting smoothly and almost freely on a manifold M. Then there is a homotopy equivalence $B\mathcal{G} \simeq EG \times_G M$, and so $\pi_n(B\mathcal{G}) \cong \pi_n(EG \times_G M)$.*

Corollary 1.52 *Let $\mathcal{X}$ be an effective (classical) orbifold with frame bundle* $\mathrm{Fr}(\mathcal{X})$*, and let $\mathcal{G}$ be any groupoid presentation of $\mathcal{X}$. Then there is a homotopy equivalence $B\mathcal{G} \simeq EO(n) \times_{O(n)} \mathrm{Fr}(\mathcal{X})$, and so $\pi_n^{\mathrm{orb}}(\mathcal{X}) \cong \pi_n(EO(n) \times_{O(n)} \mathrm{Fr}(\mathcal{X}))$.*

If $\mathcal{G}$ is an orbifold groupoid associated to the orbifold $\mathcal{X}$ with underlying space X, then the map $G_0 \to X$ gives rise to a map $p : B\mathcal{G} \to X$. For instance, in the case of the action groupoid $G \ltimes M$ above, the map $p : B\mathcal{G} \to |\mathcal{G}|$ corresponds to the familiar projection onto the orbit space, $p : EG \times_G M \to M/G$. Now, in general there is an open cover of X by sets V such that $\mathcal{G}|_{p^{-1}(V)}$ is Morita equivalent to $H \ltimes U$, where H is a finite group acting on some $U \subseteq G_0$. We can assume that U is a contractible open set in $\mathbb{R}^n$ with H acting linearly, and so

$$p^{-1}(V) \simeq B(H \ltimes U) \simeq EH \times_H U \simeq BH.$$

As a result, $p : B\mathcal{G} \to X$ is a map such that the inverse image of each point is rationally acyclic, because the reduced rational cohomology of BH always vanishes if H is finite. By the Vietoris–Begle Mapping Theorem (or the Leray spectral sequence), we conclude that p induces an isomorphism in rational homology: $p_* : H_*(B\mathcal{G}; \mathbb{Q}) \cong H_*(X; \mathbb{Q})$.

Example 1.53 We now look more closely at the case of an orbifold $\mathcal{X}$ associated to a global quotient M/G. We know that the orbifold homotopy groups are simply the groups $\pi_n(EG \times_G M)$. What is more, we have a fibration $M \to EG \times_G M \to BG$, and BG has a contractible universal cover –

namely, EG, as G is a finite, hence discrete, group. Applying the long exact sequence of homotopy groups, we see that $\pi_n^{\mathrm{orb}}(\mathcal{X}) \cong \pi_n(M)$ for $n \geq 2$, whereas for the orbifold fundamental group we have a possibly non-split group extension

$$1 \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(\mathcal{X}) \to G \to 1. \tag{1.7}$$

Note that a simple consequence of this analysis is that for a global quotient M/G, the group $\pi_1^{\mathrm{orb}}(M/G)$ must map onto the group G. This fact can be particularly useful in determining when a given orbifold is not a global quotient. For example, the weighted projective spaces $\mathbb{WP}(a_0, \dots, a_n)$ considered in Example 1.15 arise as quotients of an $\mathbb{S}^1$ action on $\mathbb{S}^{2n+1}$. Looking at the Borel construction $E\mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^{2n+1}$ and the associated long exact sequence of homotopy groups, we see that $\pi_1^{\mathrm{orb}}(\mathbb{WP}) = 0$, $\pi_2^{\mathrm{orb}}(\mathbb{WP}) = \mathbb{Z}$ and $\pi_i^{\mathrm{orb}}(\mathbb{WP}) \cong \pi_i(\mathbb{S}^{2n+1})$ for $i \geq 3$. Thus, $\mathbb{WP}(a_0, \dots, a_n)$ cannot be a global quotient except in the trivial case where all weights equal 1. An interesting case arises when all the weights are equal. The resulting orbifold has the same ineffective cyclic isotropy at every point, but is still not a global quotient. This illustrates some of the subtleties of the ineffective situation.

Based on the example of the weighted projective spaces, one can easily show the following more general result.

Proposition 1.54 *If $\mathcal{X}$ is an orbifold arising from the quotient of a smooth, almost free action of a non-trivial connected compact Lie group on a simply connected compact manifold, then $\pi_1^{\mathrm{orb}}(\mathcal{X}) = 0$ and $\mathcal{X}$ cannot be presented as a global quotient.*

One could also ask whether or not every orbifold $\mathcal{X}$ can be presented as a quotient $G \ltimes M$ if we now allow infinite groups G. We have seen that for effective orbifolds, the answer is yes. In fact, one expects that this holds more generally.

Conjecture 1.55 *If $\mathcal{G}$ is an orbifold groupoid, then it is Morita equivalent to a translation groupoid $G \ltimes M$ arising from a smooth, almost free action of a Lie group.*

For additional results in this direction, see [69].

As we have mentioned, any (weak) homotopy invariants of the classifying space $B\mathcal{G}$ associated to a groupoid presenting an orbifold $\mathcal{X}$ will be orbifold invariants. In particular, we can define the singular cohomology of an orbifold.

Definition 1.56 Let $\mathcal{X}$ be an orbifold presented by the groupoid $\mathcal{G}$, and let R be a commutative ring with unit. Then the *singular cohomology* of $\mathcal{X}$ with

coefficients in R is $H^*_{\text{orb}}(\mathcal{X}; R) = H^*(B\mathcal{G}; R)$. In particular, we define the *integral cohomology* $H^*_{\text{orb}}(\mathcal{X}; \mathbb{Z}) = H^*(B\mathcal{G}; \mathbb{Z})$.

Note that in the case of a quotient orbifold M/G, this invariant is simply the equivariant cohomology $H^*(EG \times_G M; \mathbb{Z})$, up to isomorphism. We will discuss some other cohomology theories for orbifolds in subsequent chapters.

1.5 Orbifolds as singular spaces

There are two ways to view orbifolds: one way is through groupoids and stacks, where orbifolds are viewed as smooth objects; more traditionally, one views them as singular spaces. In the latter case, one aims to remove the singularity using techniques from algebraic geometry. There are two well-known methods for accomplishing this, which we shall describe in the setting of complex orbifolds. The main reference for this section is the excellent book by Joyce [75], which we highly recommend for further information and examples. Throughout this section, we identify the orbifold $\mathcal{X}$ with its underlying space X.

Definition 1.57 Let X be a complex orbifold, and $f : Y \to X$ a holomorphic map from a smooth complex manifold Y to X. The map f is called a *resolution* if $f : f^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ is biholomorphic and $f^{-1}(X_{\text{sing}})$ is an analytic subset of Y. A resolution f is called *crepant* if $f^*K_X = K_Y$.

Here we require the canonical bundle K_X to be an honest bundle, rather than just an orbibundle; the following condition will guarantee this.

Definition 1.58 An n-dimensional complex orbifold X is *Gorenstein* if all the local groups G_x are subgroups of $SL_n(\mathbb{C})$.

Indeed, we have seen that K_X is an orbibundle with fibers of the form $\mathbb{C}/G_x$, where G_x acts through the determinant. It follows that the Gorenstein condition is necessary for a crepant resolution to exist. These notions must first be understood locally, since a crepant resolution of an orbifold X is locally isomorphic to crepant resolutions of its local singularities (see Example 1.14).

Example 1.59 We now pass to the important special case when $G \subset SL_2(\mathbb{C})$. In this case, G is conjugate to a finite subgroup of $SU(2)$, and the quotient singularities are classically understood (first classified by Klein in 1884). We briefly outline the theory.

There is a one-to-one correspondence between non-trivial finite subgroups G of $SU(2)$ and the Dynkin diagrams Q of type A_n ($n \geq 1$), D_n ($n \geq 4$), E_6,

E_7, and E_8. The Dynkin diagrams that are listed are precisely those which contain no double or triple edges.

Each singularity $\mathbb{C}^2/G$ admits a unique crepant resolution (Y, f). The inverse image $f^{-1}(0)$ of the singular point is a union of a finite number of rational curves in Y. They correspond naturally to the vertices in Q, all have self-intersection -2, and two curves intersect transversely in a single point if and only if the corresponding vertices are joined by an edge in the diagram; otherwise they do not intersect.

These curves provide a basis for $H_2(Y; \mathbb{Z})$, which can be identified with the root lattice of the diagram. The intersection form with respect to this basis is the negative of the Cartan matrix of Q. Homology classes in $H_2(Y; \mathbb{Z})$ with self-intersection -2 can be identified with the set of roots of the diagram. There are one-to-one correspondences between the curves and the non-trivial conjugacy classes in G, as well as with the non-trivial representations of G. Indeed, one can regard the conjugacy classes as a basis for $H_2(Y; \mathbb{Z})$, and the representations as a basis for $H^2(Y; \mathbb{Z})$. These correspondences are part of the so-called McKay correspondence (see [108], [130]).

We now explicitly list all the finite subgroups of $SU(2)$ that give rise to these singular spaces.

- (A_n) $G = \mathbb{Z}/(n+1)\mathbb{Z}$ with the generator g acting as $g(z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$, where $\lambda^{n+1} = 1$.
- (D_n) G, a generalized quaternion group of order $4n$ generated by elements S and T, where $S^{2n} = 1$ and we have the relations $T^2 = S^n$ and $TST^{-1} = S^{-1}$. The action is given by $S(z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$ with $\lambda^{2n} = 1$ and $T(z_1, z_2) = (-z_2, z_1)$.
- (E_6) Binary tetrahedral group of order 24.
- (E_7) Binary octahedral group of order 48.
- (E_8) Binary icosahedral group of order 120.

The situation for general singularities $\mathbb{C}^m/G$ can be quite complicated, but for $m = 3$, Roan [131] has proved the following.

Theorem 1.60 *Let G be any finite subgroup of $SL_3(\mathbb{C})$. Then the quotient singularity $\mathbb{C}^3/G$ admits a crepant resolution.*

Note that for $m = 3$ (and higher), finitely many different crepant resolutions can exist for the same quotient. In dimensions $m > 3$, singularities are not that well understood (see [130] for more on this). The following is the easiest "bad situation".

Example 1.61 Let G be the subgroup $\{\pm I\} \subset SL_4(\mathbb{C})$. Then $\mathbb{C}^4/G$ admits no crepant resolution.

Let us now consider a complex orbifold X satisfying the Gorenstein condition (note for example that this automatically holds for Calabi–Yau orbifolds). For each singular point, there are finitely many possible local crepant resolutions, although it may be that none exist when the dimension is greater than 3. If G is an isotropy group for X and $\mathbb{C}^m/G$ admits no crepant resolutions, then X cannot have a crepant resolution. Assume, then, that these local crepant resolutions all exist. A strategy for constructing a crepant resolution for X in its entirety is to glue together all of these local resolutions. Indeed, this works if the singularities are isolated: one can choose crepant resolutions for each singular point and glue them together to obtain a crepant resolution for X. The case of non-isolated singularities is a lot trickier. However, Roan's result mentioned above does lead to a global result.

Theorem 1.62 *Let X be a complex three-dimensional orbifold with orbifold groups in $SL_3(\mathbb{C})$. Then X admits a crepant resolution.*

We should mention that constructing crepant resolutions in some instances yields spaces of independent interest. For example, if X is a Calabi–Yau orbifold and (Y, f) is a crepant resolution of X, then Y has a family of Ricci-flat Kähler metrics which make it into a Calabi–Yau manifold. In the particular case where X is the quotient $\mathbb{T}^4/(\mathbb{Z}/2\mathbb{Z})$ (Example 1.9), then the Kummer construction (see [13]) gives rise to a crepant resolution that happens to be the K3 surface.

We now switch to a different way of handling spaces with singularities.

Definition 1.63 Let X be a complex analytic variety of dimension m. A *one-parameter family of deformations* of X is a complex analytic variety Z of dimension $m+1$, together with a proper holomorphic map $f : Z \to D$, where D is the unit disc in $\mathbb{C}$. These must be such that the central fiber $X_0 = f^{-1}(0)$ is isomorphic to X. The rest of the fibers $X_t = f^{-1}(t) \subset Z$ are called *deformations* of X.

If the deformations X_t are non-singular for $t \neq 0$, they are called *smoothings* of X; by a *small deformation* of X we mean a deformation X_t where t is small. The variety X is *rigid* if all small deformations X_t of X are biholomorphic to X.

A singular variety may admit a family of non-singular deformations, so this gives a different approach for replacing singular spaces with non-singular ones. Moreover, whereas a variety X and its resolution Y are birationally equivalent (hence very similar as algebro-geometric objects), the deformations X_t can be very different from X.

For later use, we record the definition of a *desingularization*, which combines deformation and resolution.

Definition 1.64 A *desingularization* of a complex orbifold X is a resolution of a deformation $f : T_t \to X_t$. We call it a *crepant desingularization* if K_{X_t} is defined and $f^* K_{X_t} = K_{T_t}$.

What can we say about the deformations of $\mathbb{C}^m/G$? We begin again with the case $m = 2$.

Example 1.65 The deformations of $\mathbb{C}^2/G$ are well understood. The singularity can be embedded into $\mathbb{C}^3$ as a hypersurface via the following equations, according to our earlier classification of the group G:

(A_n) $x^2 + y^2 + z^{n+1} = 0$ for $n \geq 1$,
(D_n) $x^2 + y^2 z + z^{n-1} = 0$ for $n \geq 4$,
(E_6) $x^2 + y^3 + z^4 = 0$,
(E_7) $x^2 + y^3 + yz^3 = 0$,
(E_8) $x^2 + y^3 + z^5 = 0$.

We obtain a deformation by setting the corresponding equations equal to t. These are the only deformations. Furthermore, the crepant resolution of the singularity deforms with it. Consequently, its deformations are diffeomorphic to the crepant resolution. However, not all holomorphic 2-spheres in the crepant resolution remain holomorphic in the deformations under these diffeomorphisms.

For $m \geq 3$, the codimension of the singularities in $\mathbb{C}^m/G$ plays a big role. Note that if $G \subset SL_m(\mathbb{C})$, then we see that the singularities of $\mathbb{C}^m/G$ are of codimension at least two, as no non-trivial element can fix a codimension one subspace in $\mathbb{C}^m$. Now by the *Schlessinger Rigidity Theorem* (see [75, p. 132]), if $G \subset SL_m(\mathbb{C})$ and the singularities of $\mathbb{C}^m/G$ are all of codimension at least three, $\mathbb{C}^m/G$ must be rigid. Hence we see that non-trivial deformations X_t of $X = \mathbb{C}^m/G$ can only exist when the singularities are of codimension two.

2

Cohomology, bundles and morphisms

As we discussed in Chapter 1, many invariants for manifolds can easily be generalized to classical effective orbifolds. In this chapter we will outline this in some detail, seeking natural extensions to all orbifolds. Extra care is required when dealing with ineffective orbifolds, which is why we will cast all of our constructions in the framework of *orbifold groupoids*.

2.1 De Rham and singular cohomology of orbifolds

We begin by making a few basic observations about orbifold groupoids. Suppose that $\mathcal{G}$ is such a groupoid. We saw in Proposition 1.44 that each arrow $g : x \to y$ in G_1 extends to a diffeomorphism $g : U_x \to U_y$ between neighborhoods of x and y.

Lemma 2.1 *If $\phi : \mathcal{G} \to \mathcal{H}$ is an equivalence of orbifold groupoids, then $\phi_0 : G_0 \to H_0$ is a local diffeomorphism.*

Proof We can write ϕ_0 as the composition $t \circ \pi_1 \circ \lambda$, where the map λ is

$$\begin{aligned} \lambda : G_0 &\to H_1 \times_{H_0} G_0 \\ y &\mapsto (u(\phi_0(y)), \phi_0(y)). \end{aligned}$$

Recall that u is the unit map $G_0 \to G_1$. The map λ is an immersion, and $t \circ \pi_1$ is a submersion by assumption. Since $\dim G_0 = \dim(H_1 \times_{H_0} G_0) = \dim H_0$, both $t \circ \pi_1$ and λ are local diffeomorphisms. □

Consider the tangent bundle $TG_0 \to G_0$ of the smooth manifold G_0. Each arrow $g : x \to y$ induces an isomorphism $Dg : T_x G_0 \to T_y G_0$. In other words, TG_0 comes equipped with a fiberwise linear action of the arrows. A vector bundle over G_0 with this property is called a vector bundle for the orbifold

groupoid $\mathcal{G}$, or $\mathcal{G}$-*vector bundle*. In Section 2.3, we will discuss such bundles in greater generality. To emphasize the compatibility with the arrows, we write $T\mathcal{G}$ and refer to it as the *tangent bundle* of the orbifold groupoid $\mathcal{G}$. Using this bundle, we can define many other bundles compatible with the groupoid multiplication, including the cotangent bundle $T^*\mathcal{G}$, wedge products $\bigwedge^* T^*\mathcal{G}$, and symmetric tensor products $\mathrm{Sym}^k\, T^*\mathcal{G}$. In particular, it makes sense to talk about Riemannian metrics (non-degenerate symmetric 2-tensors) and symplectic forms (non-degenerate closed 2-forms) on an orbifold groupoid. All of these notions, appropriately translated from groupoids into the chart/atlas formalism, exactly match the definitions of the tangent orbibundle and its associates given earlier.

In this setting, we can define a de Rham complex as follows:

$$\Omega^p(\mathcal{G}) = \{\omega \in \Omega^p(G_0) \mid s^*\omega = t^*\omega\}. \tag{2.1}$$

We call such forms ω satisfying $s^*\omega = t^*\omega$ $\mathcal{G}$-*invariant*. By naturality, the usual exterior derivative

$$d : \Omega^p(\mathcal{G}) \to \Omega^{p+1}(\mathcal{G})$$

takes $\mathcal{G}$-invariant p-forms to $\mathcal{G}$-invariant $(p+1)$-forms. Suppose that $g : x \to y$ is an arrow, and extend it to a diffeomorphism $g : U_x \to U_y$ as above. The condition $s^*\omega = t^*\omega$ can be reinterpreted as $g^*\omega|_{U_y} = \omega|_{U_x}$. In particular, if $\omega_y \neq 0$, then $\omega_x \neq 0$. Therefore, we can think of the support $\mathrm{supp}(\omega)$ as a subset of the orbit space $|\mathcal{G}|$. We say that ω has *compact support* if $\mathrm{supp}(\omega) \subseteq |\mathcal{G}|$ is compact. If ω has compact support, then so does $d\omega$. We use $\Omega_c^p(\mathcal{G})$ to denote the subspace of compactly supported p-forms. Define the *de Rham cohomology of* $\mathcal{G}$ to be

$$H^*(\mathcal{G}) = H^*(\Omega^*(\mathcal{G}), d) \tag{2.2}$$

and the *de Rham cohomology of* $\mathcal{G}$ *with compact supports* to be

$$H_c^*(\mathcal{G}) = H^*(\Omega_c^*(\mathcal{G}), d). \tag{2.3}$$

Recall that the restriction of $\mathcal{G}$ to a small neighborhood U_x is isomorphic to a translation groupoid $G_x \ltimes U_x$. Locally, $\omega \in \Omega^*(\mathcal{G})$ can be viewed as a G_x-invariant differential form.

A groupoid homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ induces chain maps

$$\phi^* : \{\Omega^*(\mathcal{H}), d\} \to \{\Omega^*(\mathcal{G}), d\},$$
$$\phi^* : \{\Omega_c^*(\mathcal{H}), d\} \to \{\Omega_c^*(\mathcal{G}), d\}.$$

Hence, it induces the homomorphisms

$$\phi^* : H^*(\mathcal{H}) \to H^*(\mathcal{G}) \text{ and } \phi^* : H_c^*(\mathcal{H}) \to H_c^*(\mathcal{G}).$$

Lemma 2.2 *If $\phi : \mathcal{G} \to \mathcal{H}$ is an equivalence, ϕ induces an isomorphism on the de Rham chain complex, and hence an isomorphism on de Rham cohomology.*

Proof By Lemma 2.1, ϕ_0 is a local diffeomorphism. Suppose that $\omega \in \Omega^*(\mathcal{G})$. We can use ϕ_0 to push forward ω to $\mathrm{im}(\phi_0)$. By assumption, for any $z \in H_0$ there is an arrow $h : z \to x$ for some $x \in \mathrm{im}(\phi_0)$, and h can be extended to a local diffeomorphism. Hence we can extend $(\phi_0)_*\omega$ to z by $(\phi_0)_*\omega_z = h^*\omega_x$. Suppose that $h' : z \to y$ for some $y \in \mathrm{im}(\phi_0)$ is another arrow connecting z to the image. Then $h'h^{-1}$ is an arrow from y to x, so by definition $h'h^{-1} = \phi_1(g)$ for some $g \in G_1$. Therefore, $(h')^*(h^{-1})^*\omega = \omega$, which shows that $(h')^*\omega_x = h^*\omega_y$. Therefore, there is a unique $\mathcal{H}$-invariant extension of $(\phi_0)_*\omega$ to H_0, denoted by $\phi_*\omega$. It is routine to check that $s^*\phi_*\omega = t^*\phi_*\omega$. It is obvious that ϕ_* commutes with d and $\phi^*\phi_* = \phi_*\phi^* = \mathrm{Id}$. $\square$

This lemma implies that $\Omega^*(\mathcal{G})$ (and therefore $H^*(\mathcal{G})$) is invariant under orbifold Morita equivalence, and so we can view it as an invariant of the orbifold structure. However, Satake observed that $H^*(\mathcal{G})$ is isomorphic to the singular cohomology $H^*(|\mathcal{G}|; \mathbb{R})$ of the quotient space, and hence is independent of the orbifold structure (the same applies to $H_c^*(\mathcal{G})$). We will discuss this more fully below.

We also have integration theory and Poincaré duality on orbifold groupoids. An orbifold groupoid $\mathcal{G}$ of dimension n is called *orientable* if $\wedge^n T^*\mathcal{G}$ is trivial, and a trivialization is called an *orientation* of $\mathcal{G}$. The groupoid $\mathcal{G}$ together with an orientation is called an *oriented* orbifold groupoid. It is clear that orientability is preserved under orbifold Morita equivalence, so it is intrinsic to the orbifold structure. For oriented orbifolds, we can define integration as follows.

Recall that a function $\rho : |\mathcal{G}| \to \mathbb{R}$ is smooth if its pullback to G_0 is smooth. Let $\{U_i\}$ be an open cover of $|\mathcal{G}|$ by charts. That is, for each U_i, the restriction of $\mathcal{G}$ to each component of the inverse image of U_i in G_0 is of the form $G_x \ltimes U_x$ for some $x \in G_0$. For now, we fix a particular chart U_x/G_x for U_i. A compactly supported orbifold n-form ω on U_i is by definition a compactly supported G_x-invariant n-form ω on U_x. We define

$$\int_{U_i} \omega = \frac{1}{|G_x|} \int_{U_x} \omega.$$

Each arrow $g : x \to y$ in G_1 induces a diffeomorphism $g : U_x \to U_y$ between components of the inverse image of U_i. It is not hard to show that

$$\frac{1}{|G_y|} \int_{U_y} \omega = \frac{1}{|G_x|} \int_{U_x} g^*\omega = \frac{1}{|G_x|} \int_{U_x} \omega.$$

As a result, the value of the integral is independent of our choice of the component U_x.

In general, let ω be a compactly supported $\mathcal{G}$-invariant n-form. Choose a smooth partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$, and define

$$\int_{\mathcal{G}} \omega = \sum_i \int_{U_i} \rho_i \, \omega. \tag{2.4}$$

As usual, this is independent of the choice of the cover and the partition of unity $\{\rho_i\}$. It is also invariant under Morita equivalence, so it makes sense to integrate forms over an orbifold $\mathcal{X}$ by integrating them on any groupoid presentation.

Using integration, we can define a *Poincaré pairing*

$$\int : H^p(\mathcal{G}) \otimes H_c^{n-p}(\mathcal{G}) \to \mathbb{R} \tag{2.5}$$

given by

$$\langle \alpha, \beta \rangle = \int_{\mathcal{G}} \alpha \wedge \beta. \tag{2.6}$$

This Poincaré pairing is non-degenerate if $\mathcal{X}$ admits a finite good cover $\mathcal{U}$. A *good cover* $\mathcal{U}$ has the property that each $U \in \mathcal{U}$ is of the form $\mathbb{R}^n/G$ and all the intersections are of this form as well. Any compact orbifold has a finite good cover. All the machinery in [29], such as the Mayer–Vietoris arguments, generalizes without any difficulty to orbifolds that admit a finite good cover.

One of the main applications of Poincaré duality for smooth manifolds is the definition of the Poincaré dual of a submanifold. Namely, for any oriented submanifold, we can construct a Thom form supported on its normal bundle, and think of that form as the Poincaré dual of the submanifold. To carry out this construction in the orbifold context, we have to choose our notion of *suborbifold* or subgroupoid carefully.

Definition 2.3 A homomorphism of orbifold groupoids $\phi : \mathcal{H} \to \mathcal{G}$ is an *embedding* if the following conditions are satisfied:

- $\phi_0 : H_0 \to G_0$ is an immersion.
- Let $x \in \mathrm{im}(\phi_0) \subset G_0$ and let U_x be a neighborhood such that $\mathcal{G}|_{U_x} \cong G_x \ltimes U_x$. Then the $\mathcal{H}$-action on $\phi_0^{-1}(x)$ is transitive, and there exists an open neighborhood $V_y \subseteq H_0$ for each $y \in \phi_0^{-1}(x)$ such that $\mathcal{H}|_{V_y} \cong H_y \ltimes V_y$ and
$$\mathcal{H}|_{\phi_0^{-1}(U_x)} \cong G_x \ltimes (G_x/\phi_1(H_y) \times V_y).$$
- $|\phi| : |\mathcal{G}| \to |\mathcal{H}|$ is proper.

$\mathcal{H}$ together with ϕ is called a *subgroupoid* of $\mathcal{G}$.

Remark 2.4 Suppose that $\phi : \mathcal{H} \to \mathcal{G}$ is a subgroupoid. Let $x = \phi(y)$ for $y \in H_0$. Then

$$U_x \cap \operatorname{im}(\phi) = \bigcup_{g \in G_x} g\ \phi_0(V_y),$$

where V_y is a neighborhood of y in H_0.

This definition is motivated by the following key examples.

Example 2.5 Suppose that $\mathcal{G} = G \ltimes X$ is a global quotient groupoid. An important object is the so-called *inertia groupoid* $\wedge\mathcal{G} = G \ltimes (\sqcup_g X^g)$. Here X^g is the fixed point set of g, and G acts on $\sqcup_g X^g$ as $h : X^g \to X^{hgh^{-1}}$ given by $h(x) = hx$. The groupoid $\wedge\mathcal{G}$ admits a decomposition as a disjoint union: let $\wedge(\mathcal{G})_{(h)} = G \ltimes (\sqcup_{g\in(h)} X^g)$. If S is a set of conjugacy class representatives for G, then

$$\wedge\mathcal{G} = \bigsqcup_{h \in S} (\wedge\mathcal{G})_{(h)}.$$

By our definition, the homomorphism $\phi : (\wedge\mathcal{G})_{(h)} \to \mathcal{G}$ induced by the inclusion maps $X^g \to X$ is an embedding. Hence, $\wedge\mathcal{G}$ and the homomorphism ϕ together form a (possibly non-disjoint) union of suborbifolds. We will sometimes abuse terminology and say that the inertia groupoid is a suborbifold.

Example 2.6 Let $\mathcal{G}$ be the global quotient groupoid defined in the previous example. We would like to define an appropriate notion of the *diagonal* Δ of $\mathcal{G} \times \mathcal{G}$. The correct definition turns out to be $\Delta = (G \times G) \ltimes (\sqcup_g \Delta_g)$, where $\Delta_g = \{(x, gx), x \in X\}$ and (h, k) takes (x, g, gx) to (hx, kgh^{-1}, kgx). Our definition of a suborbifold includes this example.

More generally, we define the *diagonal* Δ to be the groupoid fibered product $\mathcal{G} \times_{\mathcal{G}} \mathcal{G}$. One can check that $\Delta = \mathcal{G} \times_{\mathcal{G}} \mathcal{G}$ is locally of the desired form, and hence a subgroupoid of $\mathcal{G} \times \mathcal{G}$.

Now that we know how to talk about suborbifolds in terms of subgroupoids, we can talk about transversality.

Definition 2.7 Suppose that $f : \mathcal{H}_1 \to \mathcal{G}$ and $g : \mathcal{H}_2 \to \mathcal{G}$ are smooth homomorphisms. We say that $f \times g$ is *transverse to the diagonal* $\Delta \subset \mathcal{G} \times \mathcal{G}$ if locally $f \times g$ is transverse to every component of Δ. We say that f and g are transverse to each other if $f \times g$ is transverse to the diagonal Δ.

Example 2.8 Suppose that $f : \mathcal{H}_1 \to \mathcal{G}$ and $g : \mathcal{H}_2 \to \mathcal{G}$ are smooth and transverse to each other. Then it follows from the definitions that the groupoid fibered product $p_1 \times p_2 : \mathcal{H}_1 \times_{\mathcal{G}} \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2$ is a suborbifold if the

underlying map is topologically closed. But in fact there is a finite-to-one map from the orbifold fibered product to the ordinary fibered product, and the ordinary fibered product is closed. Hence, so is $p_1 \times p_2$.

Definition 2.9 Suppose that $\phi : \mathcal{H} \to \mathcal{G}$ is a homomorphism and $i : \mathcal{K} \to \mathcal{G}$ is a suborbifold. Furthermore, assume that ϕ and i are transverse. Then the *inverse image* of $\mathcal{K}$ in $\mathcal{H}$ is $\phi^{-1}(\mathcal{K}) = \mathcal{H} \times_{\mathcal{G}} \mathcal{K}$. If $\mathcal{H}$ and $\mathcal{K}$ are both suborbifolds, then their *orbifold intersection* is defined to be $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$.

By the transversality assumption, $\phi^{-1}(\mathcal{K})$ is smooth and $p_1 : \phi^{-1}(\mathcal{K}) \to \mathcal{H}$ is a suborbifold. We can go on to formulate more of the theory of transversality using the language of orbifold fibered products. However, we note at the outset that one cannot always perturb any two homomorphisms into transverse maps. In many ways, the obstruction bundle (see Section 4.3) measures this failure of transversality.

Suppose that $\phi : \mathcal{H} \to \mathcal{G}$ is an oriented suborbifold. Then $T\mathcal{H}$ is a subbundle of $\phi^* T\mathcal{G}$. We call the quotient $N_{\mathcal{H}|\mathcal{G}} = \phi^* T\mathcal{G}/T\mathcal{H}$ the *normal bundle* of $\mathcal{H}$ in $\mathcal{G}$. Just as in the smooth manifold case, there is an open embedding from an open neighborhood of the zero section of $N_{\mathcal{H}|\mathcal{G}}$ onto an open neighborhood of the image of $\mathcal{H}$ in $\mathcal{G}$. Choose a Thom form Θ on $N_{\mathcal{H}|\mathcal{G}}$. We can view Θ as a closed form of $\mathcal{G}$, and it is Poincaré dual to $\mathcal{H}$ in the sense that

$$\int_{\mathcal{G}} \Theta \wedge \alpha = \int_{\mathcal{H}} \phi^* \alpha \tag{2.7}$$

for any compactly supported form α. The proofs of these statements are identical to the smooth manifold case, so we omit them. We often use $\eta_{\mathcal{H}}$ to denote Θ when it is viewed as a closed form on $\mathcal{G}$.

When $\mathcal{G}$ is compact, η_{Δ} is equivalent to Poincaré duality in the following sense. Choose a basis α_i of $H^*(\mathcal{G})$. Using the Künneth formula, we can make a decomposition

$$[\eta_{\Delta}] = \sum_{i,j} a_{ij}\ \alpha_i \otimes \alpha_j.$$

Let $(a^{ij}) = (a_{ij})^{-1}$ be the inverse matrix. It is well known in the case of smooth manifolds that $a^{ij} = \langle \alpha_i, \alpha_j \rangle$, and the usual proof works for orbifolds as well.

As we have remarked, the de Rham cohomology of an orbifold is the same as the singular cohomology of its orbit space. Therefore, it does not contain any information about the orbifold structure. Another drawback is that it is only defined over the real numbers. We will now define a more general singular cohomology for orbifolds that allows for arbitrary coefficients. This is best accomplished via the classifying space construction. In the last chapter (see

page 25), we saw that the (weak) homotopy type of the classifying space $B\mathcal{G}$ was invariant under Morita equivalence; therefore, we defined orbifold homotopy groups by setting

$$\pi_n^{\mathrm{orb}}(\mathcal{X}, x) = \pi_n(B\mathcal{G}, \tilde{x}),$$

where $\mathcal{G}$ was an orbifold groupoid presentation of $\mathcal{X}$ and $\tilde{x} \in G_0$ is a lift of the basepoint $x \in X$. Since by Whitehead's Theorem (see [145, p. 399]) a weak homotopy equivalence induces a homology isomorphism, we also define the *singular cohomology* of $\mathcal{X}$ with coefficients in a commutative ring R by

$$H^*_{\mathrm{orb}}(\mathcal{X}; R) = H^*(B\mathcal{G}; R),$$

where $\mathcal{G}$ is an orbifold groupoid presentation of $\mathcal{X}$. When the orbifold is given as a groupoid $\mathcal{G}$, we will also write $H^*_{\mathrm{orb}}(\mathcal{G}; R)$ for $H^*(B\mathcal{G}; R)$. These invariants, while sensitive to the orbifold structure, can be difficult to compute.

Example 2.10 Consider the point orbifold $\bullet^G$; here the classifying space is the usual classifying space of the finite group G, denoted BG. This space has a contractible universal cover, so its higher homotopy groups are zero, while $\pi_1^{\mathrm{orb}}(\bullet^G) = G$. On the other hand, we have $H^*_{\mathrm{orb}}(\bullet^G; \mathbb{Z}) \cong H^*(G; \mathbb{Z})$, the group cohomology of G.

Example 2.11 More generally, if Y/G is a quotient orbifold, where G is a compact Lie group, then we have seen in Chapter 1 that $B(G \ltimes Y) \simeq EG \times_G Y$, the Borel construction on Y. Hence in this case $H^*_{\mathrm{orb}}(\mathcal{G}; \mathbb{Z})$ is the usual equivariant cohomology $H^*(EG \times_G Y; \mathbb{Z})$.

The cohomology and homotopy groups thus defined are clearly invariants of the orbifold. However, if the cohomology is computed with rational coefficients we are back in a situation similar to that of the de Rham cohomology. As discussed in Chapter 1, if $X = |\mathcal{G}|$, then we have a map $B\mathcal{G} \to X$ with fibers BG_x. These spaces are rationally acyclic, and hence by the Vietoris–Begle Theorem we obtain:

Proposition 2.12 *There is an isomorphism of cohomology groups*

$$H^*(B\mathcal{G}; \mathbb{Q}) \cong H^*(X; \mathbb{Q}).$$

We can now express Satake's Theorem as a de Rham Theorem for orbifolds, namely:

Theorem 2.13 $H^*_{\mathrm{orb}}(\mathcal{G}; \mathbb{R}) \cong H^*(\mathcal{G})$.

It is well known that an oriented orbifold $\mathcal{X}$ admits a fundamental class over the rational numbers. The proposition above implies that $B\mathcal{G}$ is a rational

Poincaré duality space. We also see that the information on the orbifold structure is contained precisely in the torsion occurring in $H^*(B\mathcal{G};\mathbb{Z})$. Indeed, computing the torsion classes of $H^*(B\mathcal{G};\mathbb{Z})$ is an important problem; for example $H^3(B\mathcal{G};\mathbb{Z})$ classifies the set of gerbes.

2.2 The orbifold fundamental group and covering spaces

Given an orbifold $\mathcal{X}$, perhaps the most accessible invariant is the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{X}, x)$, originally introduced by Thurston for the study of 3-manifolds. We have already provided a definition and some important properties of this invariant. Our goal here is to connect it to covering spaces, as can be done with the ordinary fundamental group.

Definition 2.14 Let $\mathcal{G}$ be an orbifold groupoid. A *left $\mathcal{G}$-space* is a manifold E equipped with an action by $\mathcal{G}$. Such an action is given by two maps: an *anchor* $\pi : E \to G_0$, and an *action* $\mu : G_1 \times_{G_0} E \to E$. The latter map is defined on pairs (g, e) with $\pi(e) = s(g)$, and written $\mu(g, e) = g \cdot e$. It satisfies the usual identities for an action: $\pi(g \cdot e) = t(g)$, $1_x \cdot e = e$, and $g \cdot (h \cdot e) = (gh) \cdot e$ for $x \xrightarrow{h} y \xrightarrow{g} z$ in G_1 and $e \in E$ with $\pi(e) = x$.

Intuitively, each arrow $g : x \to y$ induces a map $g : E_x \to E_y$ of fibers compatible with the multiplication of arrows. For example, the tangent bundle $T\mathcal{G}$ and its associated bundles considered at the beginning of the chapter are all $\mathcal{G}$-spaces. Of course, there is also a dual notion of right $\mathcal{G}$-spaces; a right $\mathcal{G}$-space is the same thing as a left $\mathcal{G}^{\mathrm{op}}$-space, where $\mathcal{G}^{\mathrm{op}}$ is the *opposite groupoid* obtained by exchanging the roles of the target and source maps.

Definition 2.15 For two $\mathcal{G}$-spaces $E = (E, \pi, \mu)$ and $E' = (E', \pi', \mu')$, a map of $\mathcal{G}$-spaces $\alpha : E \to E'$ is a smooth map which commutes with the structure, i.e., $\pi'\alpha = \pi$ and $\alpha(g \cdot e) = g \cdot \alpha(e)$. We sometimes call such maps $\mathcal{G}$-*equivariant*.

For each $\mathcal{G}$, the set of $\mathcal{G}$-spaces and $\mathcal{G}$-equivariant maps forms a category. Moreover, if $\phi : \mathcal{H} \to \mathcal{G}$ is a homomorphism of groupoids, then we can pull back a $\mathcal{G}$-space E by taking a fibered product:

$$\begin{tikzcd} E \times_{G_0} H_0 \arrow[r] \arrow[d] & H_0 \arrow[d, "\phi_0"] \\ E \arrow[r, "\pi"] & G_0 \end{tikzcd}.$$

There is an obvious action of H_1 on $E \times_{G_0} H_0$, and we write $\phi^* E$ for the resulting $\mathcal{H}$-space. It is clear that we can also pull back maps between two $\mathcal{G}$-spaces, so that ϕ^* is a functor from $\mathcal{G}$-spaces to $\mathcal{H}$-spaces. If ϕ is an equivalence, then we can push an $\mathcal{H}$-space forward to obtain a $\mathcal{G}$ space in the same way we pushed forward differential forms earlier. Hence, when ϕ is an equivalence, it induces an equivalence of categories between the category of $\mathcal{G}$-spaces and the category of $\mathcal{H}$-spaces.

If (E, π, μ) is a $\mathcal{G}$-space, we can associate to it an orbifold groupoid $\mathcal{E} = \mathcal{G} \ltimes E$ with objects $E_0 = E$ and arrows $E_1 = E \times_{G_0} G_1$. As this is a straightforward generalization of the group action case, we call this the *action groupoid* or *translation groupoid* associated to the action of the groupoid $\mathcal{G}$ on E. There is an obvious homomorphism of groupoids $\pi_E : \mathcal{E} \to \mathcal{G}$. Note that the fiber of $E_0 \to |\mathcal{E}|$ is $\pi_E^{-1}(x)/G_x$ for any $x \in E_0$. It is easy to see that $\mathcal{E}$ is an orbifold groupoid as well. We call E a *connected* $\mathcal{G}$-space if the quotient space $|\mathcal{E}|$ is connected.

Now we focus on covering spaces.

Definition 2.16 Let E be a $\mathcal{G}$-space. If $E \to G_0$ is a connected covering projection, then we call the associated groupoid $\mathcal{E}$ an *orbifold cover* or *covering groupoid* of $\mathcal{G}$. Let $\mathrm{Cov}(\mathcal{G})$ be the subcategory of orbifold covers of $\mathcal{G}$; a groupoid homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ induces a pullback

$$\phi^* : \mathrm{Cov}(\mathcal{G}) \to \mathrm{Cov}(\mathcal{H}).$$

As we showed before, if ϕ is an equivalence of groupoids, then ϕ^* is an equivalence of categories.

Suppose that U_x/G_x is an orbifold chart for $x \in G_0$ and $\pi^{-1}(U_x)$ is a disjoint union of open sets such that each component is diffeomorphic to U_x. Then the restriction of the map $E_0 \to |\mathcal{E}|$ is $\pi^{-1}(U_x) \to \pi^{-1}(U_x)/G_x$. Let $\widetilde{U}$ be a component of $\pi^{-1}(U_x)$. Then, $\mathcal{E}|_{\widetilde{U}}$ can be expressed as an orbifold chart $\widetilde{U}/\Gamma$, where $\Gamma \subseteq G_x$ is the subgroup preserving $\widetilde{U}$. The map $|\mathcal{E}| \to |\mathcal{G}|$ can be locally described as the map $\widetilde{U}/\Gamma \cong U_x/\Gamma \to U_x/G_x$, where $\widetilde{U}$ is identified with U_x via π. This recovers Thurston's original definition of covering orbifolds.

Among the covers of $\mathcal{G}$, there is a (unique up to isomorphism) universal cover $\pi : U \to G_0$, in the sense that for any other cover $E \to G_0$ there is a map $p : U \to E$ of $\mathcal{G}$-spaces commuting with the covering projections.

Proposition 2.17 *$\mathcal{E} \to B\mathcal{E}$ induces an equivalence of categories between orbifold covering spaces of $\mathcal{G}$ and the covering spaces (in the usual sense) of $B\mathcal{G}$.*

Proof It is easy to check that $B\mathcal{E} \to B\mathcal{G}$ is a covering space if $\mathcal{E} \to \mathcal{G}$ is a groupoid covering space. To prove the opposite, consider a covering space $E \to B\mathcal{G}$. Since $G_0 \to B\mathcal{G}$ is a subset, $E|_{G_0} \to G_0$ is clearly a covering space. We also need to construct an action of G_1 on E. Recall that there is also a map $G_1 \times [0, 1] \to B\mathcal{G}$ with the identifications $(g, 0) \cong s(g)$ and $(g, 1) \cong t(g)$. Therefore, $E_{(g,0)} = E_{s(g)}$ and $E_{(g,1)} = E_{t(g)}$. However, $E|_{G_1\times[0,1]}$ is a covering space. In particular, it has the unique path lifting property. The lifting of the path $g \times [0, 1]$ defines a map $E_{(g,0)} \to E_{(g,1)}$. It is easy to check that this defines an action of G_1 on $E|_{G_0}$. Hence, $E|_{G_0}$ can be viewed as a groupoid covering space of $\mathcal{G}$. □

Let $A(U, \pi)$ denote the group of deck translations of the universal cover. As in the case of ordinary covers, we have the following theorem.

Theorem 2.18 *The group $A(U, \pi)$ of deck translations of the universal orbifold cover of $\mathcal{G}$ is isomorphic to the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{G}) \cong \pi_1(B\mathcal{G})$.*

More generally, we see that orbifold covers of $\mathcal{G}$ will be in one-to-one correspondence with conjugacy classes of subgroups in $\pi_1^{\mathrm{orb}}(\mathcal{G})$.

Example 2.19 (Hurwitz cover) Orbifold covers arise naturally as holomorphic maps between Riemann surfaces. Suppose that $f : \Sigma_1 \to \Sigma_2$ is a holomorphic map between two Riemann surfaces Σ_1, Σ_2. Usually, f is not a covering map. Instead, it ramifies in finitely many points $z_1, \ldots, z_k \in \Sigma_2$. Namely, $f : \Sigma_1 - \cup_i f^{-1}(z_i) \to \Sigma_2 - \{z_1, \ldots, z_k\}$ is an honest covering map. Suppose that the preimage of z_i is $y_{i1}, \ldots, y_{ij_i}$. Let m_{ip} be the ramification order at y_{ip}. That is, under some coordinate system near y_{ip}, the map f can be written as $x \to x^{m_{ip}}$. We assign an orbifold structure on Σ_1 and Σ_2 as follows (see also Example 1.16). We first assign an orbifold structure at y_{ip} with order m_{ip}. Let m_i be the largest common factor of the m_{ip}s. Then we assign an orbifold structure at z_i with order m_i. One readily verifies that under these assignments, $f : \Sigma_1 \to \Sigma_2$ becomes an orbifold cover. Viewed in this way, $f : \Sigma_1 \to \Sigma_2$ is referred to as a *Hurwitz cover* or *admissible cover*.

This example can be generalized to nodal orbifold Riemann surfaces. Recall that a nodal orbifold Riemann surface $(\widetilde{\Sigma}, \mathbf{z}, \mathbf{m}, \mathbf{n})$ is a nodal curve (nodal Riemann surface), together with orbifold structure given by a faithful action of $\mathbb{Z}/m_i$ on a neighborhood of the marked point z_i and a faithful action of $\mathbb{Z}/n_j$ on a neighborhood of the jth node, such that the action is complementary on the two different branches. That is to say, a neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{xy = 0\} \subset \mathbb{C}^2$) has an orbifold

chart by a branched covering map $(x, y) \to (x^{n_j}, y^{n_j})$, with $n_j \geq 1$, and with group action $e^{2\pi i/n_j}(x, y) = (e^{2\pi i/n_j}x, e^{-2\pi i/n_j}y)$. An orbifold cover of a nodal orbifold Riemann surface is called a *Hurwitz nodal cover*. Hurwitz nodal covers appear naturally as the degenerations of Hurwitz covers.

Example 2.20 If $\mathcal{X} = Y/G$ is a global quotient and $Z \to Y$ is a universal cover, then $Z \to Y \to \mathcal{X}$ is the orbifold universal cover of $\mathcal{X}$. This results in an extension of groups

$$1 \to \pi_1(Y) \to \pi_1^{\mathrm{orb}}(\mathcal{X}) \to G \to 1. \tag{2.8}$$

On the other hand, as discussed in Example 1.53, the classifying space for a global quotient is simply the Borel construction $EG \times_G Y$; and using the standard fibration $Y \to EG \times_G Y \to BG$, we recover the group extension described above by applying the fundamental group functor. Note that it is clear that a point is the orbifold universal cover of $\bullet^G$, and so $\pi_1^{\mathrm{orb}}(\bullet^G) = G$.

Definition 2.21 An orbifold is a *good orbifold* if its orbifold universal cover is smooth.

It is clear that a global quotient orbifold is good. We can use the orbifold fundamental group to characterize good orbifolds more precisely. Let $x \in \mathcal{X}$ and let $U = \widetilde{U}/G_x$ be an orbifold chart at x. We choose U small enough so that $\widetilde{U}$ is diffeomorphic to a ball. Suppose that $f : \mathcal{Y} \to \mathcal{X}$ is an orbifold universal cover. By definition, $f^{-1}(U)$ is a disjoint union of components of the form $\widetilde{U}/\Gamma$ for some subgroups $\Gamma \subseteq G_x$. Consider the map $\widetilde{U}/\Gamma \to \widetilde{U}/G_x$. The group of deck translations is obviously G_x/Γ, which is thus a subgroup of $\pi_1^{\mathrm{orb}}(\mathcal{X}, x)$. Therefore, we obtain a map

$$\rho_x : G_x \to G_x/\Gamma \subseteq \pi_1^{\mathrm{orb}}(\mathcal{X}, x).$$

A different choice of component in $f^{-1}(\widetilde{U}/G_x)$ yields a homomorphism ρ'_x conjugate to ρ_x by an element $g \in \pi_1^{\mathrm{orb}}(\mathcal{X}, x)$ that interchanges the corresponding components. Therefore, the conjugacy class of (ρ_x) is well defined.

Lemma 2.22 *$\mathcal{X}$ is a good orbifold if and only if ρ_x is injective for each $x \in \mathcal{X}$.*

Proof We use the notation above. $f^{-1}(U)$ (and therefore $\mathcal{Y}$) is smooth if and only if $\Gamma = 1$. The latter is equivalent to the injectivity of ρ_x. $\square$

We will now look at some additional examples. The following observation is very useful in computations. Suppose that $f : \mathcal{Y} \to \mathcal{X}$ is an orbifold universal cover. Then the restriction

$$f : \mathcal{Y} \setminus f^{-1}(\Sigma\mathcal{X}) \to \mathcal{X} \setminus \Sigma\mathcal{X}$$

is an honest cover with $G = \pi_1^{\mathrm{orb}}(\mathcal{X})$ as covering group, and where $\Sigma\mathcal{X}$ is the singular locus of $\mathcal{X}$. Therefore, $\mathcal{X} = \mathcal{Y}/G$, and there is a surjective homomorphism

$$p_f : \pi_1(\mathcal{X} \setminus \Sigma\mathcal{X}) \to G.$$

In general, there is no reason to expect that p_f will be an isomorphism. However, to compute $\pi_1^{\mathrm{orb}}(X)$, we can start with $\pi_1(\mathcal{X} \setminus \Sigma\mathcal{X})$, and then specify any additional relations that are needed.

Example 2.23 Let $G \subset GL_n(\mathbb{Z})$ denote a finite subgroup. As discussed at the beginning of Section 1.2, there is an induced action of G on $\mathbb{T}^n$ with a fixed point. The toroidal orbifold $\mathcal{G}$ associated to $\mathbb{T}^n \to \mathbb{T}^n/G$ has $EG \times_G \mathbb{T}^n$ as its classifying space; hence the orbifold fundamental group is $\pi_1(EG \times_G \mathbb{T}^n) \cong \mathbb{Z}^n \rtimes G$, a semi-direct product. Note that in this case, the orbifold universal cover (as a space) is simply $\mathbb{R}^n$. The action of G on $\mathbb{Z}^n$ is explicitly defined by matrices, so in many cases it is not hard to write an explicit presentation for this semi-direct product.

For example, consider the Kummer surface $\mathbb{T}^4/\tau$, where τ is the involution

$$\tau(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{-it_1}, e^{-it_2}, e^{-it_3}, e^{-it_4}).$$

The universal cover is $\mathbb{R}^4$. The group G of deck translations is generated by four translations λ_i by integral points, and by the involution τ given by

$$(t_1, t_2, t_3, t_4) \mapsto (-t_1, -t_2, -t_3, -t_4).$$

It is easy to check that the orbifold fundamental group admits a presentation

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \tau \mid \tau^2 = 1, \tau\lambda_i\tau^{-1} = \lambda_i^{-1}\}.$$

Note that this is a presentation for the semi-direct product $\mathbb{Z}^4 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Example 2.24 Consider the orbifold Riemann surface Σ_g of genus g and n orbifold points $\mathbf{z} = (x_1, \ldots, x_n)$ of orders $k_1, \ldots, k_n$. Then, according to [140, p. 424], a presentation for its orbifold fundamental group is given by

$$\pi_1^{\mathrm{orb}}(\Sigma_g) = \left\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \sigma_1, \ldots, \sigma_n \mid \sigma_1 \ldots \sigma_n \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1, \sigma_i^{k_i} = 1\right\}, \tag{2.9}$$

where α_i and β_i are the generators of $\pi_1(\Sigma_g)$ and σ_i are the generators of $\Sigma_g \setminus \mathbf{z}$ represented by a loop around each orbifold point. Note that $\pi_1^{\mathrm{orb}}(\Sigma_g)$ is obtained from $\pi_1(\Sigma_g \setminus \mathbf{z})$ by introducing the relations $\sigma_i^{k_i} = 1$. Consider the special case when $\Sigma = \widetilde{\Sigma}/G$, where G is a finite group of automorphisms. In this case, the

orbifold fundamental group is isomorphic to $\pi_1(EG \times_G \widetilde{\Sigma})$, which in turn fits into a group extension

$$1 \to \pi_1(\widetilde{\Sigma}) \to \pi_1^{\mathrm{orb}}(\Sigma) \to G \to 1. \tag{2.10}$$

In other words, the orbifold fundamental group is a *virtual surface group*. This will be true for any *good* orbifold Riemann surface.

2.3 Orbifold vector bundles and principal bundles

We now discuss vector bundles in the context of groupoids more fully.

Definition 2.25 A $\mathcal{G}$*-vector bundle* over an orbifold groupoid $\mathcal{G}$ is a $\mathcal{G}$-space E for which $\pi : E \to G_0$ is a vector bundle, such that the action of $\mathcal{G}$ on E is fiberwise linear. Namely, any arrow $g : x \to y$ induces a linear isomorphism $g : E_x \to E_y$. In particular, E_x is a linear representation of the isotropy group G_x for each $x \in G_0$.

The orbifold groupoid $\mathcal{E} = \mathcal{G} \ltimes E$ associated to E can be thought of as the total space (as a groupoid) of the vector bundle. The natural projection $\pi_E : \mathcal{E} \to \mathcal{G}$ is analogous to the projection of a vector bundle. It induces a projection $\pi_{|E|} : |\mathcal{E}| \to |\mathcal{G}|$, but in general this quotient is no longer a vector bundle. Instead, it has the structure of an orbibundle, so that $\pi_{|E|}^{-1}(x) = E_x/G_x$.

Definition 2.26 A *section* σ of $\mathcal{E}$ is an invariant section of $E \to G_0$. So, if $g : x \to y$ is an arrow, $g(\sigma(x)) = \sigma(y)$. We will often simply say that σ is a section of $E \to G_0$, and we write $\Gamma(\mathcal{E})$ for the set of sections.

$\Gamma(\mathcal{E})$ is clearly a vector space. Many geometric applications of vector bundles are based on the assumption that they always have plenty of local sections. Unfortunately, this may not always be the case for non-effective orbifold groupoids.

Definition 2.27 An arrow g is called a *constant arrow* (or *ineffective arrow*) if there is a neighborhood V of g in G_1 such that for any $h \in V$ $s(h) = t(h)$. We use $\mathrm{Ker}(G_1)$ to denote the space of constant arrows.

Each constant arrow g belongs to G_x for $x = s(g) = t(g)$. The restriction of the groupoid to some neighborhood U_x is a translation groupoid $U_x \times G_x \to U_x$. Then g is constant if and only if g acts on U_x trivially. Let $\mathrm{Ker}(G_x) = G_x \cap \mathrm{Ker}(G_1)$; then $\mathrm{Ker}(G_x)$ acts trivially on U_x.

Definition 2.28 A $\mathcal{G}$-vector bundle $E \to G_0$ is called a *good vector bundle* if $\mathrm{Ker}(G_x)$ acts trivially on each fiber E_x. Equivalently, $\mathcal{E} \to \mathcal{G}$ is a good vector bundle if and only if $\mathrm{Ker}(E_1) = \mathrm{Ker}(G_1) \times_{G_0} E$.

A good vector bundle always has enough local sections. Therefore, for good bundles, we can define local connections and patch them up to get a global connection. Chern–Weil theory can then be used to define characteristic classes for a good vector bundle; they naturally lie in the de Rham cohomology groups $H^*(\mathcal{G}) \cong H^*(|\mathcal{G}|; \mathbb{R})$. It seems better, however, to observe that $B\mathcal{E} \to B\mathcal{G}$ is naturally a vector bundle, so we have associated classifying maps $B\mathcal{G} \to BO(m)$ or $B\mathcal{G} \to BU(m)$. It thus makes sense to *define* the characteristic classes of $\mathcal{E} \to \mathcal{G}$ as the characteristic classes of $B\mathcal{E} \to B\mathcal{G}$. Under this definition, characteristic classes naturally lie in either $H^*(B\mathcal{G}; \mathbb{Z})$ (Chern classes) or in $H^*(B\mathcal{G}; \mathbb{F}_2)$ (Stiefel–Whitney classes). Now, the map $B\mathcal{G} \to |\mathcal{G}|$ induces an isomorphism $H^*(B\mathcal{G}; \mathbb{Q}) \to H^*(|\mathcal{G}|; \mathbb{Q})$. In this book we will think of this as the natural place for Chern classes of complex bundles, and we will be using both definitions without distinction.

Example 2.29 Suppose that a Lie group G acts smoothly, properly, and with finite isotropy on X, and let E be a G-bundle. Then E/G admits a natural orbifold structure such that $E/G \to X/G$ is an orbifold vector bundle. Conversely, if $F \to X/G$ is an orbifold vector bundle, the pullback p^*F is a G-bundle over X.

We now give some examples of good vector bundles; of course, any vector bundle over an effective groupoid is good.

Example 2.30 Suppose that $\mathcal{G}$ is an orbifold groupoid. Then the tangent bundle $T\mathcal{G}$, the cotangent bundle $T^*\mathcal{G}$, and $\bigwedge^* T^*\mathcal{G}$ are all good vector bundles.

Example 2.31 Consider the point groupoid $\bullet^G$. A $\bullet^G$-vector bundle E corresponds to a representation of G, and E is good if and only if E is a trivial representation.

Many geometric constructions (such as index theory) can be carried out in the context of good orbifold groupoid vector bundles. Moreover, any orbifold groupoid has a canonical associated effective orbifold groupoid.

Lemma 2.32 $\mathrm{Ker}(G_1)$ *consists of a union of connected components in* G_1.

Proof By definition, $\mathrm{Ker}(G_1)$ is open. We claim that it is closed. Let $g_n \to g$ for a sequence $g_n \in \mathrm{Ker}(G_1)$. We observe that $s(g) = t(g) = x$ for some x. Hence, $g \in G_x$. Moreover, $x_n = s(g_n) = t(g_n)$ converges to x. As usual, take a small

neighborhood U_x so that the restriction of $\mathcal{G}$ to U_x is equivalent to U_x/G_x. It is clear that under the equivalence g_n is identified with g for sufficiently large n. Therefore, g fixes some open subset of U_x, and hence fixes every point of U_x. □

Definition 2.33 For any orbifold groupoid $\mathcal{G}$, we define an effective orbifold groupoid $\mathcal{G}_{\text{eff}}$ with objects $G_{\text{eff},0} = G_0$ and arrows

$$G_{\text{eff},1} = G_1 \setminus (\text{Ker}(G_1) \setminus u(G_0)),$$

where $u : G_0 \to G_1$ is the groupoid unit.

Note that $E \to G_0$ is a good vector bundle if and only if it induces a vector bundle over $\mathcal{G}_{\text{eff}}$.

Example 2.34 If $G \ltimes M$ is an action groupoid associated to a quotient orbifold, then it will be effective if the action of G is effective. If $G \to G_{\text{eff}}$ is the quotient by the kernel of the action, then $G_{\text{eff}} \ltimes M$ is the associated effective orbifold groupoid.

We now introduce principal bundles.

Definition 2.35 Let K be a Lie group. A *principal K-bundle P* over $\mathcal{G}$ is a $\mathcal{G}$-space P together with a left action $K \times P \to P$ that makes $P \to G_0$ into a principal K bundle over the manifold G_0.

Let $\mathcal{P}$ be the corresponding orbifold groupoid; then $B\mathcal{P} \to B\mathcal{G}$ is a principal K-bundle in the usual sense. Hence by the homotopy classification of principal K-bundles, we have a classifying map $B\mathcal{G} \to BK$, and we can obtain characteristic classes just as before.

A particularly interesting case occurs when K is a discrete group. As the reader might expect, it is intimately related to covering spaces. $B\mathcal{P} \to B\mathcal{G}$ is a principal K-bundle, and so $B\mathcal{P}$ can be thought of as a (possibly disconnected) covering space. Choose a lifting $\hat{x}_0$ of the basepoint $\tilde{x}_0 \in G_0$; the path-lifting property defines a homomorphism

$$\rho : \pi_1^{\text{orb}}(\mathcal{G}) = \pi_1(B\mathcal{G}, \tilde{x}_0) \to K.$$

A different choice of $\hat{x}_0$ defines a conjugate homomorphism. Therefore, the conjugacy class of ρ is an invariant of P. Conversely, given a homomorphism ρ, let P_{univ} be the universal cover. Then $P_{\text{univ}} \times_\rho K$ is a principal K-bundle with the given ρ. Therefore, we obtain an exact analog of the classical theory of principal K-bundles (see [146, p. 70]):

Theorem 2.36 *The isomorphism classes of principal K-bundles over $\mathcal{G}$, where K is a discrete group, are in one-to-one correspondence with K-conjugacy classes of group homomorphisms $\pi_1^{\mathrm{orb}}(\mathcal{G}) \to K$.*

As in the classical setting we can now introduce fiber bundles.

Definition 2.37 If a $\mathcal{G}$-space $P \to G_0$ is a fibered product, we call P a *fiber bundle* over $\mathcal{G}$.

As usual, there is a close relationship between fiber bundles and principal bundles. If $P \to \mathcal{G}$ is a principal K-bundle and K acts smoothly on a manifold E, then $P \times_K E$ is a smooth fiber bundle. Conversely, if K is the structure group of the fiber bundle $E \to \mathcal{G}$, then E naturally gives rise to a principal K-bundle. A very interesting case is given by a covering $\mathcal{G}$-space $\pi : E \to \mathcal{G}$. Suppose that $|\pi^{-1}(\tilde{x}_0)| = n$. It is clear that $\mathrm{Aut}(\pi^{-1}(\tilde{x}_0)) = S_n$, the symmetric group on n letters. Therefore, E induces a principal S_n-bundle $P \to \mathcal{G}$ such that $E = P \times_{S_n} \{1, \ldots, n\}$.

Specializing to the case where $\mathcal{G}$ is a nodal orbifold Riemann surface, we obtain the following classical theorem.

Theorem 2.38 *The Hurwitz nodal covers of Σ are classified by conjugacy classes of homomorphisms $\rho : \pi_1^{\mathrm{orb}}(\mathcal{G}) \to S_n$.*

One would expect universal bundles to play an important role here. Suppose that $EK \to BK$ is the universal principal K-bundle. We define $\mathcal{B}K$ as the (non-smooth) unit groupoid with $\mathcal{B}K_0 = \mathcal{B}K_1 = BK$ and $s = t = \mathrm{Id}$. For any morphism $\phi : \mathcal{G} \to \mathcal{B}K$, the pullback $\phi_0^* EK$ is a principal K-bundle over $\mathcal{G}$. It would be good if the converse were also true.

2.4 Orbifold morphisms

The theory of orbifolds diverges from that of manifolds when one considers the notion of a morphism. Although Satake [139] defined maps between orbifolds, it can be seen that with his definition the pullback of an orbifold vector bundle may fail to be an orbifold vector bundle.

To overcome this problem, one has to introduce a different notion of a map between orbifolds. There are two versions available: the Moerdijk–Pronk *strong map* [113] and the Chen–Ruan *good map* [38]. Fortunately, it has been shown [104] that the two versions are equivalent. We will follow the first version, which is best suited to the groupoid constructions we have been using. We now

proceed to develop the basic properties of orbifold morphisms, following the treatment in [112]. Recall the following definition from Section 1.4.

Definition 2.39 Suppose that $\mathcal{H}$ and $\mathcal{G}$ are orbifold groupoids. An *orbifold morphism* from $\mathcal{H}$ to $\mathcal{G}$ is a pair of groupoid homomorphisms

$$\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G},$$

such that the left arrow is an equivalence.

As mentioned in the last chapter, not all of these morphisms ought to be viewed as distinct:

- If there exists a natural transformation between two homomorphisms $\phi, \phi' : \mathcal{K} \to \mathcal{G}$, then we consider $\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi'}{\rightarrow} \mathcal{G}$ to be equivalent to $\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G}$.
- If $\delta : \mathcal{K}' \to \mathcal{K}$ is an orbifold equivalence, the morphism

$$\mathcal{H} \xleftarrow{\epsilon\circ\delta} \mathcal{K}' \xrightarrow{\phi\circ\delta} \mathcal{G}$$

  is equivalent to $\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G}$.

Let $\mathcal{R}$ be the minimal equivalence relation among orbifold morphisms from $\mathcal{H}$ to $\mathcal{G}$ generated by the two relations above.

Definition 2.40 Two orbifold morphisms are said to be *equivalent* if they belong to the same $\mathcal{R}$-equivalence class.

We now verify a basic result.

Theorem 2.41 *The set of equivalence classes of orbifold morphisms from $\mathcal{H}$ to $\mathcal{G}$ is invariant under orbifold Morita equivalence.*

Proof Suppose that $\delta : \mathcal{H}' \to \mathcal{H}$ is an orbifold equivalence. It is clear from the definitions that an equivalence class of orbifold morphisms from $\mathcal{H}'$ to $\mathcal{G}$ induces an equivalence class of orbifold morphisms from $\mathcal{H}$ to $\mathcal{G}$ by precomposing with δ. Conversely, suppose that $\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G}$ is an orbifold morphism, and consider the groupoid fiber product $\mathcal{K}' = \mathcal{H}' \times_{\mathcal{H}} \mathcal{K}$. Then there are orbifold equivalences $p : \mathcal{K}' \to \mathcal{K}$ and $\delta' : \mathcal{K}' \to \mathcal{H}'$. We map $\mathcal{H} \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G}$ to the orbifold morphism

$$\mathcal{H}' \xleftarrow{\delta'} \mathcal{K}' \xrightarrow{\phi\circ p} \mathcal{G}.$$

A quick check shows that this maps equivalent orbifold morphisms to equivalent orbifold morphisms.

Next suppose that $\delta : \mathcal{G}' \to \mathcal{G}$ is an orbifold equivalence. Again, it is obvious that an equivalence class of orbifold morphisms to $\mathcal{G}'$ induces an equivalence class of orbifold morphisms to $\mathcal{G}$. We can use a similar method to construct an inverse to this assignment. Suppose that $\mathcal{H} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{G}$ is an orbifold morphism. Consider the groupoid fiber product $\mathcal{K}' = \mathcal{K} \times_{\mathcal{H}} \mathcal{G}'$. The projection maps give an orbifold equivalence $\mathcal{K}' \to \mathcal{K}$ and a homomorphism $\mathcal{K}' \to \mathcal{G}'$. By composing with the orbifold equivalence $\epsilon : \mathcal{K} \to \mathcal{H}$, we obtain an orbifold morphism

$$\mathcal{H} \leftarrow \mathcal{K}' \to \mathcal{G}'.$$

Again, a straightforward check shows that this transformation maps equivalence classes to equivalence classes. □

It can be shown [111, 125] that the set of Morita equivalence classes of orbifold groupoids forms a category with morphisms the equivalence classes of orbifold morphisms. We call this the *category of orbifolds*.

Example 2.42 We classify all orbifold morphisms between $\bullet^G$ and $\bullet^H$. To do so, we must first study orbifold equivalences $\epsilon : \mathcal{K} \to \bullet^G$. Suppose that $\mathcal{K}$ has objects K_0 and arrows K_1. By definition, K_0 must be a discrete set of points, and for each $x_0 \in K_0$ it is clear that the restriction of $\mathcal{K}$ to x_0 must be translation groupoid $G \ltimes \{x_0\} \cong \bullet^G$. Hence, we can locally invert ϵ by mapping the object of $\bullet^G$ to x_0. Let ϵ^{-1} be this inverse. Then the orbifold morphism $\bullet^G \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \bullet^H$ is equivalent to the orbifold morphism

$$\bullet^G = \bullet^G \xrightarrow{\phi \circ \epsilon^{-1}} \bullet^H.$$

Therefore, we have reduced our problem to the classification of homomorphisms $\psi : \bullet^G \to \bullet^H$ up to natural transformations. Such a ψ corresponds to a group homomorphism $G \to H$, and a natural transformation between ψ and ψ' is simply an element $h \in H$ such that $\psi' = h\psi h^{-1}$. Consequently, the set of equivalence classes of orbifold morphisms from $\bullet^G$ to $\bullet^H$ is in one-to-one correspondence with H-conjugacy classes of group homomorphisms $\psi : G \to H$.

We can use similar arguments to understand the local structure of an arbitrary orbifold morphism. Suppose that $F : \mathcal{G} \to \mathcal{H}$ is a morphism of orbifold groupoids given by $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ covering the map $f : |\mathcal{G}| \to |\mathcal{H}|$. Let $x \in G_0$; then locally $\tilde{f} = \phi_0 \epsilon_0^{-1} : U_x \to V_{f(x)}$, where $U_x, V_{f(x)}$ are orbifold charts. Furthermore, F induces a group homomorphism $\lambda = \phi_1 \epsilon_1^{-1} : G_x \to H_{f(x)}$. By definition, $\tilde{f}$ is λ-equivariant. Such a pair $(\tilde{f}, \lambda)$ is called a *local lifting* of f. It

induces a groupoid homomorphism $G_x \ltimes U_x \to H_{f(x)} \ltimes V_{f(x)}$ in an obvious way. In fact, the restriction of F is precisely $(\tilde{f}, \lambda)$ under the isomorphisms corresponding to the charts.

In general, it is difficult to study orbifold morphisms starting from the definitions, since we have to study *all* equivalent orbifold groupoids representing the domain orbifold. Therefore, it is necessary to develop a more practical approach. One of the most effective tools is to explore the relationship between orbifold morphisms and bundles. Indeed, this was the original motivation for the introduction of orbifold morphisms.

Suppose that $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ is an orbifold morphism. Given an $\mathcal{H}$-vector or principal bundle E, then we call the $\mathcal{G}$-bundle $\epsilon_* \phi^* E$ the *pullback* of E via the orbifold morphism. If there is a natural transformation between ϕ and ϕ', then $\epsilon_* \phi^* E \cong \epsilon_* (\phi')^* E$. We have thus proved the following theorem.

Theorem 2.43 *Each orbifold morphism $F = \{\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}\}$ pulls back isomorphism classes of orbifold vector or principal bundles over $\mathcal{H}$ to isomorphism classes of orbifold vector or principal bundles over $\mathcal{G}$, and if F' is equivalent to F, then $F^* \cong F'^*$.*

2.5 Classification of orbifold morphisms

In what follows we will present the Chen–Ruan classification of orbifold morphisms from lower dimensional orbifolds. In the process, we introduce many concrete examples of orbifold morphisms. This classification forms the foundation of the Chen–Ruan cohomology theory (and also orbifold quantum cohomology theory). Our approach is quite different from the original. It is less direct, but in many ways much cleaner.

We first introduce representability. Suppose that $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ is an orbifold morphism. Choose a local chart U_x of $\mathcal{G}$, and fix a point $z \in \epsilon_0^{-1}(x)$. Since ϵ is an equivalence, we can invert $\epsilon|_{U_z} : G_z \ltimes U_z \to G_x \ltimes U_x$. Let $\epsilon^{-1}|_{U_x}$ be the inverse. Then we obtain a homomorphism $\phi\epsilon^{-1}|_{U_x} : G_x \ltimes U_x \to G_{\phi_0(z)} \ltimes U_{\phi_0(z)}$, which induces a group homomorphism $\lambda_x : G_x \to G_{\phi_0(z)}$. If we choose a different $z' \in \epsilon_0^{-1}(x)$, then $\phi_0(z)$ is connected to $\phi_0(z')$ by an arrow g, and the corresponding local morphism is related by a conjugation with g.

Definition 2.44 We call $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ *representable* if λ_x is injective for all $x \in G_0$.

In what follows we focus on the representable orbifold morphisms, although most of the constructions also work well for non-representable ones.

Note that for a global quotient Y/G, there is a canonical orbifold principal G-bundle $Y \to Y/G$.

Theorem 2.45 *Suppose that* $F = \{\mathcal{G} \leftarrow \mathcal{K} \to G \ltimes Y\}$ *is an orbifold morphism. Then,*

1. *The pullback* $F^*Y \to \mathcal{G}$ *is a* G*-bundle with a* G*-equivariant map* $\phi : F^*Y \to Y$*. Conversely, suppose that* $E \to \mathcal{G}$ *is a smooth* G*-bundle and* $\phi : E \to Y$ *is a* G*-map. Then the quotient by* G *induces an orbifold morphism from* $\mathcal{G}$ *to* $G \ltimes Y$*.*
2. *If* F' *is equivalent to* F*, then there is a bundle isomorphism* $p : F'^*Y \to F^*Y$ *such that* $\phi p = \phi'$*.*
3. F *is representable if and only if* $E = F^*Y$ *is smooth.*

Proof All the statements are clear except the relation between the smoothness of $E = F^*Y$ and representability of F. However, this is a local property, and locally we have the representation $F : G_x \ltimes U_x \to G_y \ltimes U_y$. By a previous argument, F is equivalent to a pair $(\tilde{f}, \lambda)$, where $\lambda : G_x \to G_y$ is a group homomorphism and $\tilde{f} : U_x \to U_y$ is a λ-equivariant map. What is more, we have an embedding $G_y \ltimes U_y \to G \ltimes Y$. The groupoid presentation of the orbifold principal bundle $Y \to Y/G$ is $p : Y \times G \to Y$, where p is the projection onto the first factor and $h \in G$ acts as $h(x, g) = (hx, gh^{-1})$. Now, we use the local form of F to obtain a local form of F^*Y as a G_x-quotient of

$$U_x \times_{\tilde{f}} Y \times G \to U_x.$$

Here, $h \in G_x$ acts as

$$h(x', y, g) = (hx', \lambda(h)y, g\lambda(h)^{-1}).$$

The action above is free on the total space if and only if $\lambda(h) \neq 1$. Hence, F^*Y is smooth if and only if λ is injective, as desired. □

Corollary 2.46 *Equivalence classes of representable orbifold morphisms from* $\mathcal{G}$ *to* Y/G *are in one-to-one correspondence with equivalence classes of diagrams* $\mathcal{G} \leftarrow E \xrightarrow{\phi} Y$*, where the left arrow is a* G*-bundle projection and the right arrow is a* G*-map. The equivalence relation on the diagrams is generated by bundle isomorphisms* $\epsilon : E' \to E$ *with corresponding* G*-map* $\phi' = \phi\epsilon$*.*

The corollary reduces the classification of orbifold morphisms to an equivariant problem, at least in the case where the codomain is a global quotient.[1]

[1] When the codomain is a general groupoid, one can still understand orbifold morphisms using principal bundles; however, the structure group must be replaced by a structure groupoid. Details of this alternative perspective and helpful discussions of the relationship between orbifold groupoids and stacks appear in [69], [70], [109], and [116].

As we have seen, a principal G-bundle $E \to \mathcal{G}$ is determined by the conjugacy class of a homomorphism $\rho : \pi_1^{\mathrm{orb}}(\mathcal{G}, x_0) \to G$. We call ρ the *Chen–Ruan characteristic* of the orbifold morphism. It is a fundamental invariant in the classification of orbifold morphisms. Let us apply the corollary in some examples to see how this works.

Example 2.47 Consider the orbifold morphisms from $\mathbb{S}^1$ with trivial orbifold structure to $\bullet^G$. In other words, we want to study the loop space $\Omega(\bullet^G)$. The G-maps from E to $\bullet$ are obviously trivial; hence, we only have to classify the G-bundles E. By principal bundle theory, these are classified by the conjugacy classes of characteristics $\rho : \pi_1(\mathbb{S}^1, x_0) \to G$. However, $\pi_1(\mathbb{S}^1, x_0)$ is $\mathbb{Z}$, generated by a counterclockwise loop. Let g be the image of this generator; then ρ is determined by g. Therefore $\Omega(\bullet^G)$ is in one-to-one correspondence with conjugacy classes of elements in G.

Example 2.48 The previous example can be generalized to the loop space $\Omega(G \ltimes Y)$ of a general global quotient. In this case, $E \to \mathbb{S}^1$ is a possibly disconnected covering space, with a fixed G-map $\phi : E \to Y$. Again, E is determined by the conjugacy class of a homomorphism $\rho : \mathbb{Z} = \pi_1(\mathbb{S}^1, x_0) \to G$. Choose a lifting $\tilde{x}_0 \in E$ of the basepoint x_0. Suppose σ is a loop based at x_0 that generates $\pi_1(\mathbb{S}^1, x_0)$. Lift σ to a path $\tilde{\sigma}(t)$ in E starting at $\tilde{\sigma}(0) = \tilde{x}_0$. The end point $\tilde{\sigma}(1)$ is then $g\tilde{x}_0$, where $g = \rho([\sigma])$ is the image of the generator. Let $\gamma(t) = \phi(\sigma(t))$. Then we obtain a path $\gamma(t)$ in Y and $g \in G$ such that $g\gamma(0) = \gamma(1)$. It is clear that ϕ is uniquely determined by $\gamma(t)$. The different liftings $\tilde{x}_0$ correspond to an action $h(g, \gamma(t)) = (hgh^{-1}, h\gamma(t))$. Therefore,

$$\Omega(G \ltimes Y) = G \ltimes \{(g, \gamma(t)) \mid g\gamma(0) = \gamma(1)\},$$

where G acts as we described previously.

Let $\mathcal{G}$ be a groupoid, and consider the pullback diagram of spaces

$$\begin{tikzcd} S_{\mathcal{G}} \arrow[r] \arrow[d, "\beta"'] & G_1 \arrow[d, "{(s,t)}"] \\ G_0 \arrow[r, "\mathrm{diag}"] & G_0 \times G_0 \end{tikzcd} . \tag{2.11}$$

Then $S_{\mathcal{G}} = \{g \in G_1 \mid s(g) = t(g)\}$ is intuitively the space of "loops" in $\mathcal{G}$. The map $\beta : S_{\mathcal{G}} \to G_0$ sends a loop $g : x \to x$ to its basepoint $\beta(g) = x$. This map is proper, and one can verify that the space $S_{\mathcal{G}}$ is in fact a manifold. Suppose that $h \in G_1$; then h induces a map $h : \beta^{-1}(s(h)) \to \beta^{-1}(t(h))$ as follows. For any $g \in \beta^{-1}(s(h))$, set $h(g) = hgh^{-1}$. This action makes $S_{\mathcal{G}}$ into a left $\mathcal{G}$-space.

Definition 2.49 We define the *inertia groupoid* $\wedge\mathcal{G}$ as the action groupoid $\mathcal{G} \ltimes S_{\mathcal{G}}$.

This inertia groupoid generalizes the situation for a global quotient considered in Example 2.5. We observe that β induces a proper homomorphism $\beta : \wedge\mathcal{G} \to \mathcal{G}$. The construction of the inertia groupoid is natural, in the sense that if $\phi : \mathcal{H} \to \mathcal{G}$ is a homomorphism, it induces a homomorphism $\phi_* : \wedge\mathcal{H} \to \wedge\mathcal{G}$. When ϕ is an equivalence, so is ϕ_*. Thus, the Morita equivalence class of $\wedge\mathcal{G}$ is an orbifold invariant.

Given an orbifold groupoid $\mathcal{G}$, what we have described above are the orbifold morphisms from $\mathbb{S}^1$ into $\mathcal{G}$ such that the induced map $\mathbb{S}^1 \to |\mathcal{G}|$ takes a constant value x (also known as the *constant loops*). It is clear that such an orbifold morphism factors through an orbifold morphism to U_x/G_x. Hence, we can use our description of the loop space for a global quotient. It follows that, as a set, $| \wedge \mathcal{G}| = \{(x, (g)_{G_x}) \mid x \in |\mathcal{G}|, g \in G_x\}$. The groupoid $\wedge\mathcal{G}$ is an extremely important object in stringy topology, and is often referred to as the *inertia orbifold of* $\mathcal{G}$ or the *groupoid of twisted sectors*.

Example 2.50 Consider the orbifold morphisms from an arbitrary orbifold $\mathcal{G}$ to $\bullet^G$. Again, there is only one G-map $\phi : E \to \bullet$, and so we only have to consider the classification of G-bundles $E \to \mathcal{G}$. These correspond to conjugacy classes of characteristics $\rho : \pi_1^{\mathrm{orb}}(\mathcal{G}, x_0) \to G$. We can use this to study a particularly interesting example – the space $\overline{\mathcal{M}}_k$ of constant representable orbifold morphisms from a Riemann sphere $\mathbb{S}^2$ with k orbifold points to an arbitrary orbifold $\mathcal{G}$.

Suppose that the image of the constant morphism is $x \in |\mathcal{G}|$. Let G_x be the local group. Clearly, the morphism factors through the constant morphism to $\bullet^{G_x}$. Hence, it is determined by the conjugacy classes of representable homomorphisms $\rho : \pi_1^{\mathrm{orb}}(\mathbb{S}^2) \to G_x$. Suppose that the orbifold structures at the marked points are given by the integers $m_1, \dots, m_k$. Then, as we have seen,

$$\pi_1^{\mathrm{orb}}(\mathbb{S}^2, x_0) = \{\lambda_1, \dots, \lambda_k \mid \lambda_i^{m_i} = 1, \lambda_1 \dots \lambda_k = 1\}.$$

Then ρ is representable if and only if $\rho(\lambda_i)$ has order m_i. Let Π be the set of (isomorphism classes of) orbifold fundamental groups $\pi_1^{\mathrm{orb}}(\mathbb{S}^2, x_0)$ obtained as the orbifold structures at the k marked points in $\mathbb{S}^2$ varies, and let $\overline{\mathcal{M}}_k = \{\rho : \pi \to G_x \mid \pi \in \Pi\}$. Then $\overline{\mathcal{M}}_k$ is a $\mathcal{G}$-space in an obvious way, and we can form the action groupoid $\mathcal{G} \ltimes \overline{\mathcal{M}}_k$. We will often use $\overline{\mathcal{M}}_k$ to denote this action groupoid as well. Using the above presentation of $\pi_1^{\mathrm{orb}}(\mathbb{S}^2, x_0)$, we can identify

$$\overline{\mathcal{M}}_k = \{(g_1, \dots, g_k)_{G_x} \mid g_i \in G_x, g_1 \dots g_k = 1\}, \tag{2.12}$$

where g_i is the image of λ_i.

We can generalize the twisted sector groupoid construction $\wedge\mathcal{G}$ to obtain the groupoid $\mathcal{G}^k$ of *k-multisectors*, where $k \geq 1$ is an integer. Moreover, a construction similar to that of the constant loops can give an orbifold groupoid structure to the space of k-multisectors. Let

$$|\mathcal{G}^k| = \{(x, (g_1, \ldots, g_k)_{G_x}) \mid x \in |\mathcal{G}|, g_i \in G_x\}.$$

It is clear that $|\overline{\mathcal{M}}_k| \cong |\mathcal{G}^{k-1}|$. We construct an orbifold groupoid structure for $|\mathcal{G}^k|$ as follows. Consider the space

$$\begin{aligned} S^k_{\mathcal{G}} = \{(g_1, \ldots, g_k) \mid g_i \in G_1, s(g_1) = t(g_1) = s(g_2) = t(g_2) \\ = \cdots = s(g_k) = t(g_k)\}. \end{aligned} \tag{2.13}$$

This is a smooth manifold. We have $\beta_k : S^k_{\mathcal{G}} \to G_0$ defined by

$$\beta_k(g_1, \ldots, g_k) = s(g_1) = t(g_1) = s(g_2) = t(g_2) = \cdots = s(g_k) = t(g_k).$$

Just as with the twisted sectors, there is a fiberwise action for $h \in G_1$: the map

$$h : \beta_k^{-1}(s(h)) \to \beta_k^{-1}(t(h))$$

is given by

$$h(g_1, \ldots, g_k) = (hg_1h^{-1}, \ldots, hg_kh^{-1}).$$

This action gives $S^k_{\mathcal{G}}$ the structure of a $\mathcal{G}$-space. The orbit space of the associated translation groupoid $\mathcal{G}^k = \mathcal{G} \ltimes S^k_{\mathcal{G}}$ is precisely the one given by the formula above. The identification $\overline{\mathcal{M}}_k \cong \mathcal{G}^{k-1}$ depends on the choice of a presentation for each $\pi_1^{\mathrm{orb}}(\mathbb{S}^2, x_0)$. That is, when we switch the ordering of the marked points, we get a different identification. Hence, there is an action of the symmetric group S_n on $\mathcal{G}^n$. It is interesting to write down what happens explicitly. We shall write down the formula for interchanging two marked points. The general case is left as an exercise for readers. Suppose we switch the order of the first two marked points. The induced automorphism on $\mathcal{G}^n$ is

$$(g_1, g_2, \ldots, g_n) \to (g_2, g_2^{-1}g_1g_2, g_3, \ldots, g_n).$$

The k-sectors will become vitally important in Chapter 4 when we define and study Chen–Ruan cohomology.

Example 2.51 Another interesting example is given by the representable orbifold morphisms to a symmetric product Y^k/S_k. This reduces to studying S_k-maps ϕ from S_k-bundles E to Y^k. Let $\phi = (\phi_1, \ldots, \phi_k)$; for any $\mu \in S_k$, $\phi_i(\mu x) = \phi_{\mu(i)}(x)$. We can de-symmetrize the map as follows. Let $\mathbf{k} = \{1, \ldots, k\}$ be the set with k symbols. We define

$$\bar{\phi} : E \times \mathbf{k} \to Y$$

by $\bar{\phi}(x, i) = \phi_i(x)$. Then, for any $\mu \in S_k$, we have $\bar{\phi}(\mu x, \mu^{-1} i) = \phi_{\mu i}(\mu^{-1} x) = \phi_i(x)$. Therefore, we can quotient out by S_k to obtain a non-equivariant map (still denoted by $\bar{\phi}$)

$$\bar{\phi} : \bar{E} = (E \times \mathbf{k})/S_k \to Y.$$

It is clear that $\bar{E}$ is an associated fiber bundle of E, and hence an orbifold cover of degree k. Conversely, if we have a morphism $\bar{\phi} : \bar{E} = (E \times \mathbf{k})/S_k \to Y$, we can reconstruct $\phi = (\phi_1, \ldots, \phi_k)$ by defining $\phi_i = \bar{\phi}([x, i])$. It is clear that we recover the theory of Hurwitz covers as the theory of representable orbifold morphisms from an orbifold Riemann surface to $\bullet^{S_n}$.

3

Orbifold K-theory

3.1 Introduction

Orbifold K-theory is the K-theory associated to orbifold vector bundles. This can be developed in the full generality of groupoids, but as we have seen in Chapter 1, any effective orbifold can be expressed as the quotient of a smooth manifold by an almost free action of a compact Lie group. Therefore, we can use methods from equivariant topology to study the K-theory of effective orbifolds. In particular, using an appropriate equivariant Chern character, we obtain a decomposition theorem for *orbifold K-theory* as a ring. A byproduct of our orbifold K-theory is a natural notion of orbifold Euler number for a general effective orbifold. What we lose in generality is gained in simplicity and clarity of exposition. Given that all known interesting examples of orbifolds do indeed arise as quotients, we feel that our presentation is fairly broad and will allow the reader to connect orbifold invariants with classical tools from algebraic topology. In order to compute orbifold K-theory, we make use of equivariant Bredon cohomology with coefficients in the representation ring functor. This equivariant theory is the natural target for equivariant Chern characters, and seems to be an important technical device for the study of orbifolds.

A key physical concept in orbifold string theory is twisting by discrete torsion. An important goal of this chapter is to introduce twisting for orbifold K-theory. We introduce *twisted orbifold K-theory* using an explicit geometric model. In the case when the orbifold is a global quotient $\mathcal{X} = Y/G$, where G is a finite group, our construction can be understood as a twisted version of equivariant K-theory, where the twisting is done using a fixed element $\alpha \in H^2(G; \mathbb{S}^1)$. The basic idea is to use the associated central extension, and to consider equivariant bundles with respect to this larger group which cover the G-action on Y. A computation of the associated twisted theory can be

explicitly obtained (over the complex numbers) using ingredients from the classical theory of projective representations.

More generally we can define a twisted orbifold K-theory associated to the universal orbifold cover; in this generality it can be computed in terms of twisted Bredon cohomology. This can be understood as the E_2-term of the twisted version of a spectral sequence converging to twisted orbifold K-theory, where in all known instances the higher differentials are trivial in characteristic zero (this is a standard observation in the case of the Atiyah–Hirzebruch spectral sequence). Finally, we should also mention that orbifold K-theory seems like the ideal setting for comparing invariants of an orbifold to that of its resolutions. A basic conjecture in this direction is the following.

Conjecture 3.1 (K-Orbifold String Theory Conjecture) *If $\mathcal{X}$ is a complex orbifold and $Y \to \mathcal{X}$ is a crepant resolution, then there is a natural additive isomorphism*

$$K(Y) \otimes \mathbb{C} \cong K_{\text{orb}}(\mathcal{X}) \otimes \mathbb{C}$$

between the orbifold K-theory of $\mathcal{X}$ and the ordinary K-theory of its crepant resolution Y.

Note, for example that if $\mathcal{X}$ is a complex 3-orbifold with isotropy groups in $SL_3(\mathbb{C})$, then it admits a crepant resolution – this condition is automatically satisfied by Calabi–Yau orbifolds.

3.2 Orbifolds, group actions, and Bredon cohomology

Our basic idea in studying orbifold K-theory is to apply methods from equivariant topology. In this section, we recall some basic properties of orbifolds and describe how they relate to group actions.

We have seen that if a compact Lie group G acts smoothly, effectively, and almost freely on a manifold M, then the quotient M/G is an effective orbifold. More generally, $\mathcal{X} = M/G$ is an orbifold for any smooth Lie group action if the following conditions are satisfied:

- For any $x \in M$, the isotropy subgroup G_x is finite.
- For any $x \in M$ there is a *smooth slice* S_x at x.
- For any two points $x, y \in M$ such that $y \notin Gx$, there are slices S_x and S_y such that $GS_x \cap GS_y = \emptyset$.

If G is compact, an almost free G-action automatically satisfies the second and third conditions. Examples arising from proper actions of discrete groups will also appear in our work.

In Chapter 1, we used frame bundles to show that every effective orbifold $\mathcal{X}$ has an action groupoid presentation $G \ltimes M$, where in fact we may take $G = O(n)$ to be an orthogonal group. Furthermore, we conjectured (Conjecture 1.55) that in fact *every* orbifold has such a presentation. Therefore, it is no great loss of generality if we restrict our attention to quotient orbifolds of the form $G \ltimes M$ for (possibly non-effective) almost free actions of a Lie group G on a smooth manifold M.

We will assume for simplicity that our orbifolds are compact. In the case of quotient orbifolds M/G with G a compact Lie group, this is equivalent to the compactness of M itself (see [31, p. 38]); a fact the we will use. In order to apply methods from algebraic topology in the study of orbifolds, we recall a well-known result about manifolds with smooth actions of compact Lie groups (see [71]):

Theorem 3.2 *If a compact Lie group G acts on a smooth, compact manifold M, then the manifold is triangulable as a finite G-CW complex.*

Hence any such manifold will have a cellular G-action such that the orbit space M/G has only finitely many cells.

For the rest of this chapter, we will focus on quotient orbifolds M/G, which as we have seen are quite general. We will consider actions of both compact and discrete groups, using G to denote a compact Lie group and Γ to denote a discrete group.

In Section 2.3, we defined singular cohomology and characteristic classes for orbifolds. In the case of a quotient $G \ltimes M$, the orbifold cohomology coincided with the usual equivariant cohomology $H^*(EG \times_G M; R)$. This became the natural home for characteristic classes associated to the orbifold M/G. However, if R is a ring such that the order $|G_x|$ of each isotropy group is invertible in R, then there is an algebra isomorphism $H^*_{orb}(G \ltimes M; R) \cong H^*(M/G; R)$, obtained from a Leray spectral sequence. An appropriate ring R can be constructed from the integers by inverting the least common multiple of the orders of all the local transformation groups; the rational numbers $\mathbb{Q}$ are of course also a suitable choice. Thus if $G \ltimes M$ has all isotropy groups of odd order, we may think of its Stiefel–Whitney classes $w_i(G \ltimes M)$ as classes in $H^*(M/G; \mathbb{F}_2)$. Similarly, if $G \ltimes M$ is complex, we have Chern classes $c_i(G \ltimes M) \in H^*(M/G; R)$ for an appropriate ring R.

More generally, what we see is that with integral coefficients, the equivariant cohomology of M will have interesting *torsion* classes. Unfortunately, integral computations are notoriously difficult, especially when finite group cohomology is involved. The mod p equivariant cohomology of M will contain

interesting information about the action; in particular, its Krull dimension will be equal to the maximal p-rank of the isotropy subgroups (see [128]). However, for our geometric applications it is convenient to use an equivariant cohomology theory which has substantial *torsion-free* information. That is where K-theory[1] naturally comes in, as instead of cohomology, the basic object is a representation ring.

Less well known than ordinary equivariant cohomology is the *Bredon cohomology* associated to a group action. It is in fact the most adequate equivariant cohomology theory available. We briefly sketch its definition for the case of compact Lie groups, and refer the reader to [30], [101], [63], and [73, appendix].

Let $\mathrm{Or}(G)$ denote the homotopy category whose objects are the orbit spaces G/H for subgroups $H \subseteq G$, and whose morphisms $\mathrm{Hom}_{\mathrm{Or}(G)}(G/H, G/K)$ are G-homotopy classes of G-maps between these orbits. A *coefficient system* for Bredon cohomology is a functor $F : \mathrm{Or}(G)^{\mathrm{op}} \to Ab$. For any G-CW complex M, define

$$C_*^G(M) : \mathrm{Or}(G) \to Ab_*$$

by setting

$$C_*^G(M)(G/H) = C_*(M^H/\mathbf{W}H_0). \tag{3.1}$$

Here $C_*(-)$ denotes the cellular chain complex functor, and $\mathbf{W}H_0$ is the identity component of $\mathbf{N}H/H$. We now define

$$C_G^*(M; F) = \mathrm{Hom}_{\mathrm{Or}(G)}(C_*^G(M), F) \tag{3.2}$$

and $H_G^*(M; F) = H(C_G^*(M; F))$. One can see that for each $n \geq 0$, the group $C_G^n(M; F)$ is the direct product, over all orbits $G/H \times D^n$ of n-cells in M, of the groups $F(G/H)$. Moreover, $C_G^*(M; F)$ is determined on $\mathrm{Or}(G, M)$, the full subcategory consisting of the orbit types appearing in M. From the definitions, there will be a spectral sequence (see [63])

$$E_2 = \mathrm{Ext}^*_{\mathrm{Or}(G)}(\underline{H}_*(M), F) \Rightarrow H_G^*(M; F), \tag{3.3}$$

where $\underline{H}_*(M)(G/H) = H_*(M^H/\mathbf{W}H_0; \mathbb{Z})$.

In our applications, the isotropy groups will always be finite. Our basic example will involve the complex representation ring functor $R(-)$ on $\mathrm{Or}(G, M)$; i.e., $G/H \mapsto R(H)$. In this case, the fact that $R(H)$ is a ring for each H implies that Bredon cohomology will have a natural ring structure (constructed using the diagonal).

[1] For background on equivariant K-theory, the reader may consult [142], [101].

We will also use the rationalized functor $R_{\mathbb{Q}} = R(-) \otimes \mathbb{Q}$. For G finite, it is shown in [144] that $R_{\mathbb{Q}}$ is an injective functor; similarly, when Γ is a discrete group it is shown in [101] that $R_{\mathbb{Q}}$ is injective for proper actions with finite isotropy. This result will also hold for G-CW complexes with finite isotropy, where G is a compact Lie group. This follows by adapting the methods in [144] and is described in [63]. The key technical ingredient is the surjectivity of the homomorphism $R_{\mathbb{Q}}(H) \to \lim_{K \in F_p(H)} R_{\mathbb{Q}}(K)$, where H is any finite subgroup of G and $F_p(H)$ is the family of all proper subgroups in H. Thus, we have the following basic isomorphism: $H^*_G(M; R_{\mathbb{Q}}) \cong \mathrm{Hom}_{\mathrm{Or}(G)}(\underline{H}_*(M); R_{\mathbb{Q}})$.

Suppose that $\mathcal{X} = M/G$ is a quotient orbifold. Using equivariant K-theory, we will show that the Bredon cohomology $H^*_G(M; R_{\mathbb{Q}})$ is independent of the presentation M/G, and canonically associated with the orbifold $\mathcal{X}$ itself. A direct proof with more general coefficients would be of some interest. In the case of an effective orbifold, we can canonically associate to it the Bredon cohomology of its frame bundle; motivated by this, we introduce the following definition.

Definition 3.3 Let $\mathcal{X}$ be a effective orbifold. The *orbifold Bredon cohomology of $\mathcal{X}$ with $R_{\mathbb{Q}}$-coefficients* is $H^*_{\mathrm{orb}}(\mathcal{X}; R_{\mathbb{Q}}) = H^*_{O(n)}(\mathrm{Fr}(\mathcal{X}); R_{\mathbb{Q}})$.

3.3 Orbifold bundles and equivariant K-theory

In Chapter 2, we introduced the notion of orbifold vector bundles using the language of groupoids. That is, we saw that orbibundles on an orbifold $\mathcal{X}$ could be described as $\mathcal{G}$-vector bundles, where $\mathcal{G}$ is an orbifold groupoid presentation of $\mathcal{X}$. It is apparent that they behave naturally under vector space constructions such as sums, tensor products, exterior products, and so forth.

Definition 3.4 Given a compact orbifold groupoid $\mathcal{G}$, let $K_{\mathrm{orb}}(\mathcal{G})$ to be the Grothendieck ring of isomorphism classes of $\mathcal{G}$-vector bundles on $\mathcal{G}$. When $\mathcal{X}$ is an orbifold, we define $K_{\mathrm{orb}}(\mathcal{X})$ to be $K_{\mathrm{orb}}(\mathcal{G})$, where $\mathcal{G}$ is any groupoid presentation of $\mathcal{X}$.

Recall that under an orbifold morphism $F : \mathcal{H} \to \mathcal{G}$, one can verify that orbifold bundles over $\mathcal{G}$ pull back to orbifold bundles over $\mathcal{H}$. We have the following proposition.

Proposition 3.5 *Each orbifold morphism $F : \mathcal{H} \to \mathcal{G}$ induces a ring homomorphism $F^* : K_{\mathrm{orb}}(\mathcal{G}) \to K_{\mathrm{orb}}(\mathcal{H})$.*

In particular, for Morita equivalent groupoids $\mathcal{G}$ and $\mathcal{H}$ we see that $K_{\mathrm{orb}}(\mathcal{G}) \cong K_{\mathrm{orb}}(\mathcal{H})$. Thus, $K_{\mathrm{orb}}(\mathcal{X})$ is well defined.

Of course, an important example of an orbifold morphism is the projection map $p : M \to M/G$, where G is a compact Lie group acting almost freely on the manifold M. In this case, if E is an orbifold vector bundle over M/G, then p^*E is a smooth vector bundle over M. It is obvious that p^*E is G-equivariant. Conversely, if F is a G-equivariant bundle over M, $F/G \to M/G$ is an orbifold vector bundle over $\mathcal{X} = M/G$. Therefore, we have a canonical identification between $K_{\mathrm{orb}}(\mathcal{X})$ and $K_G(M) = K_{\mathrm{orb}}(G \ltimes M)$.

Proposition 3.6 *Let $\mathcal{X} = M/G$ be a quotient orbifold. Then the projection map $p : M \to M/G$ induces an isomorphism $p^* : K_{\mathrm{orb}}(\mathcal{X}) \to K_G(M)$.*

Corollary 3.7 *If $\mathcal{X}$ is a effective orbifold, we can identify its orbifold K-theory with the equivariant K-theory of its frame bundle.*[2]

It is possible to extend this definition of orbifold K-theory in the usual way; indeed if $\mathcal{X}$ is an orbifold, then $\mathcal{X} \times \mathbb{S}^n$ is also an orbifold and, moreover, the inclusion $i : \mathcal{X} \to \mathcal{X} \times \mathbb{S}^n$ is an orbifold morphism. Let $i_n^* : K_{\mathrm{orb}}(\mathcal{X} \times \mathbb{S}^n) \to K_{\mathrm{orb}}(\mathcal{X})$; then we can define $K_{\mathrm{orb}}^{-n}(\mathcal{X}) = \ker(i_n^*)$. However, the canonical identification outlined above shows that for a quotient orbifold this extension must agree with the usual extension of equivariant complex K-theory to a $\mathbb{Z}/2\mathbb{Z}$-graded theory (i.e., there will be Bott periodicity). Our approach here will be to study orbifold K-theory using equivariant K-theory, as it will enable us to make some meaningful computations. Note that if an orbifold $\mathcal{X}$ is presented in two different ways as a quotient, say $M/G \cong \mathcal{X} \cong M'/G'$, then we have shown that $K_{orb}^*(\mathcal{X}) \cong K_G^*(M) \cong K_{G'}^*(M')$. Another point to make is that the homomorphism $G \to G_{\mathrm{eff}}$ will induce a ring map $K_{orb}^*(\mathcal{X}_{\mathrm{eff}}) \to K_{orb}^*(\mathcal{X})$.

We also introduce the (K-theoretic) orbifold Euler characteristic.[3]

Definition 3.8 Let $\mathcal{X}$ be an orbifold. The *orbifold Euler characteristic* of $\mathcal{X}$ is

$$\chi_{\mathrm{orb}}(\mathcal{X}) = \dim_{\mathbb{Q}} K_{\mathrm{orb}}^0(\mathcal{X}) \otimes \mathbb{Q} - \dim_{\mathbb{Q}} K_{\mathrm{orb}}^1(\mathcal{X}) \otimes \mathbb{Q}$$

It remains to show that these invariants are tractable, or even well defined.

Proposition 3.9 *If $\mathcal{X} = M/G$ is a compact quotient orbifold for a compact Lie group G, then $K_{\mathrm{orb}}^*(\mathcal{X})$ is a finitely generated abelian group, and the orbifold Euler characteristic is well defined.*

[2] This has also been proposed by Morava [115], and also appears implicitly in [147].

[3] This definition extends the string-theoretic orbifold Euler characteristic which has been defined for global quotients.

Proof We know that M is a finite, almost free G-CW complex. It follows from [142] that there is a spectral sequence converging to $K_{\text{orb}}(\mathcal{X}) = K_G(M)$, with

$$E_1^{p,q} = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \bigoplus_{\overline{\sigma} \in X^{(p)}} R(G_\sigma) & \text{otherwise.} \end{cases}$$

Here, $X^{(p)}$ denotes the collection of p-cells in the underlying space X of $\mathcal{X}$, and $R(G_\sigma)$ denotes the complex representation ring of the stabilizer of σ in M. In fact, the E_2 term is simply the homology of a chain complex assembled from these terms. By our hypotheses, each G_σ is finite, and there are finitely many such cells; hence each term is finitely generated as an abelian group, and there are only finitely many of them. We conclude that E_1 satisfies the required finiteness conditions, and so must its subquotient E_∞, whence the same holds for $K^*_{\text{orb}}(\mathcal{X}) = K^*_G(M)$. □

Corollary 3.10 *With notation as before, we have*

$$\chi_{\text{orb}}(\mathcal{X}) = \sum_{\overline{\sigma} \in X} (-1)^{\dim \overline{\sigma}} \operatorname{rank} R(G_\sigma).$$

The spectral sequence used above is in fact simply the equivariant analog of the Atiyah–Hirzebruch spectral sequence. We have described the E_1-term as a chain complex assembled from the complex representation rings of the isotropy subgroups. Actually, the E_2-term coincides with the equivariant *Bredon cohomology* $H^*_G(M; R(-))$ of M described in the previous section, with coefficients in the representation ring functor. In fact this spectral sequence collapses rationally at the E_2-term (see [101, p. 28]). Consequently, $H^*_{\text{orb}}(\mathcal{X}; \mathbb{R})$, $K^*_{\text{orb}}(\mathcal{X}) \otimes \mathbb{R}$, and $H^*_G(M; R(-) \otimes \mathbb{R})$ are all additively isomorphic. What is more, the last two invariants have the same ring structure (provided that we take the $\mathbb{Z}/2\mathbb{Z}$-graded version of Bredon cohomology).

Computations for equivariant K-theory can be quite complicated. Our approach will be to study the case of global quotients arising from actions of finite and discrete groups. The key computational tool will be an equivariant Chern character, which we will define for almost free actions of compact Lie groups. This will be used to establish the additive rational equivalences outlined above. However, we note that $K_{\text{orb}}(\mathcal{X})$ can contain important *torsion* classes, and so its rationalization is a rather crude approximation.

Let us review the special case of a global quotient, where the K-theoretic invariant above is more familiar.

Example 3.11 Let G denote a finite group acting on a manifold Y and let $\mathcal{X} = Y/G$. In this case we know that there is an isomorphism $K_{\text{orb}}(\mathcal{X}) \cong K_G(Y)$.

Tensored with the rationals, the equivariant K-theory decomposes as a direct sum, and we obtain the well-known formula

$$K^*_{orb}(\mathcal{X}) \otimes \mathbb{Q} \cong \bigoplus_{\substack{(g)\\ g\in G}} K^*\left(Y^{\langle g\rangle}/\,\mathbf{C}_G(g)\right) \otimes \mathbb{Q}, \tag{3.4}$$

where (g) is the conjugacy class of $g \in G$ and $\mathbf{C}_G(g)$ denotes the centralizer of g in G. Note that this decomposition appears in [11], but can be traced back (independently) to [144], [151], and [89].

One of the key elements in the theory of orbifolds is the inertia orbifold $\wedge\mathcal{X}$ introduced in the previous chapter. In the case of a global quotient $\mathcal{X} = Y/G$, it can be shown (see [38]) that we have a homeomorphism

$$|\wedge \mathcal{X}| \cong \bigsqcup_{\substack{(g)\\ g\in G}} Y^{\langle g\rangle}/\,\mathbf{C}_G(g), \tag{3.5}$$

so we see that $K^*_{\text{orb}}(\mathcal{X}) \cong_{\mathbb{Q}} K^*(|\wedge \mathcal{X}|)$, where $|\wedge \mathcal{X}|$ is the underlying space of the inertia orbifold $\wedge\mathcal{X}$. The conjugacy classes are used to index the so-called *twisted sectors* arising in this decomposition. We will use this as a model for our more general result in the following section.

3.4 A decomposition for orbifold K-theory

We will now prove a decomposition for orbifold K-theory using the methods developed by Lück and Oliver in [101]. The basic technical result we will use is the construction of an equivariant Chern character. Cohomology will be assumed $\mathbb{Z}/2\mathbb{Z}$-graded in the usual way. We have the following theorem of Adem and Ruan [5].

Theorem 3.12 *Let $\mathcal{X} = M/G$ be a compact quotient orbifold, where G is a compact Lie group. Then there is an equivariant Chern character which defines a rational isomorphism of rings*

$$K^*_{\text{orb}}(\mathcal{X}) \cong_{\mathbb{Q}} \prod_{\substack{(C)\\ C\subseteq G \text{ cyclic}}} [H^*(M^C/\,\mathbf{C}_G(C)) \otimes \mathbb{Q}(\zeta_{|C|})]^{\mathbf{W}_G(C)},$$

where (C) ranges over conjugacy classes of cyclic subgroups, $\zeta_{|C|}$ is a primitive root of unity, and $\mathbf{W}_G(C) = \mathbf{N}_G(C)/\,\mathbf{C}_G(C)$, a necessarily finite group.

Proof As has been remarked, we can assume that M is a finite, almost free G-CW complex. Now, as in [101] and [11], the main idea of the proof is to

construct a natural Chern character for any G-space as above, and then prove that it induces an isomorphism for orbits of the form G/H, where $H \subset G$ is finite. Using induction on the number of orbit types and a Mayer–Vietoris sequence will then complete the proof.

To begin, we recall the existence (see [101], Prop. 3.4) of a ring homomorphism

$$\psi : K^*_{\mathbf{N}_G(C)}(M^C) \to K^*_{\mathbf{C}_G(C)}(M^C) \otimes R(C);$$

in this much more elementary setting, it can be defined by its value on vector bundles. Namely,

$$\psi([E]) = \sum_{V \in \mathrm{Irr}(C)} [\mathrm{Hom}_C(V, E)] \otimes [V]$$

for any $\mathbf{N}_G(C)$-vector bundle $E \to M^C$. We make use of the natural maps

$$\begin{aligned} K^*_{\mathbf{C}_G(C)}(M^C) \otimes R(C) &\to K^*_{\mathbf{C}_G(C)}(EG \times M^C) \otimes R(C) \\ &\to K^*(EG \times_{\mathbf{C}_G(C)} M^C) \otimes R(C), \end{aligned}$$

as well as the Chern map

$$\begin{aligned} K^*(EG \times_{\mathbf{C}_G(C)} M^C) \otimes R(C) &\to H^*(EG \times_{\mathbf{C}_G(C)} M^C; \mathbb{Q}) \otimes R(C) \\ &\cong H^*(M^C / \mathbf{C}_G(C); \mathbb{Q}) \otimes R(C). \end{aligned}$$

Note that the isomorphism above stems from the crucial fact that all the fibers of the projection map $EG \times_{\mathbf{C}_G(C)} M^C \to M^C / \mathbf{C}_G(C)$ are rationally acyclic, as they are classifying spaces of finite groups. Finally, we will make use of the ring map $R(C) \otimes \mathbb{Q} \to \mathbb{Q}(\zeta_{|C|})$; its kernel is the ideal of elements whose characters vanish on all generators of C. Putting all of this together and using the restriction map, we obtain a natural ring homomorphism

$$K^*_G(M) \otimes \mathbb{Q} \to H^*(M^C / \mathbf{C}_G(C); \mathbb{Q}(\zeta_{|C|}))^{\mathbf{N}_G(C)/\mathbf{C}_G(C)}. \tag{3.6}$$

Here we have taken invariants on the right hand side, as the image naturally lands there. Verification of the isomorphism on G/H is an elementary consequence of the isomorphism $K^*_G(G/H) \cong R(H)$, and the details are left to the reader. $\square$

Corollary 3.13 *Let $\mathcal{X} = M/G$ be a compact quotient orbifold. Then there is an additive decomposition*

$$K^*_{orb}(\mathcal{X}) \otimes \mathbb{Q} = K^*_G(M) \otimes \mathbb{Q} \cong \bigoplus_{\substack{(g) \\ g \in G}} K^*(M^{\langle g \rangle} / \mathbf{C}_G(g)) \otimes \mathbb{Q}.$$

Note that the (finite) indexing set consists of the G-conjugacy classes of elements in the isotropy subgroups – all of finite order. Thus, just as in the case of a global quotient, we see that the orbifold K-theory of $\mathcal{X}$ is rationally isomorphic to the ordinary K-theory of the underlying space of the twisted sectors $\wedge\mathcal{X}$.

Theorem 3.14 *Let $\mathcal{X} = M/G$ denote a compact quotient orbifold. Then there is a homeomorphism*

$$\bigsqcup_{\substack{(g)\\ g\in G}} M^{\langle g\rangle}/\mathbf{C}_G(g) \cong |\wedge\mathcal{X}|,$$

and, in particular, $K^*_{\mathrm{orb}}(\mathcal{X}) \cong_{\mathbb{Q}} K^*(|\wedge\mathcal{X}|)$.

Proof We begin by considering the situation locally. Suppose that we have a chart in M of the form $V \times_H G$, mapping onto V/H in X, where by assumption $H \subset G$ is a finite group. Then

$$\begin{aligned}(V \times_H G)^{\langle a\rangle} &= \{H(x,u) \mid H(x,ua) = H(x,u)\}\\ &= \{H(x,u) \mid uau^{-1} = h \in H,\ x \in V^{\langle h\rangle}\}.\end{aligned}$$

Let us now define an H action on $\bigsqcup_{t\in H}(V^{\langle t\rangle}, t)$ by $k(x,t) = (kx, ktk^{-1})$. We define a map

$$\phi : (V \times_H G)^{\langle g\rangle} \to \bigsqcup_{t\in H}(V^{\langle t\rangle}, t)/H$$

by $\phi(H(x,u)) = [x, ugu^{-1}]$. We check that this is well defined: indeed, if $H(x,u) = H(y,v)$ then there is a $k \in H$ with $(y,v) = k(x,u)$, so $y = kx$, $v = ku$. This means that $vgv^{-1} = kugu^{-1}k^{-1}$, and so $[y, vgv^{-1}] = [kx, kugu^{-1}k^{-1}] = [x, ugu^{-1}]$ as $k \in H$. Now suppose that $z \in \mathbf{C}_G(g)$; then $\phi(H(x,u)z) = \phi(H(x,uz)) = [x, uzgz^{-1}u^{-1}] = [x, ugu^{-1}] = \phi(H(x,u))$; hence we have a well-defined map on the orbit space

$$\overline{\phi} : (V \times_H G)^{\langle g\rangle}/\mathbf{C}_G(g) \to \bigsqcup_{t\in H}(V^{\langle t\rangle}, t)/H.$$

This map turns out to be injective. Indeed, if $(x, ugu^{-1}) = k(y, vgv^{-1})$ for some $k \in H$, then $x = ky$ and $g = u^{-1}kvgv^{-1}k^{-1}u$, hence $u^{-1}kv \in \mathbf{C}_G(g)$ and $H(x,u)(u^{-1}kv) = H(x,kv) = H(ky,kv) = H(y,v)$. The image of $\overline{\phi}$ consists of the H-equivalence classes of pairs (x,t), where $x \in V^{\langle t\rangle}$ and t is conjugate to g in G.

Putting this together and noting that $(V \times_H G)^{\langle g \rangle} = \emptyset$ unless g is conjugate to an element in H, we observe that we obtain a homeomorphism

$$\bigsqcup_{\substack{(g)\\ g\in G}} (V \times_H G)^{\langle G \rangle} / \mathbf{C}_G(g) \cong \bigsqcup_{t \in H} (V^{\langle t \rangle}, t)/H \cong \bigsqcup_{\substack{(t)\\ t\in H}} V^{\langle t \rangle} / \mathbf{C}_H(t).$$

To complete the proof of the theorem it suffices to observe that by the compatibility of charts, the local homeomorphisms on fixed-point sets can be assembled to yield the desired global homeomorphism on M. □

Remark 3.15 Alternatively, the theorem is an easy consequence of the fact that the translation groupoid $\wedge G \ltimes M = G \ltimes \sqcup_{g \in G} M^{\langle g \rangle}$ is Morita equivalent to the groupoid $\sqcup_{(g)} \mathbf{C}_G(g) \ltimes M^{\langle g \rangle}$. In fact, the inclusion of the latter into the former is an equivalence. Thus, their quotient spaces must be homeomorphic.

Remark 3.16 We can compose the result above with the ordinary Chern character on $| \wedge \mathcal{X}|$ to obtain a stringy Chern character

$$\mathrm{ch} : K^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C} \to H^*(| \wedge \mathcal{X}|; \mathbb{C}). \tag{3.7}$$

In fact, this is an isomorphism of graded abelian groups (where we take $\mathbb{Z}/2\mathbb{Z}$-graded cohomology on the right hand side). Note that $H^*(| \wedge \mathcal{X}|; \mathbb{C})$ arises naturally as the target of the stringy Chern character. At this point, we only consider the additive structure of $H^*(| \wedge \mathcal{X}|; \mathbb{C})$; in Chapter 4, we will endow it with a different grading and a stringy cup product. The resulting ring is often referred to as the *Chen–Ruan cohomology ring*.

Corollary 3.17 *We have* $\chi_{\mathrm{orb}}(\mathcal{X}) = \chi(| \wedge \mathcal{X}|)$.

Example 3.18 We will now consider the case of the weighted projective space $\mathrm{W}\mathbb{P}(p, q)$, where p and q are assumed to be distinct prime numbers. Recall that $\mathrm{W}\mathbb{P}(p, q) = \mathbb{S}^3/\mathbb{S}^1$, where $\mathbb{S}^1$ acts on the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ via $\lambda(v, w) = (\lambda^p v, \lambda^q w)$. There are two singular points, $x = [1, 0]$ and $y = [0, 1]$, with corresponding isotropy subgroups $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$. The fixed-point sets are disjoint circles in $\mathbb{S}^3$, hence the formula for the orbifold K-theory yields

$$K^*_{\mathrm{orb}}(\mathrm{W}\mathbb{P}(p, q)) \cong_{\mathbb{Q}} \mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_q) \times \Lambda(b_2), \tag{3.8}$$

where ζ_p and ζ_q are the corresponding primitive roots of unity (compare with Corollary 2.7.6 in [9]). More explicitly, we have an isomorphism

$$\begin{aligned} K^*_{\mathrm{orb}}(\mathrm{W}\mathbb{P}(p, q)) \otimes \mathbb{Q} \cong \mathbb{Q}[x]/(x^{p-1} + x^{p-2} + \cdots + x + 1) \\ \times (x^{q-1} + x^{q-2} + \cdots + x + 1)(x^2), \end{aligned}$$

from which we see that the orbifold Euler characteristic is $\chi_{\text{orb}}(\mathrm{W}\mathbb{P}(p,q)) = p+q$.

Remark 3.19 The decomposition described above is based on entirely analogous results for proper actions of discrete groups (see [101]). In particular, this includes the case of *arithmetic orbifolds*, also discussed in [3] and [76]. Let $G(\mathbb{R})$ denote a semi-simple $\mathbb{Q}$-group, and K a maximal compact subgroup. Let $\Gamma \subset G(\mathbb{Q})$ denote an arithmetic subgroup. Then Γ acts on $X = G(\mathbb{R})/K$, a space diffeomorphic to a Euclidean space. Moreover, if H is any finite subgroup of Γ, then X^H is a totally geodesic submanifold, hence also diffeomorphic to a Euclidean space. We can make use of the Borel–Serre completion $\overline{X}$ (see [25]). This is a contractible space with a proper Γ-action such that the $\overline{X}^H$ are also contractible (we are indebted to Borel and Prasad for outlining a proof of this in [24]) but having a compact orbit space $\Gamma\backslash\overline{X}$. In this case, we obtain the multiplicative formula

$$K^*_\Gamma(X)\otimes\mathbb{Q} \cong K^*_\Gamma(\overline{X})\otimes\mathbb{Q} \cong \prod_{\substack{(C)\\ C\subset\Gamma \text{ cyclic}}} H^*(B\,\mathbf{C}_\Gamma(C);\mathbb{Q}(\zeta_{|C|}))^{\mathbf{N}_\Gamma(C)}.$$

This allows us to express the orbifold Euler characteristic of $\Gamma\backslash X$ in terms of group cohomology:

$$\chi_{\text{orb}}(\Gamma\backslash X) = \sum_{\substack{(\gamma)\\ \gamma\in\Gamma \text{ of finite order}}} \chi(B\,\mathbf{C}_\Gamma(\gamma)). \tag{3.9}$$

Example 3.20 Another example of some interest is that of compact, two-dimensional, hyperbolic orbifolds. They are described as quotients of the form $\Gamma\backslash PSL_2(\mathbb{R})/SO(2)$, where Γ is a Fuchsian subgroup. The groups Γ can be expressed as extensions of the form

$$1 \to \Gamma' \to \Gamma \to G \to 1,$$

where Γ' is the fundamental group of a closed orientable Riemann surface, and G is a finite group (i.e., they are *virtual surface groups*). Geometrically, we have an action of G on a surface Σ with fundamental group Γ'; this action has isolated singular points with cyclic isotropy. The group Γ is $\pi_1(EG\times_G\Sigma)$, which coincides with the orbifold fundamental group. Assume that G acts on Σ with n orbits of cells, having respective isotropy groups $\mathbb{Z}/v_1\mathbb{Z},\ldots,\mathbb{Z}/v_n\mathbb{Z}$, and with quotient a surface W of genus equal to g. The formula then yields (compare with the description in [105, p. 563])

$$K^*_{\text{orb}}(W)\otimes\mathbb{Q} \cong \widetilde{R}(\mathbb{Z}/v_1\mathbb{Z})\otimes\mathbb{Q}\times\cdots\times\widetilde{R}(\mathbb{Z}/v_n\mathbb{Z})\otimes\mathbb{Q}\times K^*(W)\otimes\mathbb{Q}.$$

In this expression, $\widetilde{R}$ denotes the reduced representation ring, which arises because the trivial cyclic subgroup only appears once. From this we see that

$$\dim_{\mathbb{Q}} K^0_{\mathrm{orb}}(W) \otimes \mathbb{Q} = \sum_{i=1}^{n} (v_i - 1) + 2, \qquad \dim_{\mathbb{Q}} K^1_{\mathrm{orb}}(W) \otimes \mathbb{Q} = 2g,$$

and so $\chi_{\mathrm{orb}}(W) = \sum_{i=1}^{n} (v_i - 1) + \chi(W)$.

Remark 3.21 This decomposition formula is analogous to the decomposition of equivariant algebraic K-theory which appears in work of Vezzosi and Vistoli [157, p. 5] and Toen (see [150, p. 29] and [149, p. 49]) in the context of algebraic Deligne–Mumford stacks. Under suitable conditions, Toen obtains rational isomorphisms between the G-theory of a Deligne–Mumford stack and that of its inertia stack. Vezzosi and Vistoli, on the other hand, express the equivariant algebraic K-theory $K_*(X, G)$ of an affine group scheme of finite type over k acting on a Noetherian regular separated algebraic space X in terms of fixed-point data, again under suitable hypotheses (and after inverting some primes). A detailed comparison of these with the topological splitting above would seem worthwhile.

Remark 3.22 It should also be observed that the decomposition above could equally well have been stated in terms of the computation of Bredon cohomology mentioned previously, i.e., $H^*_G(M, R_{\mathbb{Q}}) \cong \mathrm{Hom}_{\mathrm{Or}(G)}(\underline{H}_*(M); R_{\mathbb{Q}})$ and the collapse at E_2 of the rationalized Atiyah–Hirzebruch spectral sequence: $K^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{Q} \cong H^*_G(M; R_{\mathbb{Q}})$. It had been previously shown that a Chern character with expected naturality properties inducing such an isomorphism cannot exist; in particular [63] contains an example where such an isomorphism is impossible. However, the example is for a circle action with stationary points, our result[4] shows that *almost free* actions of compact Lie groups do indeed give rise to appropriate equivariant Chern characters. A different equivariant Chern character for abelian Lie group actions was defined in [18], using a $\mathbb{Z}/2\mathbb{Z}$-indexed de Rham cohomology (called *delocalized equivariant cohomology*). Presumably it must agree with our decomposition in the case of almost free actions. Nistor [121] and Block and Getzler [22] have pointed out an alternative approach using cyclic cohomology.

Remark 3.23 If $\mathcal{X} = M/G$ is a quotient orbifold, then the K-theory of $EG \times_G M$ and the orbifold K-theory are related by the Atiyah–Segal Completion Theorem in [10]. Considering the equivariant K-theory $K^*_G(M)$ as a module

[4] Moerdijk has informed us that in unpublished work (1996), he and Svensson obtained essentially the same Chern character construction as that appearing in this chapter.

over $R(G)$, it states that $K^*(EG \times_G M) \cong K^*_G(M)\widehat{}$, where the completion is taken at the augmentation ideal $I \subset R(G)$.

3.5 Projective representations, twisted group algebras, and extensions

We will now extend many of the constructions and concepts used previously to an appropriately *twisted* setting. This twisting occurs naturally in the framework of mathematical physics. In this section, we will always assume that we are dealing with finite groups, unless stated otherwise. Most of the background results which we list appear in [79, Chapt. III].

Definition 3.24 Let V denote a finite-dimensional complex vector space. A mapping $\rho : G \to GL(V)$ is called a *projective representation* of G if there exists a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$ and $\rho(1) = \mathrm{Id}_V$.

Note that α defines a $\mathbb{C}^*$-valued cocycle on G, i.e., $\alpha \in Z^2(G; \mathbb{C}^*)$. Also, there is a one-to-one correspondence between projective representations of G as above and homomorphisms from G to $PGL(V)$. We will be interested in the notion of *linear equivalence* of projective representations.

Definition 3.25 Two projective representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are said to be *linearly equivalent* if there exists a vector space isomorphism $f : V_1 \to V_2$ such that $\rho_2(g) = f\rho_1(g)f^{-1}$ for all $g \in G$.

If α is the cocycle attached to ρ, we say that ρ is an α-representation on the space V. We list a few basic results regarding these structures.

Lemma 3.26 *Let ρ_i (for $i = 1, 2$) be an α_i-representation on the space V_i. If ρ_1 is linearly equivalent to ρ_2, then α_1 is equal to α_2.*

It is easy to see that given a fixed cocycle α, we can take the direct sum of any two α-representations.

Definition 3.27 We define $M_\alpha(G)$ to be the monoid of linear isomorphism classes of α-representations of G. Its associated Grothendieck group will be denoted $R_\alpha(G)$.

In order to use these constructions, we need to introduce the notion of a *twisted group algebra*. If $\alpha : G \times G \to \mathbb{C}^*$ is a cocycle, we denote by $\mathbb{C}^\alpha G$ the vector space over $\mathbb{C}$ with basis $\{\overline{g} \mid g \in G\}$ and product $\overline{x} \cdot \overline{y} = \alpha(x, y)\overline{xy}$

extended distributively. One can check that $\mathbb{C}^\alpha G$ is a $\mathbb{C}$-algebra with $\overline{1}$ as the identity element. This algebra is called the *α-twisted group algebra* of G over $\mathbb{C}$. Note that if $\alpha(x, y) = 1$ for all $x, y \in G$, then $\mathbb{C}^\alpha G = \mathbb{C}G$ is the usual group algebra.

Definition 3.28 If α and β are cocycles, then $\mathbb{C}^\alpha G$ and $\mathbb{C}^\beta G$ are *equivalent* if there exists a $\mathbb{C}$-algebra isomorphism $\psi : \mathbb{C}^\alpha G \to \mathbb{C}^\beta G$ and a mapping $t : G \to \mathbb{C}^*$ such that $\psi(\overline{g}) = t(g)\tilde{g}$ for all $g \in G$, where $\{\overline{g}\}$ and $\{\tilde{g}\}$ are bases for the two twisted algebras.

We have a basic result which classifies these twisted group algebras.

Theorem 3.29 *We have an isomorphism $\mathbb{C}^\alpha G \cong \mathbb{C}^\beta G$ between twisted group algebras if and only if α is cohomologous to β; hence if α is a coboundary, $\mathbb{C}^\alpha G \cong \mathbb{C}G$. Indeed, $\alpha \mapsto \mathbb{C}^\alpha G$ induces a bijective correspondence between $H^2(G; \mathbb{C}^*)$ and the set of equivalence classes of twisted group algebras of G over $\mathbb{C}$.*

Next we recall how these twisted algebras play a role in determining $R_\alpha(G)$.

Theorem 3.30 *There is a bijective correspondence between α-representations of G and $\mathbb{C}^\alpha G$-modules. This correspondence preserves sums and bijectively maps linearly equivalent (respectively irreducible, completely reducible) representations into isomorphic (respectively irreducible, completely reducible) modules.*

Definition 3.31 Let $\alpha \in Z^2(G; \mathbb{C}^*)$. An element $g \in G$ is said to be *α-regular* if $\alpha(g, x) = \alpha(x, g)$ for all $x \in \mathbf{C}_G(g)$.

Note that the identity element is α-regular for all α. Also, one can see that g is α-regular if and only if $\overline{g} \cdot \overline{x} = \overline{x} \cdot \overline{g}$ for all $x \in \mathbf{C}_G(g)$.

If an element $g \in G$ is α-regular, then any conjugate of g is also α-regular. Therefore, we can speak of α-regular conjugacy classes in G. For technical purposes, we also want to introduce the notion of a *standard cocycle*. A cocycle α is standard if (1) $\alpha(x, x^{-1}) = 1$ for all $x \in G$, and (2) $\alpha(x, g)\alpha(xg, x^{-1}) = 1$ for all α-regular $g \in G$ and all $x \in G$. In other words, α is standard if and only if for all $x \in G$ and for all α-regular elements $g \in G$, we have $\overline{x}^{-1} = \overline{x^{-1}}$ and $\overline{x}\,\overline{g}\,\overline{x}^{-1} = \overline{xgx^{-1}}$. It turns out that any cohomology class $c \in H^2(G; \mathbb{C}^*)$ can be represented by a standard cocycle, so from now on we will make this assumption.

The next result is basic.

Theorem 3.32 *If r_α is equal to the number of non-isomorphic irreducible $\mathbb{C}^\alpha G$-modules, then this number is equal to the number of distinct α-regular*

conjugacy classes of G. In particular, $R_\alpha(G)$ is a free abelian group of rank equal to r_α.

In what follows we will be using cohomology classes in $H^2(G;\mathbb{S}^1)$, where the G-action on the coefficients is assumed to be trivial. Note that $H^2(G;\mathbb{S}^1) \cong H^2(G;\mathbb{C}^*) \cong H^2(G;\mathbb{Q}/\mathbb{Z}) \cong H^3(G;\mathbb{Z})$. We will always use standard cocycles to represent any given cohomology class.

An element $\alpha \in H^2(G;\mathbb{S}^1)$ corresponds to an equivalence class of group extensions

$$1 \to \mathbb{S}^1 \to \widetilde{G}_\alpha \to G \to 1.$$

The group $\widetilde{G}_\alpha$ can be given the structure of a compact Lie group, where $\mathbb{S}^1 \to \widetilde{G}_\alpha$ is the inclusion of a closed subgroup. The elements in the extension group can be represented by pairs $\{(g,a) \mid g \in G, a \in \mathbb{S}^1\}$ with the product $(g_1,a_1)(g_2,a_2) = (g_1g_2, \alpha(g_1,g_2)a_1a_2)$.

Consider the case when $z \in \mathbf{C}_G(g)$; then we can compute the following commutator of lifts:

$$\begin{aligned}(z,1)(g,1)[(g,1)(z,1)]^{-1} &= (zg,\alpha(z,g))(z^{-1}g^{-1},\alpha(g,z)^{-1})\\ &= (1,\alpha(zg,(zg)^{-1})\alpha(z,g)\alpha(g,z))\\ &= (1,\alpha(z,g)\alpha(g,z)^{-1}).\end{aligned}$$

This computation is independent of the choice of lifts. It can be seen that this defines a character γ_g^α for the centralizer $\mathbf{C}_G(g)$ via the correspondence $z \mapsto \alpha(z,g)\alpha(g,z)^{-1}$. This character is trivial if and only if g is α-regular.

There is a one-to-one correspondence between isomorphism classes of representations of $\widetilde{G}_\alpha$ which restrict to scalar multiplication on the central $\mathbb{S}^1$ and isomorphism classes of α-representations of G. If $\psi : \widetilde{G}_\alpha \to GL(V)$ is such a representation, then we define an associated α-representation via $\rho(g) = \psi(g,1)$. Note that

$$\begin{aligned}\rho(gh) &= \psi(gh,1) = \alpha(g,h)^{-1}\psi(gh,\alpha(g,h)) = \alpha(g,h)^{-1}\psi((g,1)(h,1))\\ &= \alpha(g,h)^{-1}\rho(g)\rho(h).\end{aligned}$$

Conversely, given $\rho : G \to GL(V)$, we simply define $\psi(g,a) = a\rho(g)$; note that

$$\begin{aligned}\psi((g,a)(h,b)) &= \psi(gh,\alpha(g,h)ab) = ab\rho(g)\rho(h) = a\rho(g)b\rho(h)\\ &= \psi(g,a)\psi(h,b).\end{aligned}$$

Therefore, we can identify $R_\alpha(G)$ with the subgroup of $R(\widetilde{G}_\alpha)$ generated by representations that restrict to scalar multiplication on the central $\mathbb{S}^1$.

In the next section we will need an explicit understanding of the action of $\widetilde{\mathbf{C}_G(g)}_\alpha$ on $R_{\mathrm{Res}(\alpha)}(\langle g\rangle)$, where $\mathrm{Res}(\alpha)$ is the restriction of the cocycle to the subgroup $\langle g\rangle$ (this restriction is cohomologous to zero). It is easiest to describe using the formulation above. Given a representation ϕ for $\widetilde{\langle g\rangle}_\alpha$, an element $(z,a)\in\widetilde{\mathbf{C}_G(g)}_\alpha$, and $(x,b)\in\widetilde{\langle g\rangle}_\alpha$, we define $(z,a)\phi(x,b)=\phi((z,a)(x,b)(z,a)^{-1})$. Notice that this value is precisely $\gamma_x^\alpha(z)\phi(x,b)$; this is independent of the choice of lifting and defines an action of $\mathbf{C}_G(g)$. For $x,y\in\langle g\rangle$ we have $\gamma_x^\alpha(z)\gamma_y^\alpha(z)=\gamma_{xy}^\alpha(z)$. In particular, if $g^n=1$, we have $(\gamma_g^\alpha(z))^n=1$. The correspondence $x\mapsto\gamma_x^\alpha(z)$ defines a character $L^\alpha(z)$ for $\langle g\rangle$, whence the action is best described as sending an α-representation ρ to $L^\alpha(z)\rho$. Note that the evaluation $\phi\mapsto\mathrm{tr}(\phi(g,1))$ defines a $\mathbb{C}\,\mathbf{C}_G(g)$-homomorphism $u: R_{\mathrm{Res}(\alpha)}(\langle g\rangle)\otimes\mathbb{C}\to\gamma_g^\alpha$.

3.6 Twisted equivariant K-theory

We are now ready to define a twisted version of equivariant K-theory for global quotients.[5] We assume as before that G is a finite group. Now suppose we are given a class $\alpha\in Z^2(G;\mathbb{S}^1)$ and the compact Lie group extension which represents it, $1\to\mathbb{S}^1\to\widetilde{G}_\alpha\to G\to 1$; finally, let X be a finite G-CW complex.

Definition 3.33 An *α-twisted G-vector bundle* on X is a complex vector bundle $E\to X$ together with an action of $\widetilde{G}_\alpha$ on E such that $\mathbb{S}^1$ acts on the fibers through complex multiplication and the action covers the given G-action on X.

One may view such a bundle $E\to X$ as a $\widetilde{G}_\alpha$-vector bundle, where the action on the base is via the projection onto G and the given G-action. Note that if we divide out by the action of $\mathbb{S}^1$, we obtain a *projective* bundle over X. These twisted bundles can be added, forming a monoid.

Definition 3.34 The *α-twisted G-equivariant K-theory* of X, denoted by ${}^\alpha K_G(X)$, is defined as the Grothendieck group of isomorphism classes of α-twisted G-bundles over X.

As with α-representations, we can describe this twisted group as the subgroup of $K_{\widetilde{G}_\alpha}(X)$ generated by isomorphism classes of bundles that restrict to multiplication by scalars on the central $\mathbb{S}^1$. As the $\mathbb{S}^1$-action on X is trivial,

[5] By now there are many different versions of twisted K-theory; we refer the reader to [55] for a succinct survey, as well as connections to the Verlinde algebra.

we have a natural isomorphism $K_{\mathbb{S}^1}(X) \cong K(X) \otimes R(\mathbb{S}^1)$. Composing the restriction with the map $K(X) \otimes R(\mathbb{S}^1) \to R(\mathbb{S}^1)$, we obtain a homomorphism $K_{\widetilde{G}_\alpha}(X) \to R(\mathbb{S}^1)$; we can define ${}^\alpha K_G(X)$ as the inverse image of the subgroup generated by the representations defined by scalar multiplication.

Just as in non-twisted equivariant K-theory, this definition extends to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. In fact we can define ${}^\alpha K^0_G(X) = {}^\alpha K_G(X)$ and ${}^\alpha K^1_G(X) = \ker[{}^\alpha K_G(\mathbb{S}^1 \times X) \to {}^\alpha K_G(X)]$. We can also extend the description given above to express ${}^\alpha K^*_G(X)$ as a subgroup of $K^*_{\widetilde{G}_\alpha}(X)$.

We begin by considering the case $\alpha = 0$; this corresponds to the split extension $G \times \mathbb{S}^1$. Any ordinary G-vector bundle can be made into a $G \times \mathbb{S}^1$-bundle via scalar multiplication on the fibers; conversely, a $G \times \mathbb{S}^1$-bundle restricts to an ordinary G-bundle. Hence we readily see that ${}^\alpha K^*_G(X) = K^*_G(X)$.

Theorem 3.35 *Let α and β be cocycles. If $\alpha = \beta\gamma$ for a coboundary γ, there is an isomorphism*

$$\kappa_\gamma : {}^\alpha K^*_G(X) \to {}^\beta K^*_G(X).$$

As a consequence, $H^1(G; \mathbb{S}^1)$ acts as automorphisms of ${}^\alpha K^*_G(X)$.

Now we consider the case when X is a trivial G-space.

Lemma 3.36 *Let X denote a CW-complex with a trivial G-action; then there is a natural isomorphism $K(X) \otimes R_\alpha(G) \to {}^\alpha K_G(X)$.*

Proof This result is the analog of the untwisted version (see [142, p. 133]). The natural map $R(\widetilde{G}_\alpha) \to K_{\widetilde{G}_\alpha}(X)$ can be combined with the map $K(X) \to K_{\widetilde{G}_\alpha}(X)$ (which gives any vector bundle the trivial G-action) to yield a natural isomorphism $K(X) \otimes R(\widetilde{G}_\alpha) \to K_{\widetilde{G}_\alpha}(X)$ which covers the restriction to the $\mathbb{S}^1$-action; the result follows from looking at inverse images of the subgroup generated by the scalar representation. □

The inverse of the map above is given by

$$[E] \mapsto \bigoplus_{\{[M] \in \mathrm{Irr}(\widetilde{G}_\alpha)\}} [\mathrm{Hom}_{\widetilde{G}_\alpha}(M, E)] \otimes [M].$$

Note that only the M which restrict to scalar multiplication on $\mathbb{S}^1$ are relevant – these are precisely the irreducible α-representations.

Let X be a G-space and Y a G'-space, and let $h : G \to G'$ denote a group homomorphism. If $f : X \to Y$ is a continuous map equivariant with respect to this homomorphism, we obtain a map ${}^\alpha f^* : {}^\alpha K_{G'}(Y) \to {}^{h^*(\alpha)} K_G(X)$, where $h^* : H^2(G'; \mathbb{S}^1) \to H^2(G; \mathbb{S}^1)$ is the map induced by h in cohomology. Let $H \subseteq G$ be a subgroup; the inclusion defines a restriction map $H^2(G; \mathbb{S}^1) \to$

$H^2(H;\mathbb{S}^1)$. In fact, if $\widetilde{G}_\alpha$ is the group extension defined by $\alpha \in H^2(G;\mathbb{S}^1)$, then $\mathrm{Res}^G_H(\alpha)$ defines the "restricted" group extension over H, denoted $\widetilde{H}_{\mathrm{Res}(\alpha)}$; we have a restriction map ${}^\alpha K_G(X) \to {}^{\mathrm{Res}(\alpha)}K_H(X)$.

Now consider the case of an orbit G/H. We claim that ${}^\alpha K_G(G/H) = R_{\mathrm{Res}^G_H(\alpha)}(H)$. Indeed, we can identify $K_{\widetilde{G}_\alpha}(G/H) = K_{\widetilde{G}_\alpha}(\widetilde{G}_\alpha/\widetilde{H}_\alpha) \cong R(\widetilde{H}_\alpha)$, and by restricting to the representations that induce scalar multiplication on $\mathbb{S}^1$, we obtain the result.

We are now ready to state a basic decomposition theorem for our twisted version of equivariant K-theory.

Theorem 3.37 *Let G denote a finite group and X a finite G-CW complex. For any $\alpha \in Z^2(G;\mathbb{S}^1)$, we have a decomposition*

$$ {}^\alpha K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{\substack{(g)\\ g\in G}} (K^*(X^{\langle g\rangle}) \otimes \gamma^\alpha_g)^{\mathbf{C}_G(g)}. $$

Proof Fix the class $\alpha \in Z^2(G,\mathbb{S}^1)$. To any subgroup $H \subset G$, we can associate $H \mapsto R_{\mathrm{Res}(\alpha)}(H)$. Note the special case when $H = \langle g\rangle$, a cyclic subgroup. As $H^2(\langle g\rangle;\mathbb{S}^1) = 0$, the group $R_{\mathrm{Res}(\alpha)}(\langle g\rangle)$ is isomorphic to $R(\langle g\rangle)$.

Now consider $E \to X$, an α-twisted bundle over X; it restricts to a $\mathrm{Res}(\alpha)$-twisted bundle over $X^{\langle g\rangle}$. Recall that we have an isomorphism ${}^{\mathrm{Res}(\alpha)}K^*_{\langle g\rangle}(X^{\langle g\rangle}) \cong K^*(X^{\langle g\rangle}) \otimes R_{\mathrm{Res}(\alpha)}(\langle g\rangle)$. Let $u : R_{\mathrm{Res}(\alpha)}(\langle g\rangle) \to \gamma^\alpha_g$ denote the $\mathbb{C}\,\mathbf{C}_G(g)$-map $\chi \mapsto \chi(g)$ described previously, where the centralizer acts on the projective representations as described above. Then the composition

$$ \begin{aligned} {}^\alpha K^*_G(X) \otimes \mathbb{C} &\overset{\mathrm{Res}(\alpha)}{\to} K^*_{\langle g\rangle}(X^{\langle g\rangle}) \otimes \mathbb{C} \to K^*(X^{\langle g\rangle}) \otimes R_{\mathrm{Res}(\alpha)}(\langle g\rangle) \otimes \mathbb{C} \\ &\to \ K^*(X^{\langle g\rangle}) \otimes \gamma^\alpha_g \end{aligned} $$

has its image lying in the invariants under the $\mathbf{C}_G(g)$-action. Hence we can put these together to yield a map

$$ {}^\alpha K^*_G(X) \otimes \mathbb{C} \to \bigoplus_{(g)} (K^*(X^{\langle g\rangle}) \otimes \gamma^\alpha_g)^{\mathbf{C}_G(g)}. $$

One checks that this induces an isomorphism on orbits G/H; the desired isomorphism follows from using induction on the number of G-cells in X and a Mayer–Vietoris argument (as in [11]). □

Note that in the case when X is a point, we are saying that $R_\alpha(G) \otimes \mathbb{C}$ has rank equal to the number of conjugacy classes of elements in G such that the associated character γ^α_g is trivial. This of course agrees with the number of α-regular conjugacy classes, as indeed ${}^\alpha K_G(\mathrm{pt}) = R_\alpha(G)$.

Remark 3.38 It is apparent that the constructions introduced in this section can be extended to the case of a proper action on X of a discrete group Γ. The group extensions and vector bundles used for the finite group case have natural analogs, and so we can define ${}^{\alpha}K^*_{\Gamma}(X)$ for $\alpha \in H^2(\Gamma; \mathbb{S}^1)$. We will make use of this in the next section.

Example 3.39 Consider the group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; then $H^2(G; \mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$ (as can be seen from the Künneth formula). If a, b are generators for G, we have a projective representation $\mu : G \to PGL_2(\mathbb{C})$ given by

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that this gives rise to an extension $\widetilde{G} \to GL_2(\mathbb{C})$. Restricted to $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{S}^1$, we get an extension of the form $1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{D} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1$; however this is precisely the embedding of the dihedral group in $GL_2(\mathbb{C})$. Hence the extension $\widetilde{G}$ must also be non-split, and so represents the non-trivial element α in $H^2(G; \mathbb{S}^1)$. One can easily verify that there is only one conjugacy class of α-regular elements in G, comprising the trivial element. The representation μ is clearly irreducible, hence up to isomorphism is the unique irreducible α-twisted representation of G. In particular, $R_{\alpha}(G) \cong \mathbb{Z}\langle\mu\rangle$.

Example 3.40 (Symmetric product) Let $G = S_n$, the symmetric group on n letters. Assume that $n \geq 4$; it is well known that in this range $H^2(G; \mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$. Denote the non-trivial class by α. Using the decomposition formula, one can calculate (see Uribe's thesis [154] for details) ${}^{\alpha}K^*_{S_n}(M^n)$, where the group acts on the n-fold product of a manifold M by permutation of coordinates. The quotient orbifold is the symmetric product considered in Example 1.13. From this one can recover a corrected version of a formula which appears in [43] for twisted symmetric products – the error was first observed and corrected by W. Wang in [160]:

$$\sum q^n \chi({}^{\alpha}K^*_{S_n}(M^n) \otimes \mathbb{C}) = \prod_{n>0}(1 - q^{2n-1})^{-\chi(M)} + \prod_{n>0}(1 + q^{2n-1})^{\chi(M)}$$
$$\times \left[1 + \frac{1}{2} \prod_{n>0}(1 + q^{2n})^{\chi(M)} - \frac{1}{2} \prod_{n>0}(1 - q^{2n})^{\chi(M)} \right].$$

Remark 3.41 There is a growing literature in twisted K-theory; in particular, a twisting of $K_G(X)$ can be done using an element in $H^1_G(X; \mathbb{Z}/2\mathbb{Z}) \times H^3_G(X; \mathbb{Z})$ (see [55, p. 422]). Given a G-space X, we can take the classifying map $f_X : EG \times_G X \to BG$; hence given $\alpha \in H^2(BG; \mathbb{S}^1) \cong H^3(G; \mathbb{Z})$ we obtain an element in $H^3_G(X; \mathbb{Z})$ for *any* G-space X, and furthermore these elements

naturally correspond under equivariant maps. Our twisted version of K-theory ${}^{\alpha}K_G$ specializes (for any X) to the twisting by the element $f_X^*(\alpha) \in H_G^3(X;\mathbb{Z})$.

3.7 Twisted orbifold K-theory and twisted Bredon cohomology

Recall that a *discrete torsion* α of an orbifold $\mathcal{X}$ is defined to be a class $\alpha \in H^2(\pi_1^{\text{orb}}(\mathcal{X});\mathbb{S}^1)$. As we saw in Section 2.2, the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{X})$ may be defined as the group of deck translations of the orbifold universal cover $\mathcal{Y} \to \mathcal{X}$.

For example, if $\mathcal{X} = Z/G$ is a global quotient, the universal cover Y of Z is the orbifold universal cover of X. In fact, if $EG \times_G Z$ is the Borel construction for Z, then we have a fibration $Z \to EG \times_G Z \to BG$ which gives rise to the group extension $1 \to \pi_1(Z) \to \pi_1^{orb}(\mathcal{X}) \to G \to 1$; here we are identifying $\pi_1^{\text{orb}}(\mathcal{X})$ with $\pi_1(EG \times_G Z)$. Note that a class $\alpha \in H^2(G;\mathbb{S}^1)$ induces a class $f^*(\alpha)$ in $H^2(\pi_1^{\text{orb}}(\mathcal{X});\mathbb{S}^1)$.

Now suppose that $\mathcal{X} = M/G$ is a quotient manifold for a compact Lie group G and $p : \mathcal{Y} \to \mathcal{X}$ is the orbifold universal cover. Note that p is an orbifold morphism. The same argument used in pulling back orbifold bundles implies that we can pull back the orbifold principal bundle $M \to \mathcal{X}$ to obtain an orbifold principal G-bundle $\widetilde{M} \to \mathcal{Y}$. Furthermore, $\widetilde{M}$ is smooth and has a free left $\pi_1^{\text{orb}}(\mathcal{X})$-action, as well as a right G-action. These can be combined to yield a left $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$-action. It follows that

$$K_\pi^*(\widetilde{M}) \cong K_G^*(\widetilde{M}/\pi_1^{\text{orb}}(\mathcal{X})) = K_{\text{orb}}^*(\mathcal{X}).$$

Consider a group π of the form $\Gamma \times G$, where Γ is a discrete group and G is a compact Lie group. Now let Z denote a proper π-complex such that the orbit space Z/π is a compact orbifold. We now fix a cohomology class $\alpha \in H^2(\Gamma;\mathbb{S}^1)$, corresponding to a central extension Γ_α. From this we obtain an extension $\tilde{\pi}_\alpha = \widetilde{\Gamma}_\alpha \times G$. We can define the α-twisted π-equivariant K-theory of Z, denoted ${}^{\alpha}K_\pi^*(Z)$ in a manner analogous to what we did before. Namely, we consider $\tilde{\pi}_\alpha$-bundles covering the π action on Z, such that the central circle acts by scalar multiplication on the fibers. Based on this we can introduce the following definition.[6]

Definition 3.42 Let $\mathcal{X} = M/G$ denote a compact quotient orbifold where G is a compact Lie group, and let $\mathcal{Y} \to \mathcal{X}$ denote its orbifold universal

[6] Alternatively, we could have used an equivariant version of orbifold bundles and introduced the twisting geometrically. This works for general orbifolds, but we will not elaborate on this here.

cover, with deck transformation group $\Gamma = \pi_1^{\text{orb}}(\mathcal{X})$. Given an element $\alpha \in Z^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$, we define the *$\alpha$-twisted orbifold K-theory of $\mathcal{X}$* as ${}^{\alpha}K^*_{\text{orb}}(\mathcal{X}) = {}^{\alpha}K^*_{\pi}(\widetilde{M})$, where $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$.

If $\mathcal{Y}$, the orbifold universal cover of $\mathcal{X}$, is actually a *manifold*, i.e., if $\mathcal{X}$ is a *good* orbifold (see [105]), then the G-action on $\widetilde{M}$ is free, and in this case the α-twisted orbifold K-theory will simply be ${}^{\alpha}K^*_{\pi_1^{\text{orb}}(\mathcal{X})}(\mathcal{Y})$. For the case of a global quotient $X = Z/G$ and a class $\alpha \in H^2(G; \mathbb{S}^1)$, it is not hard to verify that in fact ${}^{f^*(\alpha)}K^*_{\text{orb}}(\mathcal{X}) \cong {}^{\alpha}K^*_G(Z)$, where $f : \pi_1^{\text{orb}}(\mathcal{X}) \to G$ is defined as before.

In the general case, we note that $\pi = \pi_1^{\text{orb}}(\mathcal{X}) \times G$ acts on $\widetilde{M}$ with finite isotropy. That being so, we can make use of "twisted Bredon cohomology" and a twisted version of the usual Atiyah–Hirzebruch spectral sequence. Fix $\alpha \in Z^2(\pi_1^{\text{orb}}(\mathcal{X}); \mathbb{S}^1)$, where $\mathcal{X}$ is a compact orbifold. There is a spectral sequence of the form

$$E_2 = H^*_{\pi}(\widetilde{M}; R_{\alpha}(-)) \Rightarrow {}^{\alpha}K^*_{\text{orb}}(\mathcal{X}).$$

The E_1 term will be a chain complex built out of the twisted representation rings of the isotropy groups, all of which are finite. In many cases, this twisted Atiyah–Hirzebruch spectral sequence will also collapse at E_2 after tensoring with the complex numbers. We believe that in fact this must always be the case – see Dwyer's thesis [47] for more on this. In particular, we conjecture that if (1) $\mathcal{X}$ is a compact good orbifold with orbifold universal cover the manifold Y, (2) $\Gamma = \pi_1^{\text{orb}}(\mathcal{X})$, and (3) $\alpha \in H^2(\Gamma; \mathbb{S}^1)$, then we have an additive decomposition

$${}^{\alpha}K^*_{\Gamma}(\mathcal{X}) \otimes \mathbb{C} \cong \bigoplus_{(g)} H^*(\text{Hom}_{\mathbf{C}_{\Gamma}(g)}(C_*(Y^{\langle g \rangle}), \gamma_g^{\alpha})) \cong H^*_{\text{CR}}(\mathcal{X}; \mathcal{L}_{\alpha}). \quad (3.10)$$

Here, (g) ranges over conjugacy classes of elements of finite order in Γ, $C_*(-)$ denotes the singular chains, γ_g^{α} is the character for $\mathbf{C}_{\Gamma}(g)$ associated to the twisting, and $H^*_{\text{CR}}(\mathcal{X}; \mathcal{L}_{\alpha})$ is the *twisted Chen–Ruan cohomology* defined in the next chapter.

4
Chen–Ruan cohomology

In the previous three chapters, we have steadily introduced the theory of orbifolds in the realm of topology. We have already seen some signs that, despite many similarities, the theory of orbifolds differs from the theory of manifolds. For example, the notion of orbifold morphism is much more subtle than that of continuous map. Perhaps the strongest evidence is the appearance of the cohomology of the inertia orbifold as the natural target of the Chern character isomorphism in orbifold K-theory. The situation was forcefully crystallized when Chen and Ruan introduced a new "stringy" cohomology for the inertia orbifold of an almost complex orbifold [38]. This Chen–Ruan cohomology is not a natural outgrowth of topological investigations, but rather was primarily motivated by orbifold string theory models in physics.

In 1985, Dixon, Harvey, Vafa, and Witten [44, 45] built a string theory model on several singular spaces, such as $\mathbb{T}^6/G$. We should mention that the particular model they considered was conformal field theory. In conformal field theory, one associates a stringy Hilbert space and its operators to a manifold. Replacing the manifold with an orbifold, they made the surprising discovery that the Hilbert space constructed in a traditional fashion is not consistent, in the sense that its partition function is not modular. To recover modularity, they proposed introducing additional Hilbert space factors into the stringy Hilbert space. They called these factors "twisted sectors," since they intuitively represented the contributions of the singularities in the orbifold. In this way, they were able to build a "smooth" string theory out of a singular space. Nowadays, orbifold conformal field theory is very important in mathematics, and an impressive subject in its own right. For example, it is related to some remarkable developments in algebra, such as Borcherds' work on moonshine.

However, here we are most interested in discussing the geometric consequences of this early work. The main topological invariant arising in orbifold

conformal field theory is the orbifold Euler number. It captured the attention of algebraic geometers, who were interested in describing the geometry of crepant resolutions of an orbifold using the group theoretic data encoded in the orbifold structure. This type of question is called the McKay correspondence. Physically, it is motivated by the observation that if an orbifold admits a crepant resolution, the string theory of the crepant resolution and the orbifold string theory naturally sit in the same family of string theories. Therefore, one would expect the orbifold Euler number to be the same as the ordinary Euler number of its crepant resolution. As mentioned in the Introduction, this expectation was successfully verified by Batyrev [14, 16] using motivic integration [41, 42, 86]; the work of Roan [131, 132], Batyrev and Dais [17], and Reid [130] is also noteworthy in this context. In the process, orbifold Euler numbers were extended to orbifold Hodge numbers. In physics, Zaslow [164] essentially discovered the correct stringy cohomology group for global quotients. However, this approach is limited, because orbifold conformal field theory represents the algebraic aspect of string theory and is not the most effective framework in which to study topological and geometric invariants, such as cohomology theories.

Chen and Ruan approached the problem of understanding orbifold cohomology from the sigma-model/quantum cohomology point of view, where the fundamental object is the space of morphisms from Riemann surfaces to a fixed target orbifold. From this point of view, the inertia orbifold appears naturally as the target of the evaluation map. Their key conceptual observation was that the components of the inertia orbifold should be considered as the geometric source of the twisted sectors introduced earlier in the conformal field theories. Once they realized this, they were able to construct an orbifold quantum cohomology; Chen–Ruan cohomology then arose as the classical limit of the quantum version.

A key application of Chen–Ruan cohomology is to McKay correspondence. General physical principles indicate that orbifold quantum cohomology should be equivalent to the usual quantum cohomology of crepant resolutions, when they exist. The actual process is subtle because of a non-trivial quantization process, which, fortunately, has been understood. This physical understanding led to two conjectures of Ruan [133], which we will present in Section 4.3.

The McKay correspondence is sometimes presented as an equivalence of derived categories of coherent sheaves. Indeed, work on the McKay correspondence in algebraic geometry is usually phrased in such terms; see Bridgeland, King, and Reid [32] and the papers of Kawamata [81–83] for more on this influential approach.

We follow Chen and Ruan's original treatment, beginning with a review of the theory of orbifold morphisms developed in Chapter 2.

4.1 Twisted sectors

The basic idea in quantum cohomology is to study a *target manifold* M by considering maps $\Sigma \to M$ from various Riemann surfaces Σ into M. One wants to topologize the set of (equivalence classes of) such maps to obtain a moduli space, and then use it to define a cohomology. In order to do so, it becomes necessary, among many other considerations, to introduce special *marked points* on the surfaces Σ. The *classical limit* of such a theory restricts attention to the space of constant maps. Of course, these may be identified with the manifold M itself, and so one recovers the usual cohomology of the manifold as a special case. For more on quantum cohomology, we refer the reader to McDuff and Salamon [107].

Generalizing to the orbifold case, we consider a moduli space of orbifold morphisms from marked orbifold Riemann surfaces into a target orbifold. "Classical" cohomology (instead of quantum cohomology) then corresponds to the constant maps. Fortunately, we have classified these already in Section 2.5. There, we saw that the moduli space $\overline{\mathcal{M}}_k(\mathcal{G})$ of representable constant orbifold morphisms from an orbifold sphere with k marked points to $\mathcal{G}$ may be identified with the $(k-1)$-sectors $\mathcal{G}^{k-1}$, whose orbifold groupoid is given by the $\mathcal{G}$-space

$$\begin{aligned} S_{\mathcal{G}}^{k-1} = \{(g_1, \dots, g_{k-1}) \mid g_i \in G_1, s(g_1) = t(g_1) = s(g_2) = t(g_2) \\ = \cdots = s(g_{k-1}) = t(g_{k-1})\}. \end{aligned}$$

There are two natural classes of maps among these moduli spaces. The first class consists of *evaluation* maps

$$e_{i_1,\dots,i_l} : S_{\mathcal{G}}^k \to S_{\mathcal{G}}^l \tag{4.1}$$

defined by

$$e_{i_1,\dots,i_l}(g_1, \dots, g_k) = (g_{i_1}, \dots, g_{i_l}) \tag{4.2}$$

for each cardinality l subset $\{i_1, \dots, i_l\} \subseteq \{1, \dots, k\}$. The other class of maps consists of *involutions*

$$I : S_{\mathcal{G}}^k \to S_{\mathcal{G}}^k \tag{4.3}$$

defined by

$$I(g_1, \dots, g_k) = (g_1^{-1}, \dots, g_k^{-1}). \tag{4.4}$$

Proposition 4.1 *Each evaluation map $e_{i_1,\dots,i_l}$ is a finite union of embeddings, and each involution I is an isomorphism.*

Proof We use the same symbol in each case to denote the induced morphism on the groupoid $\mathcal{G}^k$. Let U_x/G_x be a local chart in $\mathcal{G}$ around $x \in G_0$. Then its preimage in $\mathcal{G}^k$ is

$$\left(\bigsqcup_{(g_1,\ldots,g_k)\in G_x^k} U_x^{g_1} \cap \cdots \cap U_x^{g_k} \times \{(g_1,\ldots,g_k)\}\right) \Big/ G_x,$$

where U_x^g is the subspace of U_x fixed by $g \in G_x$. Likewise, it is clear that

$$\begin{aligned} e^{-1}_{i_1,\ldots,i_l}&\left(\left(\bigsqcup_{(h_1,\ldots,h_l)\in G_x^l} U_x^{h_1} \cap \cdots \cap U_x^{h_l} \times \{(h_1,\ldots,h_l)\}\right) \Big/ G_x\right) \\ &= \left(\bigsqcup_{(g_1,\ldots,g_k)\in G_x^k} U_x^{g_1} \cap \cdots \cap U_x^{g_k} \times \{(g_1,\ldots,g_k)\}\right) \Big/ G_x. \end{aligned}$$

We can rewrite this formula in terms of the components of the preimage of each component of

$$\left(\bigsqcup_{(h_1,\ldots,h_l)\in G_x^l} U_x^{h_1} \cap \cdots \cap U_x^{h_l} \times \{(h_1,\ldots,h_l)\}\right) \Big/ G_x$$

under $e_{i_1,\ldots,i_l}$ to check that the map is locally of the desired form. It is clear that the restriction of $e_{i_1,\ldots,i_l}$ to each component is of the form $U_x^{g_1} \cap \cdots \cap U_x^{g_k} \times \{(g_1,\ldots,g_k)\} \to U_x^{g_{i_1}} \cap \cdots \cap U_x^{g_{i_l}} \times \{(g_{i_1},\ldots,g_{i_l})\}$, and hence is an embedding. The induced map on orbit spaces is obviously a proper map.

Locally, I is the map induced by the identity map $U_x^g \to U_x^{g^{-1}}$. Thus, I is clearly a morphism of $\mathcal{G}$ spaces, and hence induces a homomorphism of the associated groupoids. Furthermore, $I^2 = \mathrm{Id}$, which implies that I is an isomorphism. □

Corollary 4.2 *If $\mathcal{G}$ is complex (almost complex), then so is $\mathcal{G}^k$. Moreover, the maps $e_{i_1,\ldots,i_l}$ and I are holomorphic. If $\mathcal{G}$ is a symplectic (Riemannian) orbifold, $\mathcal{G}^k$ has an induced symplectic (Riemannian) structure.*

Proof We only have to check each assertion locally, say in a chart U_x/G_x of $\mathcal{G}$. When $\mathcal{G}$ is holomorphic, each $g \in G_x$ acts as a complex automorphism of U_x. Therefore, $U_x^{g_1} \cap \cdots \cap U_x^{g_k}$ is a complex submanifold of U_x. This implies that $\mathcal{G}^k$ is a complex suborbifold[1] of $\mathcal{G}$. In particular, the inertia groupoid $\wedge\mathcal{G} = \mathcal{G}^1$ is a complex suborbifold of $\mathcal{G}$. It is clear that $e_{i_1,\ldots,i_l}$ and I are holomorphic.

[1] Or at least a finite union of such.

If $\mathcal{G}$ has a symplectic structure ω, the restriction of the symplectic form to $\mathcal{G}^k$ defines a closed 2-form $\omega_{\mathcal{G}^k}$. To show that it is non-degenerate, we choose a compatible almost complex structure J on $\mathcal{G}$. It induces a compatible metric, g, by the usual formula:

$$g(u, v) = \omega(u, Jv).$$

J and g induce an almost complex structure and a Riemannian metric on $\mathcal{G}^k$ by restriction, and the above formula still holds for the restrictions of J, g, ω. It follows that $\omega_{\mathcal{G}^k}$ is non-degenerate. $\square$

Remark 4.3 Since $e_{i_1,\ldots,i_l}$ is an embedding and I is a diffeomorphism, $e^*_{i_1,\ldots,i_l}\gamma$ and $I^*\gamma$ are compactly supported whenever γ is a compactly supported form.

Next, we study the structure of $\mathcal{G}^k$ in more detail. Suppose that $\mathcal{G} = X/G$ is a global quotient orbifold. In this case, we have $\mathcal{G}^k = (\sqcup_{(g_1,\ldots,g_k)\in G^k} X^{g_1} \cap \cdots \cap X^{g_k} \times \{(g_1, \ldots, g_k)\})/G$ globally. Note that

$$\begin{aligned} h : X^{g_1} \cap \cdots \cap X^{g_k} \times \{(g_1, \ldots, g_k)\} \to\ & X^{hg_1h^{-1}} \cap \cdots \cap X^{hg_kh^{-1}} \\ & \times \{(hg_1h^{-1}, \ldots, hg_kh^{-1})\} \end{aligned}$$

is a diffeomorphism for each $h \in G$. Up to equivalence, then, we can rewrite the groupoid $\mathcal{G}^k$ as

$$\mathcal{G}^k \sim \bigsqcup_{\substack{(g_1,\ldots,g_k)_G \\ g_i \in G}} \left(X^{g_1} \cap \cdots \cap X^{g_k} \times \{(g_1, \ldots, g_k)_G\}\right) \Big/ \mathbf{C}(g_1) \cap \cdots \cap \mathbf{C}(g_k), \tag{4.5}$$

where $(g_1, \ldots, g_k)_G$ represents the conjugacy class of the k-tuple $(g_1, \ldots, g_k)$ under conjugation by G. In particular, as we have seen,

$$\wedge(X/G) \sim \bigsqcup_{\substack{(g)_G \\ g\in G}} X^g/\mathbf{C}(g).$$

It is clear that $\wedge(X/G)$ is not connected, in general. Furthermore, the different components may have different dimensions, so it is important to study them individually.

Let us try to parameterize the components of $\mathcal{G}^k$. Recall that

$$|\mathcal{G}^k| = \{(x, (g_1, \ldots, g_k)_{G_x}) \mid x \in |\mathcal{G}|, g_i \in G_x\}.$$

We use $\mathbf{g}$ to denote the k-tuple $(g_1, \ldots, g_k)$. Suppose that p and q are two points in the same linear orbifold chart U_x/G_x. Let $\tilde{p}$, $\tilde{q}$ be preimages of p, q. Then we may identify G_p with $(G_x)_{\tilde{p}}$ and G_q with $(G_x)_{\tilde{q}}$, and thereby

view both local groups as subgroups of G_x. We say that $(\mathbf{g}_1)_{G_p} \approx (\mathbf{g}_2)_{G_q}$ if $\mathbf{g}_1 = h\mathbf{g}_2h^{-1}$ for some element $h \in G_x$. This relation is well defined, since other choices of preimages will result in conjugate subgroups of G_x. For two arbitrary points p and q in $\mathcal{G}$, we say $(\mathbf{g})_{G_p} \approx (\mathbf{g}')_{G_q}$ if there is a finite sequence $(p_0, (\mathbf{g}_0)_{G_{p_0}}), \ldots, (p_k, (\mathbf{g}_k)_{G_{p_k}})$ such that:

1. $(p_0, (\mathbf{g}_0)_{G_{p_0}}) = (p, (\mathbf{g})_{G_p})$,
2. $(p_k, (\mathbf{g}_k)_{G_{p_k}}) = (q, (\mathbf{g}')_{G_q})$, and
3. for each i, the points p_i and p_{i+1} are both in the same linear chart, and $(\mathbf{g}_i)_{G_{p_i}} \approx (\mathbf{g}_{i+1})_{G_{p_{i+1}}}$.

This defines an equivalence relation on $(\mathbf{g})_{G_p}$. The reader should note that it is possible that $(\mathbf{g})_{G_p} \cong (\mathbf{g}')_{G_p}$ while $(\mathbf{g})_{G_p} \not\approx (\mathbf{g}')_{G_p}$ when $|\mathcal{G}|$ has a non-trivial fundamental group.

Let T_k be the set of equivalence classes of elements of $|\mathcal{G}^k|$ under $\approx$. Abusing notation, we often use $(\mathbf{g})$ to denote the equivalence class of $(\mathbf{g})_{G_q}$. Let

$$|\mathcal{G}^k|_{(\mathbf{g})} = \{(p, (\mathbf{g}')_{G_p}) | \mathbf{g}' \in G_p^k, (\mathbf{g}')_{G_p} \in (\mathbf{g})\}. \tag{4.6}$$

Since each linear chart is equivariantly contractible, its quotient space is contractible. So these subsets are exactly the connected components of $|G^k|$. Let $\mathcal{G}^k_{(\mathbf{g})}$ be the corresponding $\mathcal{G}$-component of the orbifold groupoid, i.e., the full subgroupoid on the preimage of $|\mathcal{G}^k|_{(\mathbf{g})}$ under the quotient map. It is clear that $\mathcal{G}^k$ is decomposed as a disjoint union of $\mathcal{G}$-connected components

$$\mathcal{G}^k = \bigsqcup_{(\mathbf{g}) \in T_k} \mathcal{G}^k_{(\mathbf{g})}. \tag{4.7}$$

In particular,

$$\wedge\mathcal{G} = \bigsqcup_{(g) \in T_1} \mathcal{G}^1_{(g)}. \tag{4.8}$$

Let $T_k^o \subset T^k$ be the subset of equivalence classes $(g_1, \ldots, g_k)$ with the property $g_1 \ldots g_k = 1$. Then

$$\overline{\mathcal{M}}_k(\mathcal{G}) = \bigsqcup_{(\mathbf{g}) \in T_k^o} \mathcal{G}^k_{(\mathbf{g})}.$$

There is also an identification

$$\mathcal{G}^k = \overline{\mathcal{M}}_{k+1}(\mathcal{G})$$

given by

$$(g_1, \ldots, g_k) \to (g_1, \ldots, g_k, (g_1 \ldots g_k)^{-1}).$$

Definition 4.4 $\mathcal{G}^1_{(g)}$ for $g \neq 1$ is called a *twisted sector*. For $\mathbf{g} = \{g_1, \ldots, g_k\}$, the groupoid $\mathcal{G}^k_{(\mathbf{g})}$ is called a *k-multi-sector*, or *k-sector* for short. Furthermore, we call $\mathcal{G}^1_{(1)} \cong \mathcal{G}$ the *non-twisted sector*.

We have following obvious but useful lemma.

Lemma 4.5 *Let N_p be the subgroup of G_p generated by $\mathbf{g}$ for $(p, (\mathbf{g})_{G_p}) \in |\mathcal{G}^k|$. Then N_p is isomorphic to N_q if $(p, (\mathbf{g})_{G_p})$ and $(q, (\mathbf{g})_{G_q})$ belong to the same component of $|\mathcal{G}^k|$.*

Proof This is a local statement. By the definition, locally, N_p and N_q are conjugate to each other. Hence, they are isomorphic. $\square$

4.2 Degree shifting and Poincaré pairing

For the rest of the chapter, we will assume that $\mathcal{G}$ is an almost complex orbifold with an almost complex structure J. As we saw above, $\wedge\mathcal{G}$ and $\mathcal{G}^k$ naturally inherit almost complex structures from the one on $\mathcal{G}$, and the evaluation and involution maps $e_{i_1,\ldots,i_l}$ and I are naturally pseudo-holomorphic, meaning that their differentials commute with the almost complex structures. Furthermore, we assume that $|\mathcal{G}|$ admits a finite good cover. In this case, it is easy to check that $|\wedge\mathcal{G}|$ also admits a finite good cover. Therefore, each sector $\mathcal{G}_{(g)}$ will satisfy Poincaré duality. From here on, we often omit superscripts on sectors when there is no chance for confusion.

An important feature of the Chen–Ruan cohomology groups is degree shifting, as we shall now explain. To each twisted sector, we associate a rational number. In the original physical literature, it was referred to as the *fermionic degree shifting number*. Here, we simply call it the *degree shifting number*. Originally, this number came from Kawasaki's orbifold index theory (see [85]). We define these numbers as follows. Let g be any point of $S_{\mathcal{G}}$ and set $p = s(g) = t(g)$. Then the local group G_p acts on T_pG_0. The almost complex structure on $\mathcal{G}$ gives rise to a representation $\rho_p : G_p \to GL(n, \mathbb{C})$ (here, $n = \dim_{\mathbb{C}} \mathcal{G}$). The element $g \in G_p$ has finite order. We can write $\rho_p(g)$ as a diagonal matrix

$$\mathrm{diag}(e^{2\pi i m_{1,g}/m_g}, \ldots, e^{2\pi i m_{n,g}/m_g}),$$

where m_g is the order of $\rho_p(g)$, and $0 \leq m_{i,g} < m_g$. This matrix depends only on the conjugacy class $(g)_{G_p}$ of g in G_p. We define a function $\iota : |\wedge\mathcal{G}| \to \mathbb{Q}$

by

$$\iota(p, (g)_{G_p}) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}. \tag{4.9}$$

It is straightforward to show the following lemma.

Lemma 4.6 *The function $\iota : | \wedge \mathcal{G}| \to \mathbb{Q}$ is locally constant. Its constant value on each component, which will be denoted by $\iota_{(g)}$, satisfies the following conditions:*

- *The number $\iota_{(g)}$ is integral if and only if $\rho_p(g) \in SL(n, \mathbb{C})$.*
- *For each (g),*

$$\iota_{(g)} + \iota_{(g^{-1})} = \mathrm{rank}(\rho_p(g) - I),$$

where I is the identity matrix. This is the "complex codimension" $\dim_{\mathbb{C}} \mathcal{G} - \dim_{\mathbb{C}} \mathcal{G}_{(g)} = n - \dim_{\mathbb{C}} \mathcal{G}_{(g)}$ of $\mathcal{G}_{(g)}$ in $\mathcal{G}$. As a consequence, $\iota_{(g)} + \dim_{\mathbb{C}} \mathcal{G}_{(g)} < n$ when $\rho_p(g) \neq I$.

Definition 4.7 The rational number $\iota_{(g)}$ is called a *degree shifting number*.

In the definition of the Chen–Ruan cohomology groups, we will shift up the degrees of the cohomology classes coming from $\mathcal{G}_{(g)}$ by $2\iota_{(g)}$. The reason for this is as follows. By the Kawasaki index theorem,

$$\mathrm{virdim}\, \overline{\mathcal{M}}_3(\mathcal{G}) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}.$$

To formally carry out an integration

$$\int_{\overline{\mathcal{M}}_3(\mathcal{G})} e_1^*(\alpha_1) \wedge e_2^*(\alpha_2) \wedge e_3^*(\alpha_3),$$

we need the condition

$$\deg(\alpha_1) + \deg(\alpha_2) + \deg(\alpha_3) = \mathrm{virdim}\, \overline{\mathcal{M}}_3(\mathcal{G}) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}.$$

Hence, we require

$$\deg(\alpha_1) + 2\iota_{(g_1)} + \deg(\alpha_2) + 2\iota_{(g_2)} + \deg(\alpha_3) + 2\iota_{(g_3)} = 2n.$$

Namely, we can think that the degree of α_i has been "shifted up" by $2\iota_{(g_i)}$.

An orbifold groupoid $\mathcal{G}$ is called an *SL-orbifold groupoid* if $\rho_p(g) \in SL(n, \mathbb{C})$ for all $p \in G_0$ and $g \in G_p$. Recall from Chapter 1 that this corresponds to the Gorenstein condition in algebraic geometry. For such an orbifold, all degree-shifting numbers will be integers.

We observe that although the almost complex structure J is involved in the definition of degree-shifting numbers $\iota_{(g)}$, they do not depend on J, since the parameter space of almost complex structures $SO(2n,\mathbb{R})/U(n,\mathbb{C})$ is locally connected.

Definition 4.8 We define the *Chen–Ruan cohomology groups* $H^d_{\mathrm{CR}}(\mathcal{G})$ of $\mathcal{G}$ by

$$\begin{aligned} H^d_{\mathrm{CR}}(\mathcal{G}) &= \bigoplus_{(g)\in T_1} H^d(\mathcal{G}^1_{(g)})[-2\iota_{(g)}] \\ &= \bigoplus_{(g)\in T_1} H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)}). \end{aligned} \tag{4.10}$$

Here each $H^*(\mathcal{G}^1_{(g)})$ is the singular cohomology with real coefficients or, equivalently, the de Rham cohomology, of $\mathcal{G}^1_{(g)}$. Note that in general the Chen–Ruan cohomology groups are rationally graded.

Suppose $\mathcal{G}$ is a complex orbifold with an integrable complex structure J. We have seen that each twisted sector $\mathcal{G}^1_{(g)}$ is also a complex orbifold with the induced complex structure. We consider the Dolbeault cohomology groups of (p,q)-forms (in the orbifold sense). When $\mathcal{G}$ is closed, the harmonic theory of [12] can be applied to show that these groups are finite-dimensional, and there is a Kodaira–Serre duality between them. When $\mathcal{G}$ is a closed Kähler orbifold (so that each $\mathcal{G}_{(g)}$ is also Kähler), these groups are related to the singular cohomology groups of $\mathcal{G}$ and $\mathcal{G}_{(g)}$ as in the smooth case, and the Hodge decomposition theorem holds for these cohomology groups.

Definition 4.9 Let $\mathcal{G}$ be a complex orbifold. We define, for $0\le p,q\le \dim_{\mathbb{C}}\mathcal{G}$, the *Chen–Ruan Dolbeault cohomology groups*

$$H^{p,q}_{\mathrm{CR}}(\mathcal{G}) = \bigoplus_{(g)} H^{p-\iota_{(g)},q-\iota_{(g)}}(\mathcal{G}^1_{(g)}).$$

Remark 4.10 We can define compactly supported Chen–Ruan cohomology groups $H^*_{\mathrm{CR},c}(\mathcal{G})$ and $H^{*,*}_{\mathrm{CR},c}(\mathcal{G})$ in the obvious fashion.

Recall the involution $I:\mathcal{G}^1_{(g)}\to\mathcal{G}^1_{(g^{-1})}$; it is an automorphism of $\wedge\mathcal{G}$ as an orbifold such that $I^2=\mathrm{Id}$. In particular, I is a diffeomorphism.

Proposition 4.11 (Poincaré duality) *Suppose that* $\dim_{\mathbb{R}}\mathcal{G}=2n$. *For any* $0\le d\le 2n$, *define a pairing*

$$\langle\ ,\ \rangle_{\mathrm{CR}} : H^d_{\mathrm{CR}}(\mathcal{G})\times H^{2n-d}_{\mathrm{CR},c}(\mathcal{G})\to\mathbb{R} \tag{4.11}$$

as the direct sum of the pairings

$$\langle\ ,\ \rangle_{(g)} : H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)})\times H_c^{2n-d-2\iota_{(g^{-1})}}(\mathcal{G}^1_{(g^{-1})})\to\mathbb{R},$$

where

$$\langle \alpha, \beta \rangle_{(g)} = \int_{\mathcal{G}^1_{(g)}} \alpha \wedge I^*(\beta)$$

for $\alpha \in H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)})$, $\beta \in H_c^{2n-d-2\iota_{(g^{-1})}}(\mathcal{G}^1_{(g^{-1})})$. *Then the pairing* $\langle\ ,\ \rangle_{\mathrm{CR}}$ *is non-degenerate.*

Note that $\langle\ ,\ \rangle_{\mathrm{CR}}$ equals the ordinary Poincaré pairing when restricted to the non-twisted sector $H^*(\mathcal{G})$.

Proof By Lemma 4.6, we have

$$2n - d - 2\iota_{(g^{-1})} = \dim \mathcal{G}^1_{(g)} - d - 2\iota_{(g)}.$$

Furthermore, $I|_{\mathcal{G}^1_{(g)}} : \mathcal{G}^1_{(g)} \to \mathcal{G}^1_{(g^{-1})}$ is a diffeomorphism. Under this diffeomorphism, $\langle\ ,\ \rangle_{(g)}$ is isomorphic to the ordinary Poincaré pairing on $\mathcal{G}^1_{(g)}$, and so is non-degenerate. Hence, $\langle\ ,\ \rangle_{\mathrm{CR}}$ is also non-degenerate. □

If we forget about the degree shifts, the Chen–Ruan cohomology group is just $H^*(\wedge\mathcal{G})$ with a non-degenerate pairing given by

$$\langle \alpha, \beta \rangle = \int_{\wedge\mathcal{G}} \alpha \wedge I^*\beta.$$

For the case of Chen–Ruan Dolbeault cohomology, the following proposition is straightforward.

Proposition 4.12 *Let* $\mathcal{G}$ *be an n-dimensional complex orbifold. There is a Kodaira–Serre duality pairing*

$$\langle\ ,\ \rangle_{\mathrm{CR}} : H^{p,q}_{\mathrm{CR}}(\mathcal{G}) \times H^{n-p,n-q}_{\mathrm{CR},c}(\mathcal{G}) \to \mathbb{C}$$

defined as in the previous proposition by a sum of pairings on the sectors. When $\mathcal{G}$ *is closed and Kähler, the following is true:*

- $H^r_{\mathrm{CR}}(\mathcal{G}) \otimes \mathbb{C} = \oplus_{r=p+q} H^{p,q}_{\mathrm{CR}}(\mathcal{G})$,
- $H^{p,q}_{\mathrm{CR}}(\mathcal{G}) = \overline{H^{q,p}_{\mathrm{CR}}(\mathcal{G})}$,

and the two pairings (Poincaré and Kodaira–Serre) coincide.

Theorem 4.13 *The Chen–Ruan cohomology group, together with its Poincaré pairing, is invariant under orbifold Morita equivalence.*

Proof The theorem follows easily from the fact that: (1) an equivalence (hence Morita equivalence) of orbifold groupoids induces an equivalence of the inertia orbifolds; (2) integration is invariant under Morita equivalence; and (3) ι is locally constant. □

4.3 Cup product

The most interesting part of Chen–Ruan cohomology is its product structure, which is new in both mathematics and physics. Roughly speaking, the Chen–Ruan cup product is the classical limit of a Chen–Ruan *orbifold quantum product*. For this reason, its definition reflects the general machinery of quantum cohomology, and may look a little bit strange to traditional topologists. It remains a very interesting open question to find an alternative definition of these products along more traditional topological lines, and also to better understand the important role of the obstruction bundle.

Marked orbifold Riemann surfaces are the key ingredients in defining the Chen–Ruan product. Recall from Example 1.16 that a closed two-dimensional orbifold is described by the following data:

- a closed Riemann surface Σ with complex structure j, and
- a finite subset $\mathbf{z} = (z_1, \ldots, z_k)$ of points on Σ, each with a multiplicity m_i (let $\mathbf{m} = (m_1, \ldots, m_k)$).

The corresponding orbifold structure on Σ coincides with the usual manifold structure, except that at each z_i, a chart is given by the ramified covering $z \to z^{m_i}$. Note that we allow m_i to be 1, in which case z_i is a smooth point. However, all the singular points are in $\mathbf{z}$. We call the z_i *marked points*, and refer to $(\Sigma, j, \mathbf{z}, \mathbf{m})$ as a *marked orbifold Riemann surface*. Of course, if all the multiplicities m_i are 1, we recover the usual notion of a marked Riemann surface.

The construction of the cup product follows the usual procedure in quantum cohomology. Namely, we will first define a *three-point function*, and then use it and the Poincaré pairing to obtain a product. In our case, the three-point function is an integral over the moduli space $\overline{\mathcal{M}}_3(\mathcal{G})$. In order to write down the form to integrate, we will need to construct an *obstruction bundle* and obtain its Euler form. We start with an approach phrased in terms of $\bar{\partial}$ operators on orbifolds, and then reinterpret the results using our knowledge of orbifold Riemann surfaces.

Recall that an element of $\overline{\mathcal{M}}_3(\mathcal{G})$ is a constant representable orbifold morphism f_y from $\mathbb{S}^2$ to $\mathcal{G}$, where $\mathrm{im}(f) = y \in G_0$ and the marked orbifold Riemann surface $\mathbb{S}^2$ has three marked points, z_1, z_2, and z_3, with multiplicities m_1, m_2, and m_3, respectively. In this case, there are three evaluation maps

$$e_i : \overline{\mathcal{M}}_3(\mathcal{G}) \subset \mathcal{G}^3 \to \wedge\mathcal{G}.$$

The three-point function is defined by the formula

$$\langle \alpha, \beta, \gamma \rangle = \int_{\overline{\mathcal{M}}_3(\mathcal{G})} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge \mathbf{e}(E_3), \tag{4.12}$$

for $\alpha, \beta \in H^*(\wedge\mathcal{G})$, $\gamma \in H_c^*(\wedge\mathcal{G})$, and $\mathbf{e}(E_3)$ the Euler form of a certain orbibundle. To define this bundle E_3, we consider the elliptic complex

$$\bar{\partial}_y : \Omega^0(f_y^*T\mathcal{G}) \to \Omega^{0,1}(f_y^*T\mathcal{G}).$$

We want to calculate the index of $\bar{\partial}_y$, and eventually use it to obtain a vector bundle over the moduli space as f_y varies. Recall that we identified $\overline{\mathcal{M}}_3(\mathcal{G})$ with $\sqcup_{(\mathbf{g})\in T_o^3}\mathcal{G}^3_{(\mathbf{g})}$. Let $\mathbf{g} = (g_1, g_2, g_3)$ be the tuple corresponding to f_y. The index of the operator $\bar{\partial}_y$ is then

$$\text{index } \bar{\partial}_y = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}. \tag{4.13}$$

So the index varies from component to component on $\overline{\mathcal{M}}_3(\mathcal{G})$, depending on the degree shifting numbers $\iota_{(g_i)}$. We define E_3 by defining its restriction over each component. We have $f_y \in \mathcal{G}^3_{(\mathbf{g})}$, and using the index theory of families of elliptic operators, one can show that $\ker(\bar{\partial}_y)$ can be canonically identified with $T_{f_y}\mathcal{G}^3_{(\mathbf{g})}$, and so has constant dimension over $\mathcal{G}^3_{(\mathbf{g})}$. Therefore, $\operatorname{coker}(\bar{\partial}_y)$ also has constant dimension, and forms an orbifold vector bundle $E_{(\mathbf{g})} \to \mathcal{G}^3_{(\mathbf{g})}$ as f_y varies. We define E_3 by letting its restriction to $\mathcal{G}^3_{(\mathbf{g})}$ be $E_{(\mathbf{g})} \to \mathcal{G}^3_{(\mathbf{g})}$.

Our situation is simple enough that we can write down the kernel and cokernel explicitly. Once we have an explicit presentation, the required properties are straightforward. We will need the following well-known fact (see [140], for example).

Proposition 4.14 *Let $(\Sigma, \mathbf{z}, \mathbf{m})$ be a marked complex orbifold Riemann surface, where $\mathbf{z} = (z_1, \ldots, z_k)$ and $\mathbf{m} = (m_1, \ldots, m_k)$, such that*

- *the genus $g_\Sigma \geq 1$, or*
- *$g_\Sigma = 0$ with $k \geq 3$, or*
- *$g_\Sigma = 0$ with $k = 2$ and $m_1 = m_2$.*

Then $(\Sigma, \mathbf{z}, \mathbf{m})$ is a good orbifold. Namely, it has a smooth universal cover.

For $\mathbf{g} \in T_o^3$, consider the pullback $e^*T\mathcal{G}$ of the tangent bundle over $\mathcal{G}^3_{(\mathbf{g})}$, where $e : \mathcal{G}^3_{(\mathbf{g})} \to \mathcal{G}$ is the restriction of the evaluation map sending a tuple of arrows to their base object. Let $\mathbf{g} = (g_1, g_2, g_3) \in \mathcal{G}^3_{(\mathbf{g})}$. Then g_1, g_2, and g_3 are elements of the local group $G_{e(\mathbf{h})}$, and they obviously satisfy the relations $g_1g_2g_3 = 1$ and $g_i^{m_i} = 1$, where m_i is the order of g_i. Let N be the subgroup of $G_{e(\mathbf{h})}$ generated by these three elements. By Lemma 4.5, N is independent (up

to isomorphism) of the choice of $\mathbf{g}$, so long as $\mathbf{g}$ remains within the component $\mathcal{G}^3_{(\mathbf{g})}$. Clearly, this sets up an action of the group N on $e^*T\mathcal{G}$ that fixes $\mathcal{G}_{(\mathbf{g})}$.

Consider an orbifold Riemann sphere with three orbifold points,

$$(\mathbb{S}^2, (x_1, x_2, x_3), (m_1, m_2, m_3)),$$

such that the multiplicities match the orders of the generators of the group N in the previous paragraph. We write $\mathbb{S}^2$ for brevity. Recall from Section 2.2 that

$$\pi_1^{\mathrm{orb}}(\mathbb{S}^2) = \{\lambda_1, \lambda_2, \lambda_3 \mid \lambda_i^{k_i} = 1, \lambda_1\lambda_2\lambda_3 = 1\},$$

where λ_i is represented by a loop around the marked point x_i. There is an obvious surjective homomorphism

$$\pi : \pi_1^{\mathrm{orb}}(\mathbb{S}^2) \to N. \tag{4.14}$$

Its kernel, $\ker\pi$, is a subgroup of finite index. Suppose that $\widetilde{\Sigma}$ is the orbifold universal cover of $\mathbb{S}^2$. By Proposition 4.14, $\widetilde{\Sigma}$ is smooth. Let $\Sigma = \widetilde{\Sigma}/\ker\pi$. Then Σ is compact, and there is a cover $p : \Sigma \to \mathbb{S}^2 = \Sigma/N$. Since N contains the relations $g_i^{m_i} = 1$, the surface Σ must be smooth.

Now let U_y/G_y be an orbifold chart at $y \in G_0$. The constant orbifold morphism f_y from before can be lifted to an ordinary constant map

$$\tilde{f}_y : \Sigma \to U_y.$$

Hence, $\tilde{f}_y^* T\mathcal{G} = T_y\mathcal{G}$ is a trivial bundle over Σ. We can also lift the elliptic complex to Σ:

$$\bar{\partial}_\Sigma : \Omega^0(\tilde{f}_y^* T\mathcal{G}) \to \Omega^{0,1}(\tilde{f}_y^* T\mathcal{G}).$$

The original elliptic complex is just the N-invariant part of the current one. However, $\ker(\bar{\partial}_\Sigma) = T_y\mathcal{G}$ and $\mathrm{coker}(\bar{\partial}_\Sigma) = H^{0,1}(\Sigma) \otimes T_y\mathcal{G}$. Now we vary y and obtain the bundle $e^*_{(\mathbf{g})}T\mathcal{G}$ corresponding to the kernels, and $H^{0,1}(\Sigma) \otimes e^*_{(\mathbf{g})}T\mathcal{G}$ corresponding to the cokernels, where we are using the evaluation map $e_{(\mathbf{g})} : \mathcal{G}_{(\mathbf{g})} \to \mathcal{G}$ to pull back. N acts on both bundles, and it is clear that $(e^*_{(\mathbf{g})}T\mathcal{G})^N = T\mathcal{G}_{(\mathbf{g})}$, justifying our previous claim. The obstruction bundle $E_{(\mathbf{g})}$ we want is the invariant part of $H^{0,1}(\Sigma) \otimes e^*_{(\mathbf{g})}T\mathcal{G}$, i.e., $E_{(\mathbf{g})} = (H^{0,1}(\Sigma) \otimes e^*_{(\mathbf{g})}T\mathcal{G})^N$. Since we do not assume that $\mathcal{G}$ is compact, $\mathcal{G}_{(\mathbf{g})}$ could be a non-compact orbifold in general.

Now, we are ready to define our three-point function. Suppose that $\alpha \in H^{d_1}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$, $\beta \in H^{d_2}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$, and $\gamma \in H^*_{\mathrm{CR},c}(\mathcal{G}_{(g_3)};\mathbb{C})$.

Definition 4.15 We define the *three-point function* $\langle\ ,\ ,\ \rangle$ by

$$\langle \alpha, \beta, \gamma \rangle = \sum_{(\mathbf{g}) \in T_3^0} \int_{\mathcal{G}_{(\mathbf{g})}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge \mathbf{e}(E_{(\mathbf{g})}).$$

Note that $e_3^*\gamma$ is compactly supported. Therefore, the integral is finite.

Definition 4.16 We define the *Chen–Ruan* or *CR cup product* using the Poincaré pairing and the three-point function, via the relation

$$\langle \alpha \cup \beta, \gamma \rangle_{\mathrm{CR}} = \langle \alpha, \beta, \gamma \rangle.$$

Due to the formula

$$\dim \mathcal{G}_{(\mathbf{g})} - \operatorname{rank} E_{(\mathbf{g})} = \operatorname{index}(\bar{\partial}) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)},$$

a simple computation shows that the orbifold degrees satisfy $\deg_{\mathrm{orb}}(\alpha \cup \beta) = \deg_{\mathrm{orb}}(\alpha) + \deg_{\mathrm{orb}}(\beta)$. If α and β are compactly supported Chen–Ruan cohomology classes, we can define $\alpha \cup \beta \in H^*_{\mathrm{CR},c}(\mathcal{G})$ in the same fashion. Suppose that $\alpha \in H^*(\mathcal{G}^1_{(g_1)})$ and $\beta \in H^*(\mathcal{G}^1_{(g_2)})$. Then $\alpha \cup \beta \in H^*_{\mathrm{CR}}(\mathcal{G}) = \bigoplus_{(g)\in T_1} H^*(\mathcal{G}^1_{(g)})$. Therefore, we should be able to decompose $\alpha \cup \beta$ as a sum of its components in $H^*(\mathcal{G}^1_{(g)})$. Such a decomposition would be very useful in computations. To achieve this decomposition, first note that when $g_1 g_2 g_3 = 1$, the conjugacy class of (g_1, g_2, g_3) is uniquely determined by the conjugacy class of the pair (g_1, g_2). We can use this to obtain the following lemma.

Lemma 4.17 (Decomposition) *Let α and β be as above. Then*

$$\alpha \cup \beta = \sum_{\substack{(h_1,h_2)\in T_2 \\ h_i \in (g_i)}} (\alpha \cup \beta)_{(h_1,h_2)},$$

where $(\alpha \cup \beta)_{(h_1,h_2)} \in H^*(\mathcal{G}_{(h_1 h_2)})$ is defined by the relation

$$\langle (\alpha \cup \beta)_{(h_1,h_2)}, \gamma \rangle = \int_{\mathcal{G}_{(h_1,h_2)}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge \mathbf{e}(E_{(\mathbf{g})})$$

for $\gamma \in H^*_c(\mathcal{G}_{((h_1 h_2)^{-1})})$.

Remark 4.18 Recall that for the global quotient $\mathcal{X} = Y/G$, additively, $H^*_{\mathrm{CR}}(\mathcal{X}) = H^*(\wedge\mathcal{X}) = (\bigoplus_g H^*(Y^g))^G$. Fantechi and Göttsche [52] and Kaufmann [80] (in the more abstract setting of Frobenius manifolds) observed that we can put a product on the larger space $H^*(Y, G) = \bigoplus_g H^*(Y^g)$ such that, as a ring, Chen–Ruan cohomology is its invariant subring under the natural G-action.

We describe this straightforward identification. To do so, we need only lift all of our constructions from $Y^g/C(g)$ to the level of Y^g. Let $Y^{g_1,\ldots,g_k} = Y^{g_1} \cap \cdots \cap Y^{g_k} \times \{(g_1, \ldots, g_k)\}$. First, we observe that, as an orbifold,

$$\mathcal{X}_{(g_1,g_2,(g_1g_2)^{-1})} = \left(\bigsqcup_{(h_1,h_2)=g(g_1,g_2)g^{-1}} Y^{h_1,h_2,(h_1h_2)^{-1}} \right) \Big/ G.$$

Hence, orbifold integration on $\mathcal{X}_{(g_1,g_2,(g_1g_2)^{-1})}$ satisfies

$$\int_{\mathcal{X}_{(g_1,g_2,(g_1g_2)^{-1})}} = \frac{1}{|G|}\int_{Y^{g_1,g_2,(g_1g_2)^{-1}}}.$$

The evaluation map

$$e_{i_1,\dots,i_l} : Y^{g_1,\dots,g_k}/\,\mathbf{C}(g_1)\cap\cdots\cap\mathbf{C}(g_k) \to Y^{g_{i_1},\dots,g_{i_l}}/\,\mathbf{C}(g_{i_1})\cap\cdots\cap\mathbf{C}(g_{i_l})$$

is obviously the quotient of the inclusion $Y^{g_1,\dots,g_k} \to Y^{g_{i_1},\dots,g_{i_l}}$ (still denoted by $e_{i_1,\dots,i_l}$), and similarly for the involution maps I.

We now consider the Poincaré pairing. It is clear that we just have to pair Y^g with $Y^{g^{-1}}$, and the same construction works without change. For the three-point function, we have to lift the obstruction bundle. This is clearly possible from our definition: recall that $E_{(h_1,h_2,(h_1h_2)^{-1})} = (e^*_{(h_1,h_2,(h_1h_2)^{-1})}T(Y/G)\otimes H^{0,1}(\Sigma))^{\langle h_1,h_2\rangle}$. Thus, the obstruction bundle is naturally the quotient of a vector bundle $E_{h_1,h_2,(h_1h_2)^{-1}} = (e^*_{(h_1,h_2,(h_1h_2)^{-1})}TY\otimes H^{0,1}(\Sigma))^{\langle h_1,h_2\rangle}$. Here, $\langle h_1, h_2\rangle$ is the subgroup generated by $\{h_1, h_2\}$. In summary, we obtain a three-point function for $\alpha, \beta, \gamma \in H^*(Y, G)$, defined by

$$\langle \alpha, \beta, \gamma\rangle = \frac{1}{|G|}\sum_{h_1,h_2}\int_{Y^{h_1,h_2,(h_1h_2)^{-1}}} e_1^*\alpha\wedge e_2^*\beta\wedge e_3^*\gamma\wedge \mathbf{e}(E_{h_1,h_2,(h_1h_2)^{-1}}).$$

Using the formula $\langle \alpha\cup\beta, \gamma\rangle = \langle\alpha, \beta, \gamma\rangle$, we obtain a product on $H^*(Y, G)$. Moreover, by construction $H^*_{\mathrm{CR}}(\mathcal{X}) = H^*(Y, G)^G$ as a ring. Suppose that $\alpha \in H^*(Y^{g_1})$, $\beta \in H^*(Y^{g_2})$, $\gamma \in H^*(Y^{(g_1g_2)^{-1}})$, then

$$\langle \alpha, \beta, \gamma\rangle = \frac{1}{|G|}\int_{Y^{g_1,g_2,(g_1g_2)^{-1}}} e_1^*\alpha\wedge e_2^*\beta\wedge e_3^*\gamma\wedge \mathbf{e}(E_{g_1,g_2,(g_1g_2)^{-1}}).$$

As Fantechi and Göttsche and Kaufmann observed, the product on $H^*(Y, G)$ is no longer commutative, since $E_{g_1,g_2,(g_1g_2)^{-1}} \neq E_{g_2,g_1,(g_2g_1)^{-1}}$ in general.

We summarize the constructions of this section.

Theorem 4.19 *Let $\mathcal{G}$ be an almost complex orbifold groupoid with almost complex structure J and* $\dim_{\mathbb{C}}\mathcal{G} = n$. *The cup product defined above preserves the orbifold degree, i.e., $\cup : H^p_{\mathrm{CR}}(\mathcal{G};\mathbb{C})\otimes H^q_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) \to H^{p+q}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$ for any $0\le p, q\le 2n$ such that $p+q\le 2n$, and has the following properties:*

1. *The total Chen–Ruan cohomology group $H^*_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) = \bigoplus_{0\le d\le 2n} H^d_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$ is a ring with unit $e^0_{\mathcal{G}} \in H^0(\mathcal{G};\mathbb{C})$ under the Chen–Ruan cup product $\cup$, where $e^0_{\mathcal{G}}$ represents the constant function* 1 *on $\mathcal{G}$.*
2. *The cup product $\cup$ is invariant under deformations of J.*
3. *When $\mathcal{G}$ has integral degree shifting numbers, the total Chen–Ruan cohomology group $H^*_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$ is integrally graded, and we have* supercommutativity*:*

$$\alpha_1\cup\alpha_2 = (-1)^{\deg\alpha_1\dot{\deg}\alpha_2}\alpha_2\cup\alpha_1.$$

4. *Restricted to the non-twisted sectors, i.e., the ordinary cohomology $H^*(\mathcal{G};\mathbb{C})$, the cup product $\cup$ equals the ordinary cup product on $\mathcal{G}$.*

Now we define the cup product $\cup$ on the total Chen–Ruan Dolbeault cohomology group of $\mathcal{G}$ when $\mathcal{G}$ is a complex orbifold. We observe that in this case all the objects we have been dealing with are holomorphic, i.e., $\overline{\mathcal{M}}_k(\mathcal{G})$ is a complex orbifold, each $E_{(\mathbf{g})} \to \mathcal{G}^1_{(\mathbf{g})}$ is a holomorphic orbifold bundle, and the evaluation maps are holomorphic.

Definition 4.20 For any $\alpha_1 \in H^{p,q}_{CR}(\mathcal{G};\mathbb{C})$, $\alpha_2 \in H^{p',q'}_{CR}(\mathcal{G};\mathbb{C})$, we define the three-point function and Chen–Ruan cup product in the same fashion as Definition 4.16.

Note that since the top Chern class of a holomorphic orbifold bundle can be represented by a closed (r, r)-form, where r is the rank, it follows that $\alpha_1 \cup \alpha_2$ lies in $H^{p+p',q+q'}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$.

The following theorem can be similarly proved.

Theorem 4.21 *Let $\mathcal{G}$ be an n-dimensional closed complex orbifold with complex structure J. The orbifold cup product*

$$\cup : H^{p,q}_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) \otimes H^{p',q'}_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) \to H^{p+p',q+q'}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$$

defined above has the following properties:

1. *The total Chen–Ruan Dolbeault cohomology group is a ring with unit $e^0_{\mathcal{G}} \in H^{0,0}_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$ under $\cup$, where $e^0_{\mathcal{G}}$ is the class represented by the constant function 1 on $\mathcal{G}$.*
2. *The cup product $\cup$ is invariant under deformations of J.*
3. *When $\mathcal{G}$ has integral degree shifting numbers, the total Chen–Ruan Dolbeault cohomology group of $\mathcal{G}$ is integrally graded, and we have supercommutativity*

$$\alpha_1 \cup \alpha_2 = (-1)^{\deg\alpha_1 \cdot \deg\alpha_2} \alpha_2 \cup \alpha_1.$$

4. *Restricted to the non-twisted sectors, i.e., the ordinary Dolbeault cohomology $H^{*,*}(\mathcal{G};\mathbb{C})$, the cup product $\cup$ equals the ordinary wedge product on $\mathcal{G}$.*
5. *When $\mathcal{G}$ is Kähler and closed, the cup product $\cup$ coincides with the orbifold cup product over the Chen–Chuan cohomology groups $H^*_{\mathrm{CR}}(\mathcal{G};\mathbb{C})$ under the relation*

$$H^r_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) = \oplus_{p+q=r} H^{p,q}_{\mathrm{CR}}(\mathcal{G};\mathbb{C}).$$

Theorem 4.22 *The Chen–Ruan product is invariant under Morita equivalence and hence depends on only the orbifold structure, not the presentation.*

The proof of this second theorem follows easily from the fact that $\overline{\mathcal{M}}_3(\mathcal{G}) = \mathcal{G}^2$ is invariant under Morita equivalence. The Chen–Ruan product is also associative. We refer the interested reader to the proof in [37].

The expected relationship between an orbifold's Chen–Ruan cohomology and the cohomology ring of its crepant resolution is summarized in the following two conjectures due to Ruan [133]. A complex analytic variety X with only quotient singularities carries a natural orbifold structure. We also use X to denote this orbifold structure. The singularities may be resolved in algebro-geometric fashion.

Definition 4.23 A crepant resolution $\pi : Y \to X$ is called *hyperkähler*, respectively *holomorphic symplectic*, if Y is hyperkähler, respectively holomorphic symplectic.

Conjecture 4.24 (Cohomological Hyperkähler Resolution Conjecture) *Suppose that $\pi : Y \to X$ is a hyperkähler or holomorphic symplectic resolution. Then, $H^*_{\mathrm{CR}}(X;\mathbb{C})$ is isomorphic as a ring to $H^*(Y;\mathbb{C})$.*

An important example in which the conjecture has been verified is the Hilbert scheme of points of an algebraic surface. This is a crepant resolution of the symmetric product (defined in Chapter 1) of the algebraic surface. Special cases of this conjecture were proved by Lehn and Sorger [92] for symmetric products of $\mathbb{C}^2$, by Lehn and Sorger [93], Fantechi and Göttsche [52], Uribe [153] for symmetric products of $K3$ or T^4, by Li, Qin, and Wang [100] for symmetric products of the cotangent bundle $T^*\Sigma$ of a Riemann surface, and by Qin and Wang [127] for the minimal resolutions of Gorenstein surface singularities $\widetilde{\mathbb{C}^2/\Gamma}$. We will discuss these examples in Chapter 5.

The hyperkähler condition is meant to ensure the vanishing of Gromov–Witten invariants. When Y is not hyperkähler, there is another conjecture.

Conjecture 4.25 (Cohomological Crepant Resolution Conjecture) *Suppose that $\pi : Y \to X$ is a crepant resolution. Then, $H^*_{\mathrm{CR}}(X;\mathbb{C})$ is isomorphic as a ring to* Ruan cohomology $H^*_\pi(Y;\mathbb{C})$*, where the product $\alpha \cup_\pi \beta$ in Ruan cohomology is defined as $\alpha \cup \beta$ plus a correction coming from the Gromov–Witten invariants of exceptional rational curves.*

For the explicit definition of Ruan cohomology, the reader is referred to [133]. The cohomological crepant resolution conjecture was proved for twofold symmetric products of algebraic surfaces by Li and Qin [95].

4.4 Some elementary examples

Before we discuss more sophisticated examples, such as symmetric products, let us compute some elementary ones.

Example 4.26 The easiest example is $\mathcal{G} = \bullet^G$. In this case, a sector looks like $\mathcal{G}_{(g)} = \bullet^{\mathbf{C}(g)}$. Hence, Chen–Ruan cohomology is generated by conjugacy classes of elements of G. We choose a basis $\{x_{(g)}\}$ for the Chen–Ruan cohomology group, where $x_{(g)}$ is given by the constant function 1 on $\mathcal{G}_{(g)}$. All the degree shifting numbers are zero. The Poincaré pairing is

$$\langle x_{(g)}, x_{(g^{-1})}\rangle = \frac{1}{|\,\mathbf{C}(g)|}.$$

Let us consider the cup product. First, we observe that the multisectors correspond to intersections of centralizers: $\mathcal{G}_{(g_1, g_2, (g_1 g_2)^{-1})} = \bullet^{\mathbf{C}(g_1)\cap\mathbf{C}(g_2)}$.

By the Decomposition Lemma,

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{\substack{(h_1,h_2)\\ h_1\in(g_1),h_2\in(g_2)}} d_{(h_1,h_2)} x_{(h_1 h_2)},$$

where (h_1, h_2) is the conjugacy class of the pair, and the coefficient $d_{(h_1,h_2)}$ is defined by the equation

$$d_{(h_1,h_2)}\langle x_{(h_1 h_2)}, x_{(h_1 h_2)^{-1}}\rangle = \frac{1}{|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|}.$$

Using the formula for the Poincaré pairing, we obtain $d_{(h_1,h_2)} = |\,\mathbf{C}(h_1 h_2)|/|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|$.

On the other hand, recall that the center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$ is generated by elements $\tau_{(g)} = \sum_{h\in(g)} h$. We can write down the multiplication formula for $\tau_{(g_1)} \star \tau_{(g_2)} = \sum_{h\in(g_1)} h \sum_{t\in(g_1)} t$ and rewrite the result in terms of the generators $\tau_{(g)}$. It is convenient to group the h, t in terms of conjugacy classes of pairs. If $h_1 = s h_2 s^{-1}, t_1 = s t_2 s^{-1}$, then $h_1 t_1 = s h_2 t_2 s^{-1}$. Namely, their multiplications are conjugate. Therefore, we can write

$$\tau_{(g_1)} \star \tau_{(g_2)} = \sum_{(h_1,h_2),h_i\in(g_i)} \lambda_{(h_1,h_2)} \tau_{(h_1 h_2)}.$$

When we run through the $s \in G$, we obtain only $|G|/|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|$ many distinct pairs of h, t conjugate to (h_1, h_2). The conjugacy class $(h_1 h_2)$ contains $|G|/|\,\mathbf{C}(h_1 h_2)|$ many elements. Therefore, it generates $(|G|/|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|)/(|G|/|\,\mathbf{C}(h_1 h_2)|) = |\,\mathbf{C}(h_1 h_2)|/|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|$ many $\tau_{(h_1 h_2)}$. Explicitly,

$$\lambda_{(h_1,h_2)} = \frac{|\,\mathbf{C}(h_1 h_2)|}{|\,\mathbf{C}(h_1)\cap\mathbf{C}(h_2)|} = d_{(h_1,h_2)}.$$

Therefore, we obtain an explicit ring isomorphism $H_{\mathrm{CR}}(\bullet^G;\mathbb{C}) \cong Z(\mathbb{C}G)$ by sending $x_{(g)} \to \tau_{(g)}$.

Example 4.27 Suppose that $G \subset SL(n,\mathbb{C})$ is a finite subgroup. Then $\mathcal{G} = G \ltimes \mathbb{C}^n$ is an orbifold groupoid presentation of the global quotient $\mathbb{C}^n/G$. The twisted sectors correspond to fixed point sets: i.e., $\mathcal{G}_{(g)} = (\mathbb{C}^n)^g / \mathbf{C}(g)$, where $(\mathbb{C}^n)^g$ is the subspace fixed by g. So

$$H^{p,q}(\mathcal{G}_{(g)};\mathbb{C}) = \begin{cases} 0, & \text{if } p \text{ or } q \text{ greater than zero,} \\ \mathbb{C}, & \text{if } p = q = 0. \end{cases}$$

Therefore, $H^{p,q}_{\mathrm{CR}}(\mathcal{G}) = 0$ for $p \neq q$, and $H^{p,p}_{\mathrm{CR}}(\mathcal{G})$ is a vector space generated by the conjugacy classes of elements g with $\iota_{(g)} = p$. Consequently, there is a natural additive decomposition:

$$H^*_{\mathrm{CR}}(\mathcal{G};\mathbb{C}) = Z(\mathbb{C}G) = \bigoplus_p H_p, \tag{4.15}$$

where H_p is generated by the conjugacy classes of elements g with $\iota_{(g)} = p$. The ring structure is also easy to describe. Let $x_{(g)}$ be the generator corresponding to the constant function 1 on the twisted sector $\mathcal{G}_{(g)}$. We would like a formula for $x_{(g_1)} \cup x_{(g_2)}$. As we showed before, the multiplication of conjugacy classes can be described in terms of the center $Z(\mathbb{C}G)$ of the group algebra. But in this case, we have further restrictions. Let us first describe the moduli space $\mathcal{G}_{(h_1,h_2,(h_1h_2)^{-1})}$ and its corresponding three-point function. It is clear that

$$\mathcal{G}_{(h_1,h_2,(h_1h_2)^{-1})} = \left((\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2}\right) / \mathbf{C}(h_1,h_2).$$

To have a non-zero product, we need

$$\iota_{(h_1h_2)} = \iota_{(h_1)} + \iota_{(h_2)}.$$

In that case, we need to compute

$$\int_{((\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2})/\mathbf{C}(h_1,h_2)} e_3^*(\mathrm{vol_c}((\mathbb{C}^n)^{h_1h_2})) \wedge \mathbf{e}(E), \tag{4.16}$$

where $\mathrm{vol_c}(X_{h_1h_2})$ is the compactly supported, $\mathbf{C}(h_1h_2)$-invariant, top form with volume 1 on $(\mathbb{C}^n)^{h_1h_2}$. We also view this volume form as a form on $(\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2}/(\mathbf{C}(h_1) \cap \mathbf{C}(h_2))$. However,

$$(\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2} \subset (\mathbb{C}^n)^{h_1h_2}$$

is a submanifold. It follows that the integral in (4.16) is zero unless

$$(\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2} = (\mathbb{C}^n)^{h_1h_2}.$$

When this happens, we call the pair (h_1, h_2) *transverse*. For transverse pairs, the obstruction bundle is clearly trivial. Let

$$I_{g_1,g_2} = \{(h_1, h_2) \mid h_i \in (g_i), \iota_{(h_1)} + \iota_{(h_2)} = \iota_{(h_1 h_2)}, (h_1, h_2) \text{ is transverse}\}.$$

Finally, applying the Decomposition Lemma, we find that

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2)\in I_{g_1,g_2}} d_{(h_1,h_2)} x_{(h_1 h_2)}.$$

A computation similar to the one in the last example yields $d_{(h_1,h_2)} = |\mathbf{C}(h_1 h_2)|/|\mathbf{C}(h_1) \cap \mathbf{C}(h_2)|$.

Example 4.28 The examples we have computed so far are global quotients. Weighted projective spaces (which first appeared as Example 1.15) provide the easiest examples of non-global quotient orbifolds. Consider the weighted projective space $\mathbf{W}\mathbb{P}(w_1, w_2)$, where w_1 and w_2 are coprime integers. For instance, Thurston's famous teardrop is $\mathbf{W}\mathbb{P}(1, n)$. Although it is not a global quotient, the orbifold $\mathbf{W}\mathbb{P}(w_1, w_2)$ can be presented as the quotient of $\mathbb{S}^3$ by $\mathbb{S}^1$, where $\mathbb{S}^1$ acts on the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ by

$$e^{i\theta}(z_1, z_2) = (e^{iw_1\theta} z_1, e^{iw_2\theta} z_2).$$

$\mathbf{W}\mathbb{P}(w_1, w_2)$ is an orbifold $\mathbb{S}^2$, with two singular points $x = [1, 0]$ and $y = [0, 1]$ of order w_1 and w_2, respectively. These give rise to $(w_1 - 1) + (w_2 - 1)$ twisted sectors, each indexed by a non-identity element of one of the isotropy subgroups – since the isotropy subgroups are abelian, conjugacy classes are singletons. Each twisted sector coming from x is an orbifold point with isotropy $\mathbb{Z}/w_1\mathbb{Z}$, and the sectors over y are points with $\mathbb{Z}/w_2\mathbb{Z}$ isotropy. The degree shifting numbers are i/w_2, j/w_1 for $1 \le i \le w_2 - 1$ and $1 \le j \le w_1 - 1$. Hence, the Betti numbers of the non-zero Chen–Ruan cohomology groups are

$$h^0 = h^2 = h^{\frac{2i}{w_2}} = h^{\frac{2j}{w_1}} = 1.$$

Note that the Chen–Ruan cohomology classes corresponding to the twisted sectors have rational degrees. Let $\alpha \in H_{\mathrm{CR}}^{\frac{2}{w_1}}(\mathcal{G}; \mathbb{C})$ and $\beta \in H_{\mathrm{CR}}^{\frac{2}{w_2}}(\mathcal{G}; \mathbb{C})$ be the generators corresponding to the constant function 1 on the sectors corresponding to generators of the two cyclic isotropy subgroups. An easy computation shows that Chen–Ruan cohomology is generated by the elements $\{1, \alpha^j, \beta^i\}$ with relations

$$\alpha^{w_1} = \beta^{w_2}, \alpha^{w_1+1} = \beta^{w_2+1} = 0.$$

The Poincaré pairing is given by

$$\langle \beta^i, \alpha^j \rangle_{CR} = 0,$$
$$\langle \beta^{i_1}, \beta^{i_2} \rangle_{CR} = \delta_{i_1, d_2 - i_2},$$

and

$$\langle \alpha^{j_1}, \alpha^{j_2} \rangle_{CR} = \delta_{j_1, d_1 - j_2}.$$

for $1 \leq i_1, i_2, i < w_2 - 1$ and $1 \leq j_1, j_2, j < w_1 - 1$.

4.5 Chen–Ruan cohomology twisted by a discrete torsion

A large part of the ongoing research in the orbifold field concerns various twisting processes. These twistings in orbifold theories are intimately related to current developments in twisted K-theory, as we mentioned in Chapter 3. In this book, we will discuss twisting by a discrete torsion, as this part of the story has been understood relatively well. Physically, discrete torsion measures the freedom with which one can choose certain phase factors. These are to be used to weight the path integral over each twisted sector, but must be chosen so as to maintain the consistency of the string theory.

The twisting process is interesting for many reasons. For example, the following conjecture of Vafa and Witten connects twisting with geometry. Recall from the end of Chapter 1 that there are two algebro-geometric methods to remove singularities: resolution and deformation. Both play important roles in the theory of Calabi–Yau 3-folds. A smooth manifold Y obtained from an orbifold $\mathcal{X}$ via a sequence of resolutions and deformations is called a *desingularization* of $\mathcal{X}$. In string theory, we additionally require all the resolutions to be crepant. It is known that such a smooth desingularization may not exist in dimensions higher than 3. In this case, we allow our desingularization to be an orbifold. In any case, the Chen–Ruan cohomology of $\mathcal{X}$ should correspond to that of the crepant resolution. Vafa and Witten [155] proposed that discrete torsions count the number of distinct topological types occurring among the desingularizations. However, this proposal immediately ran into trouble, because the number of desingularizations is sometimes much larger than the number of discrete torsions. Specifically, Joyce [75] constructed five different desingularizations of $\mathbb{T}^6/(\mathbb{Z}/4\mathbb{Z})$, while $H^2(\mathbb{Z}/4\mathbb{Z}; U(1)) = 0$. Accounting for these “extra” desingularizations is still an unresolved question.

Suppose that $f : \mathcal{Y} \to \mathcal{X}$ is an orbifold universal cover, and let $G = \pi_1^{\mathrm{orb}}(\mathcal{X})$ be the orbifold fundamental group. Then G acts on $\mathcal{Y}$ such that $\mathcal{X} = \mathcal{Y}/G$. Any

non-identity element $g \in G$ acts on $\mathcal{Y}$ as an orbifold morphism. The *orbifold fixed point set* $\mathcal{Y}^g$ of $\mathcal{Y}$ under g is the fiber product of the morphisms Id, $g : \mathcal{Y} \to \mathcal{Y}$. The set $\mathcal{Y}^g$ is thus a smooth suborbifold of $\mathcal{Y}$. What is more, $\mathcal{X}_{(g)} = \mathcal{Y}^g / \mathbf{C}(g)$ is obviously a twisted sector of $\mathcal{X}$, where $\mathbf{C}(g)$ is the centralizer of g in G. It is clear that $\mathcal{Y}^{h^{-1}gh}$ is diffeomorphic to $\mathcal{Y}^g$ under the action of h. However, some twisted sectors of $\mathcal{X}$ may not arise in such a fashion. We call this kind of sector a *dormant sector*.

Let $e : \mathcal{X}_{(g)} \to \mathcal{X}$ be the evaluation map. We can view $\mathcal{Y} \to \mathcal{X}$ as an orbifold principal G-bundle over $\mathcal{X}$. Hence, we can pull back to get an orbifold principal G-bundle $\mathcal{Z} = e^*\mathcal{Y} \to \mathcal{X}_{(g)}$ over $\mathcal{X}_{(g)}$. Then $\mathcal{X}_{(g)}$ is dormant if and only if the G-action on $\mathcal{Z}$ has no kernel. Moreover, $\mathcal{Z}$ is a G-invariant suborbifold of $\mathcal{Y}$ (possibly disconnected). We call $\mathcal{Z}$ a $\pi_1^{\mathrm{orb}}(\mathcal{X})$-*effective* suborbifold. The idea will be to treat the dormant sectors the same as the non-twisted sector.

Recall that a discrete torsion α is a 2-cocycle, i.e., $\alpha \in Z^2(G; U(1))$. For each $g \in G$, the cocycle α defines a function $\gamma_g^\alpha : G \to U(1)$ by $\gamma_g^\alpha(h) = \alpha(g, h)\alpha(ghg^{-1}, g)^{-1}$. When restricted to $\mathbf{C}(g)$, we recover the character used in the last chapter, which was defined to be $\alpha(g, h)\alpha(h, g)^{-1}$.

We can use γ_g^α to define a flat complex orbifold line bundle

$$L_{(g)}^\alpha = \mathcal{Y}^g \times_{\gamma_g^\alpha} \mathbb{C} \tag{4.17}$$

over $\mathcal{X}_{(g)}$. For a dormant sector or the non-twisted sector, we always assign a trivial line bundle. Let $\mathcal{L}^\alpha = \{L_{(g)}^\alpha\}$.

Definition 4.29 We define the *α-twisted Chen–Ruan cohomology group* to be

$$H_{\mathrm{CR}}^*(\mathcal{X}; \mathcal{L}^\alpha) = \bigoplus_{(g) \in T_1} H^*(|\mathcal{X}_{(g)}|; L_{(g)}^\alpha)[-2\iota_{(g)}], \tag{4.18}$$

where $|\mathcal{X}_{(g)}|$ is the underlying space of the twisted sector.

An obvious question is whether $H_{\mathrm{CR}}^*(\mathcal{X}; \mathcal{L}^\alpha)$ carries a natural Poincaré pairing and cup product in the same way as the untwisted cohomology $H_{\mathrm{CR}}^*(\mathcal{X}; \mathbb{C})$. A necessary condition is summarized in the following notion.

Definition 4.30 Suppose that $\mathcal{X}$ is an orbifold (almost complex or not). An *inner local system* $\mathcal{L} = \{L_{(g)}\}_{g \in T_1}$ is an assignment of a flat complex orbifold line bundle

$$L_{(g)} \to \mathcal{X}_{(g)}$$

to each sector $\mathcal{X}_{(g)}$, satisfying the following four compatibility conditions:

1. $L_{(1)}$ is a trivial orbifold line bundle with a fixed trivialization.
2. There is a non-degenerate pairing $L_{(g)} \otimes I^*L_{(g^{-1})} \to \mathcal{X}_{(1)} \times \mathbb{C} = L_{(1)}$.

3. There is a multiplication
$$\theta : e_1^* L_{(h_1} \otimes e_2^* L_{(h_2)} \to e_3^* L_{(h_1 h_2)}$$
over $\mathcal{X}_{(h_1,h_2)}$ for $(h_1, h_2) \in T_2$.
4. The multiplication θ is associative in the following sense. Let $(h_1, h_2, h_3) \in T_3$, and set $h_4 = h_1 h_2 h_3$. For each i, the evaluation maps $e_i : \mathcal{X}_{(h_1,h_2,h_3)} \to \mathcal{X}_{(h_i)}$ factor through
$$P = (P_1, P_2) : \mathcal{X}_{(h_1,h_2,h_3)} \to \mathcal{X}_{(h_1,h_2)} \times \mathcal{X}_{(h_1 h_2,h_3)}.$$
Let $e_{12} : \mathcal{X}_{(h_1,h_2,h_3)} \to \mathcal{X}_{(h_1 h_2)}$. We first use P_1 to define
$$\theta : e_1^* L_{(h_1)} \otimes e_2^* L_{(h_2)} \to e_{12}^* L_{(h_1 h_2)}.$$
Then, we can use P_2 to define a product
$$\theta : e_{12}^* L^*_{(h_1 h_2)} \otimes L^*_{(h_3)} \to e_4^* L^*_{(h_1 h_2 h_3)}$$
on the pullbacks of the dual line bundles. Taking the composition, we define
$$\theta(\theta(e_1^* L_{(h_1)}, e_2^* L_{(h_2)}), e_3^* L_{(h_3)}) : e_1^* L_{(h_1)} \otimes e_2^* L_{(h_2)} \otimes e_3^* L_{(h_3)} \to e_4^* L^*_{(h_4)}.$$
On the other hand, the evaluation maps e_i also factor through
$$P' : \mathcal{X}_{(h_1,h_2,h_3)} \to \mathcal{X}_{(h_1,h_2 h_3)} \times \mathcal{X}_{(h_2,h_3)}.$$
In the same way, we can define another triple product
$$\theta(e_1^* L_{(h_1)}, \theta(e_2^* L_{(h_2)}, e_3^* L_{(h_3)})) : e_1^* L_{(h_1)} \otimes e_2^* L_{(h_2)} \otimes e_3^* L_{(h_3)} \to e_4^* L^*_{(h_4)}.$$
Consequently, we require the associativity
$$\theta(\theta(e_1^* L_{(h_1)}, e_2^* L_{(h_2)}), e_3^* L_{(h_3)}) = \theta(e_1^* L_{(h_1)}, \theta(e_2^* L_{(h_2)}, e_3^* L_{(h_3)})).$$

If $\mathcal{X}$ is a complex orbifold, we will assume that each $L_{(g)}$ is holomorphic.

Definition 4.31 Given an inner local system $\mathcal{L}$, we define the *$\mathcal{L}$-twisted Chen–Ruan cohomology groups* to be
$$H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}) = \bigoplus_{(g)} H^{*-2\iota_{(g)}}(\mathcal{X}_{(g)}; L_{(g)}). \tag{4.19}$$

Suppose that $\mathcal{X}$ is a closed complex orbifold and $\mathcal{L}$ is an inner local system. We define *$\mathcal{L}$-twisted Chen–Ruan Dolbeault cohomology groups* to be
$$H^{p,q}_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}) = \bigoplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(\mathcal{X}_{(g)}; L_{(g)}). \tag{4.20}$$

One can check that the construction of the Poincaré pairing and cup product go through without change for $H^*_{\mathrm{CR}}(\mathcal{X};\mathcal{L})$. Hence, we have the following two propositions.

Proposition 4.32 *Suppose that $\mathcal{L}$ is an inner local system. Then $H^*_{\mathrm{CR}}(\mathcal{X};\mathcal{L})$ carries a Poincaré pairing and an associative cup product in the same way as $H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$.*

Proposition 4.33 *If $\mathcal{X}$ is a Kähler orbifold, we have the Hodge decomposition*

$$H^k_{\mathrm{CR}}(\mathcal{X};\mathcal{L}) = \bigoplus_{k=p+q} H^{p,q}_{\mathrm{CR}}(\mathcal{X};\mathcal{L}).$$

To obtain a product structure on cohomology twisted by a discrete torsion, we need only prove the following theorem.

Theorem 4.34 *For a discrete torsion α, the collection of line bundles $\mathcal{L}^\alpha$ forms an inner local system.*

Proof As an orbifold, the inertia orbifold $\sqcup_{(g)\in T_1}\mathcal{X}_{(g)}$ is the quotient of the disjoint union of $(\sqcup_{g\in\pi_1^{\mathrm{orb}}(\mathcal{X})}Y^g)$ and $\pi_1^{\mathrm{orb}}(\mathcal{X})$-effective suborbifolds by the action of $\pi_1^{\mathrm{orb}}(\mathcal{X})$. We work directly on $\sqcup_{g\in\pi_1^{\mathrm{orb}}(\mathcal{X})}Y^g$ to simplify the notation [80], since for a $\pi_1^{\mathrm{orb}}(\mathcal{X})$-effective suborbifold $\mathcal{Z}$, the line bundle is always trivial. In this case, we denote its fiber by $\mathbb{C}_1$ and treat it the same as the non-twisted sector. For a fixed point set Y^g, the line bundle is a trivial bundle denoted by $Y^g\times\mathbb{C}_g$. Next, we want to build the pairing and product, but we must do so in a fashion invariant under the action of $\pi_1^{\mathrm{orb}}(\mathcal{X})$. We first describe the action of $G=\pi_1^{\mathrm{orb}}(\mathcal{X})$ on our line bundles. Let $1_h\in\mathbb{C}_h$ be the identity. For each $g\in G$, we define $g:\mathbb{C}_h\to\mathbb{C}_{ghg^{-1}}$ by $g(1_h)=\gamma_g^\alpha(h)1_{ghg^{-1}}$. To show that this defines an action, we need to check that $gk(1_h)=g(k(1_h))$; this is the content of the following lemma.

Lemma 4.35 $\gamma^\alpha_{gk}(h)=\gamma^\alpha_g(khk^{-1})\gamma^\alpha_k(h)$.

Proof of Lemma 4.35 Recall that the cocycle condition for α is

$$\alpha(x,y)\alpha(xy,z)=\alpha(x,yz)\alpha(y,z). \tag{4.21}$$

Using this, we calculate:

$$\gamma^\alpha_{gk}(h)=\alpha(gk,h)\alpha(gkhk^{-1}g^{-1},gk)^{-1},$$
$$\alpha(gkhk^{-1}g^{-1},gk)^{-1}=\alpha(gkhk^{-1}g^{-1},g)^{-1}\alpha(gkhk^{-1},k)^{-1}\alpha(g,k),$$

and

$$\alpha(gkhk^{-1}, k)^{-1} = \alpha(g, kh)^{-1}\alpha(khk^{-1}, k)^{-1}\alpha(g, khk^{-1}).$$

Putting this together and applying the cocycle condition to $\alpha(gk, h)$, we obtain

$$\begin{aligned}\gamma^{\alpha}_{gk}(h) &= \alpha(k, h)\alpha(gkhk^{-1}g^{-1}, g)^{-1}\alpha(khk^{-1}, k)^{-1}\alpha(g, khk^{-1})\\ &= \gamma^{\alpha}_{g}(khk^{-1})\gamma^{\alpha}_{k}(h).\end{aligned}$$

□

The product $\mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}$ is defined by $1_g \cdot 1_h = \alpha_{g,h} 1_{gh}$. The associativity of the product follows from the cocycle condition (4.21). Note that the product gives $1_g \cdot 1_{g^{-1}} = \alpha(g, g^{-1})1_1$. This is non-degenerate, since $\alpha(g, g^{-1}) \in U(1)$.

We still have to check that the product is invariant under the $\pi_1^{\mathrm{orb}}(\mathcal{X})$-action, i.e.,

$$g(1_h) \cdot g(1_k) = \alpha(h, k)g(1_{hk}).$$

Using the definition of the action, this is equivalent to the formula

$$\gamma^{\alpha}_{g}(h)\gamma^{\alpha}_{g}(k)\alpha(ghg^{-1}, gkg^{-1}) = \alpha(h, k)\gamma^{\alpha}_{g}(hk),$$

which in turn is equivalent to the next lemma.

Lemma 4.36 *We have*

$$\begin{aligned}&\alpha(g, h)\alpha(ghg^{-1}, g)^{-1}\alpha(g, k)\alpha(gkg^{-1}, g)\alpha(ghg^{-1}, gkg^{-1})\\ &\quad = \alpha(h, k)\alpha(g, hk)\alpha(ghkg^{-1}, g)^{-1}.\end{aligned}$$

Proof of Lemma 4.36 Again, we need only calculate with the cocycle condition (4.21):

$$\alpha(g, h)\alpha(gh, k) = \alpha(g, hk)\alpha(h, k),$$

$$\alpha(ghg^{-1}, g)^{-1}\alpha(g, k)\alpha(ghg^{-1}, gk) = \alpha(gh, k),$$

and

$$\alpha(ghg^{-1}, gkg^{-1})\alpha(ghg^{-1}, gk)^{-1}\alpha(gkg^{-1}, g)^{-1} = \alpha(ghkg^{-1}, g)^{-1}.$$

Multiplying all three equations together, we obtain the lemma. □

Finally, dividing by the action of $\pi_1^{\mathrm{orb}}(\mathcal{X})$, we obtain the theorem. □

Suppose that α and α' differ by a coboundary, i.e., $\alpha'(g, h) = \alpha(g, h)\rho(g)\rho(h)\rho(gh)^{-1}$. Then $\gamma^{\alpha'}_{g} = \gamma^{\alpha}_{g}$, and furthermore, $1_g \to \rho(g)1_g$ maps the pairing and product coming from α to those of α'.

Proposition 4.37 *The inner local system $\mathcal{L}^{\alpha'}$ is isomorphic to $\mathcal{L}^{\alpha}$ in the above sense. In particular, $H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}^{\alpha'})$ is isomorphic to $H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}^{\alpha})$.*

Corollary 4.38 *$H^1(\pi_1^{\mathrm{orb}}(\mathcal{X}); U(1))$ acts on $H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}^{\alpha})$ by automorphisms.*

One should mention that inner local systems are more general than discrete torsion. For example, inner local systems capture all of Joyce's desingularizations of $T^6/\mathbb{Z}_4$, whereas discrete torsion does not. Gerbes [103, 104] provide another interesting approach to twisting, although we will not discuss this here.

We conclude this chapter by revisiting our earlier calculations in the presence of twisting.

Example 4.39 Let us reconsider (see Example 4.26) the case where $\mathcal{X} = \bullet^G$ is a point with a trivial action of the finite group G. Suppose that $\alpha \in Z^2(G; U(1))$ is a discrete torsion. We want to compute $H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}^{\alpha})$. The twisted sector $\mathcal{X}_{(g)}$ is a point with isotropy $\mathbf{C}(g)$. It is obvious that $H^0(\mathcal{X}_{(g)}; L^{\alpha}_g) = 0$ unless the character $\gamma^{\alpha}_g : \mathbf{C}(g) \to U(1)$ is trivial. Recall that a conjugacy class (g) is α-*regular* if and only if $\gamma^{\alpha}_g \equiv 1$ on the centralizer. Hence, precisely the α-regular classes will contribute. Therefore, the α-twisted Chen–Ruan cohomology is generated by α-regular conjugacy classes of elements of G. This is also the case for the center of the twisted group algebra $\mathbb{C}^{\alpha}G$, which was defined in Section 3.5. Working from the definitions, it is clear that the Chen–Ruan product corresponds precisely to multiplication in the twisted group algebra. Therefore, as a ring $H^*_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}^{\alpha})$ is isomorphic to $Z(\mathbb{C}^{\alpha}G)$.

Example 4.40 Suppose that $G \subset SL(n, \mathbb{C})$ is a finite subgroup. Then, $\mathbb{C}^n/G$ is an orbifold (see Example 4.27). Suppose that $\alpha \in Z^2(G; U(1))$ is a discrete torsion. For any $g \in G$, the fixed-point set $(\mathbb{C}^n)^g$ is a vector subspace, and $\mathcal{X}_{(g)} = (\mathbb{C}^n)^g/\mathbf{C}(g)$. By definition, $L^{\alpha}_{(g)} = (\mathbb{C}^n)^g \times_{\gamma^{\alpha}_g} \mathbb{C}$. Therefore, $H^*(\mathcal{X}_{(g)}; L^{\alpha}_{(g)})$ is the subspace of $H^*((\mathbb{C}^n)^g; \mathbb{C})$ invariant under the twisted action of $\mathbf{C}(g)$:

$$h \circ \beta = \gamma^{\alpha}_g(h) h^* \beta$$

for any $h \in \mathbf{C}(g)$ and $\beta \in H^*((\mathbb{C}^n)^g; \mathbb{C})$. However, $H^i((\mathbb{C}^n)^g; \mathbb{C}) = 0$ for $i \geq 1$. Moreover, if γ^{α}_g is non-trivial, $H^0(\mathcal{X}_{(g)}; L^{\alpha}_{(g)}) = 0$. Therefore, $H^{p,q}_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}_{\alpha}) = 0$ for $p \neq q$ and $H^{p,p}_{\mathrm{CR}}(\mathcal{X}; \mathcal{L}_{\alpha})$ is a vector space generated by the conjugacy classes of α-regular elements g with $\iota_{(g)} = p$. Consequently, we have a natural decomposition

$$H^*_{CR}(\mathcal{X}; \mathcal{L}_{\alpha}) = Z(\mathbb{C}^{\alpha}G) = \sum_p H_p, \tag{4.22}$$

where H_p is generated by the conjugacy classes of α-regular elements g with $\iota_{(g)} = p$. The ring structure is also easy to describe. For each α-regular g, let $x_{(g)}$ be the generator corresponding to the degree zero cohomology class of the twisted sector $\mathcal{X}_{(g)}$. The cup product is then exactly the same as in the untwisted case, except that we replace conjugacy classes by α-conjugacy classes, and multiplication in the group algebra by multiplication in the twisted group algebra.

5
Calculating Chen–Ruan cohomology

From the construction of Chen–Ruan cohomology, it is clear that the only non-topological datum is the obstruction bundle. This phenomenon is also reflected in calculations. That is, it is fairly easy to compute Chen–Ruan cohomology so long as there is no contribution from the obstruction bundle, but when the obstruction bundle does contribute, the calculation becomes more subtle. In such situations it is necessary to develop new technology. During the last several years, many efforts have been made to perform such calculations. So far, major success has been achieved in two special cases: *abelian orbifolds* (such as toric varieties) and *symmetric products*. For both these sorts of orbifolds, we have elegant – and yet very different – solutions.

5.1 Abelian orbifolds

An orbifold is *abelian* if and only if each local group G_x is an abelian group. Abelian orbifolds constitute a large class of orbifolds, and include toric varieties and complete intersections of toric varieties. Such orbifolds were the first class of examples to be studied extensively. Immediately after Chen and Ruan introduced their cohomology, Poddar [123] identified the twisted sectors of toric varieties and their complete intersections. There followed a series of works on abelian orbifolds by Borisov and Mavlyutov [28], Park and Poddar [122], Jiang [74], and Borisov, Chen, and Smith [26]. Chen and Hu [35] introduced an elegant de Rham model for abelian orbifolds that enabled them to compute the Chen–Ruan cohomology of such orbifolds routinely. They then applied this de Rham model to such problems as Kirwan surjectivity and wall-crossing formulae. Here, we will present their de Rham model, closely following their exposition. We refer the reader to their paper for the applications.

5.1.1 The de Rham model

Recall that the inertia orbifold $\wedge\mathcal{G}$ is a suborbifold of $\mathcal{G}$ via the embedding $e : \wedge\mathcal{G} \to \mathcal{G}$, where on the objects $(\wedge\mathcal{G})_0 = \{g \in \mathcal{G}_1 \mid s(g) = t(g)\}$ the embedding is given by $e(g) = s(g) = t(g)$. We can consider $e^*T\mathcal{G}$ and the normal bundle $N_{\wedge\mathcal{G}|\mathcal{G}}$. Let $g \in \wedge\mathcal{G}_0$. Then g acts on the fiber $e^*T_x\mathcal{G}_0$, where $x = s(g) = t(g)$. We decompose $e^*T_x\mathcal{G}_0 = \bigoplus_j E_j$ as a direct sum of eigenspaces, where E_j has eigenvalue $e^{2\pi i \frac{m_j}{m}}$ (m the order of g), and we order the indices so that $m_i \leq m_j$ if $i \leq j$. Incidentally, $\iota_{(g)} = \sum m_j/m$ is the degree shifting number. Suppose that $v \in \mathcal{G}_1$ is an arrow with $s(v) = x$. Then, viewed as an arrow in $(\wedge\mathcal{G})_1$, v connects g with vgv^{-1}. The differential of the local diffeomorphism associated to v maps $(E_j)_g$ to an eigenspace with the same eigenvalue. When the eigenvalues have multiplicity greater than 1, this map might not preserve the splitting into one-dimensional eigenspaces. To simplify notation, we assume that it does preserve the splitting for each v. In that case, the E_j form a line bundle over $\wedge\mathcal{G}$ for each j. The arguments of this section can be extended to the general case without much extra difficulty. In the first step of our calculation, we wish to formally construct a Thom form using fractional powers of the Thom forms θ_j of the E_j. The result should be compactly supported in a tubular neighborhood of $\wedge\mathcal{G}$.

Definition 5.1 Suppose that $\mathcal{G}_{(g)}$ is a twisted sector. The *twisted factor* $t(g)$ of $\mathcal{G}_{(g)}$ is defined to be the formal product

$$t(g) = \prod_{j=1}^{m} \theta_j^{\frac{m_j}{m}}.$$

Here, we use the convention that $\theta_j^0 = 1$ for any j, and that θ_j^1 is the ordinary Thom form of the bundle E_j. Furthermore, we define $\deg(t(g)) = 2\iota_{(g)}$. For any (invariant) form $\omega \in \Omega^*(\mathcal{G}_{(g)})$, the formal product $\omega t(g)$ is called a *twisted form* (or *formal form*) associated with $\mathcal{G}_{(g)}$.

We define the *de Rham complex of twisted forms* by setting

$$\Omega^p_{CR}(\mathcal{G}) = \left\{ \omega_1 t(g_1) + \cdots + \omega_k t(g_k) \mid \sum_i \deg(\omega_i)\deg(t(g_i)) = p \right\}.$$

The coboundary operator d is given by the formula

$$d(\omega_i t(g_i)) = d(\omega_i) t(g_i).$$

It is easy to check that $\{\Omega^*_{\mathrm{CR}}(\mathcal{G}), d\}$ is a chain complex; somewhat provocatively, we denote its cohomology in the same way as Chen–Ruan cohomology:

$$H^*(\{\Omega^*_{\mathrm{CR}}(\mathcal{G}), d\}) = H^*_{\mathrm{CR}}(\mathcal{G}; \mathbb{R}).$$

Note that there are homomorphisms

$$i_{(g)} : H^*(\mathcal{G}_{(g)}; \mathbb{R}) \to H_{\mathrm{CR}}^{*+2\iota(g)}(\mathcal{G}; \mathbb{R}).$$

Summing over the sectors, we obtain an additive isomorphism between the Chen–Ruan cohomology groups as defined in the last chapter and the cohomology of $\Omega^*_{\mathrm{CR}}(\mathcal{G})$. Define the *wedge product* formally by setting

$$\omega_1 t(g_1) \wedge \omega_2 t(g_2) = \omega_1 \wedge \omega_2 t(g_1) t(g_2).$$

Making sense of this formal definition requires the following key lemma.

Lemma 5.2 *$\omega_1 \wedge \omega_2 t(g_1) t(g_2)$ can be naturally identified with an element of $\Omega^*_{\mathrm{CR}}(\mathcal{G})$.*

Proof Consider the orbifold intersection of $\mathcal{G}_{(g_1)}$ and $\mathcal{G}_{(g_2)}$. This was defined to be the fibered product $\mathcal{G}_{(g_1)}\ {}_e\!\times_e \mathcal{G}_{(g_2)}$. Such intersections are possibly disconnected, and sit inside $\mathcal{G}^2 = \wedge\mathcal{G}\ {}_e\!\times_e \wedge\mathcal{G}$. The latter has components of the form $\mathcal{G}_{(h_1,h_2)}$; the components corresponding to our intersection are labeled by those equivalence classes of pairs (h_1, h_2) such that h_i is in the equivalence class (g_i) for $i = 1, 2$. Note that although all local groups are abelian (and so conjugacy classes are singletons), the equivalence classes (g_i) and (h_1, h_2) could still contain multiple elements if the orbifold $\mathcal{G}$ is not simply connected.

We have embeddings $e_1, e_2 : \mathcal{G}^2 \to \wedge\mathcal{G}$. Let $\mathcal{G}^2_{(h_1,h_2)}$ be a component of the intersection. The obvious map $e_{12} : \mathcal{G}^2_{(h_1,h_2)} \to \mathcal{G}_{(h_1h_2)}$ is also an embedding. Now we use the fact that the subgroup generated by h_1 and h_2 is abelian in order to simultaneously diagonalize their actions. The normal bundle $N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}}$ splits as

$$N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}} = N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_1)}} \oplus N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_2)}} \oplus N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_1h_2)}} \oplus N',$$

for some complement N'. Of course, $\mathcal{G}_{(h_i)} = \mathcal{G}_{(g_i)}$ for $i = 1, 2$ by assumption. Let $h_3 = h_1h_2$. We further decompose each of the normal bundles into eigenbundles:

$$N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_i)}} = \bigoplus_{j=1}^{k_i} L_{ij}$$

for $i \in \{1, 2, 3\}$, and

$$N' = \bigoplus_{j=1}^{k} L'_j.$$

The splitting of the normal bundles $N_{\mathcal{G}_{(g_i)}|\mathcal{G}}$ we considered earlier restricts to $\mathcal{G}^2_{(h_1,h_2)}$ in a manner compatible with this new splitting. For example,

$$\begin{aligned}(N_{\mathcal{G}_{(g_1)}|\mathcal{G}})|_{\mathcal{G}^2_{(h_1,h_2)}} &= N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_2)}} \oplus N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_3)}} \oplus N' \\ &= \left(\bigoplus_{j=1}^{k_2} L_{2j}\right) \oplus \left(\bigoplus_{j=1}^{k_3} L_{3j}\right) \oplus \left(\bigoplus_{j=1}^{k} L'_j\right).\end{aligned}$$

It follows that, near $\mathcal{G}^2_{(h_1,h_2)}$,

$$t(g_1) = t_2(h_1)t_3(h_1)t'(h_1),$$

where, for instance, $t_2(h_1)$ is defined as an appropriate formal product of Thom forms for the eigenbundles of $N_{\mathcal{G}^2_{(h_1,h_2)}} \mid \mathcal{G}_{(h_2)}$. Similarly,

$$t(g_2) = t_1(h_2)t_3(h_2)t'(h_2) \text{ and } t(h_3) = t_1(h_3)t_2(h_3)t'(h_3).$$

We are led to consider the formal equation

$$\frac{t(h_1)t(h_2)}{t(h_3)} = \frac{t_2(h_1)t_1(h_2)}{t_1(h_3)t_2(h_3)} t_3(h_1)t_3(h_2) \frac{t'(h_1)t'(h_2)}{t'(h_3)}. \tag{5.1}$$

Chen and Hu [35] note several interesting things about these expressions.

1. Recall that $h_3 = h_1h_2$. It not hard to see that the first fraction simplifies to 1 when restricted to $\mathcal{G}^2_{(h_1,h_2)}$.
2. The term $t_3(h_1)t_3(h_2)$ formally corresponds to the Thom form of $N_{\mathcal{G}^2_{(h_1,h_2)}|\mathcal{G}_{(h_3)}}$. Thus, we "upgrade" it from a formal form to an ordinary differential form.
3. Finally, to understand the term $t'(h_1)t'(h_2)/t'(h_3)$, we consider each L'_j separately. If h_i acts on L'_j as multiplication by $e^{2\pi i \mu_{ij}}$ and the Thom form of L'_j is $[\theta'_j]$, then the exponent of $[\theta'_j]$ in $t(h_i)$ is μ_{ij}. As $h_3 = h_1h_2$, the sum $\mu_{1j} + \mu_{2j}$ is either μ_{3j} or $\mu_{3j} + 1$. We conclude that

$$\frac{[\theta'_j]^{\mu_{1j}}[\theta'_j]^{\mu_{2j}}}{[\theta'_j]^{\mu_{3j}}} = \begin{cases} 1 & \text{if } \mu_{1j} + \mu_{2j} = \mu_{3j}, \\ [\theta'_j], & \text{if } \mu_{1j} + \mu_{2j} = \mu_{3j} + 1. \end{cases}$$

   In either case, the right hand side of the equation can be upgraded to an ordinary differential form when restricted to $\mathcal{G}^2_{(h_1,h_2)}$. Let

$$\Theta_{(h_1,h_2)} = \left.\frac{t'(h_1)t'(h_2)}{t'(h_3)}\right|_{\mathcal{G}^2_{(h_1,h_2)}}$$

   be this restriction. Then in fact we have $[\Theta_{(h_1,h_2)}] = \mathbf{e}(E'_{(h_1,h_2)})$, where

$$E'_{(h_1,h_2)} = \bigoplus_{\theta_{1j}+\theta_{2j}=\theta_{3j}+1} L'_j. \tag{5.2}$$

It follows that near $\mathcal{G}^2_{(h_1,h_2)}$

$$\omega_1 \wedge \omega_2 t(g_1)t(g_2) = \big(e_1^*\omega_1 \wedge e_2^*\omega_2 \wedge \Theta_{(h_1,h_2)} \wedge t_3(h_1)t_3(h_2)\big)\, t(h_3) \qquad (5.3)$$

is a twisted form associated with $\mathcal{G}_{(h_3)}$. By summing up over all the components of the intersection $\mathcal{G}_{(g_1)}\, {}_e\!\times_e \mathcal{G}_{(g_2)}$, we obtain $\omega_1 t(g_1) \wedge \omega_2 t(g_2)$ as an element of $\Omega^*_{\mathrm{CR}}(\mathcal{G})$. In fact, we can say more:

$$\begin{aligned} d(\omega_1 t(g_1) \wedge \omega_2 t(g_2)) &= d(\omega_1 t(g_1)) \wedge \omega_2 t(g_2) \\ &\quad + (-1)^{\deg(\omega_1)\deg(\omega_2)} \omega_1 t(g_1) \wedge d(\omega_2 t(g_2)). \end{aligned}$$

□

This key lemma implies the following corollary.

Corollary 5.3 *The operation* $\wedge$ *induces an associative ring structure on* $H^*(\{\Omega^*_{\mathrm{CR}}(\mathcal{G}), d\}) = H^*_{\mathrm{CR}}(\mathcal{G}; \mathbb{R})$.

We can extend integration to twisted forms $\omega t(g)$ by setting $\int_{\mathcal{G}} \omega t(g) = 0$ unless $t(g)$ is a Thom form. In the latter case, we use the ordinary integration introduced previously. To demonstrate the power of this setup, let us check Poincaré duality. Define the Poincaré pairing on twisted forms by

$$\langle \omega_1 t(g_1), \omega_2 t(g_2) \rangle = \int_{\mathcal{G}} \omega_1 t(g_1) \wedge \omega_2 t(g_2).$$

Note that over each component $\mathcal{G}^2_{(g_1,g_2)}$, the product $t(g_1)t(g_2)$ is strictly formal unless $g_2 = g_1^{-1}$. Moreover, $t(g)t(g^{-1})$ is the ordinary Thom form of $N_{\mathcal{G}_{(g,g^{-1})}|\mathcal{G}}$. Hence, using equation (5.3), the only non-zero term is

$$\begin{aligned} \langle \omega_1 t(g), \omega_2 t(g^{-1}) \rangle = \int_{\mathcal{G}} \omega_1 t(g) \wedge \omega_2 t(g^{-1}) &= \int_{\mathcal{G}^2_{(g,g^{-1})}} e_1^*\omega_1 \wedge e_2^*\omega_2 \\ &= \int_{\mathcal{G}_{(g)}} \omega_1 \wedge I^*\omega_2, \end{aligned}$$

in agreement with our earlier definition in Section 4.2.

Next, we show that the ring structure on $H^*_{\mathrm{CR}}(\mathcal{G})$ induced by the wedge product is the same as the Chen–Ruan product we defined before. Recall that we have identified $\overline{\mathcal{M}}_3(\mathcal{G})$ as the disjoint union of the 3-sectors $\mathcal{G}^3_{(g_1,g_2,g_3)}$ such that $g_1g_2g_3 = 1$. Let $(\mathbf{g}) = (g_1, g_2, g_3)$ with $g_1g_2g_3 = 1$. Since g_3 is determined, we can identify $\mathcal{G}^3_{(\mathbf{g})}$ with $\mathcal{G}^2_{(g_1,g_2)}$.

Theorem 5.4 *Under the above identification, the obstruction bundle* $E_{(\mathbf{g})}$ *(as defined in Section 4.2) corresponds to* $E'_{(g_1,g_2)}$ *(defined as in equation (5.2)).*

Proof Let $y \in \mathcal{G}^3_{(\mathbf{g})}$. By our abelian assumption, the matrices representing the actions of the elements in the subgroup $\langle \mathbf{g} \rangle$ can be simultaneously diagonalized. We make a decomposition:

$$T_y\mathcal{G} = T_y\mathcal{G}^3_{(\mathbf{g})} \oplus (N_{\mathcal{G}^3_{(\mathbf{g})}|\mathcal{G}})_y = T_y\mathcal{G}^3_{(\mathbf{g})} \oplus \bigoplus_{j=1}^{m} (E_j)_y.$$

With respect to this decomposition, we have g_i acting as

$$\mathrm{diag}(1, \dots, 1, e^{2\pi i \theta_{i1}}, \dots, e^{2\pi i \theta_{im}}),$$

where $\theta_{ij} \in \mathbb{Q} \cap [0, 1)$ and $i = 1, 2, 3$.

The fiber of $E_{(\mathbf{g})}$ at y is then

$$\begin{aligned}
(E_{(\mathbf{g})})_y &= (H^{0,1}(\Sigma) \otimes T_y\mathcal{G})^{\langle \mathbf{g} \rangle} \\
&= (H^{0,1}(\Sigma) \otimes T_y\mathcal{G}^3_{(\mathbf{g})})^{\langle \mathbf{g} \rangle} \oplus \bigoplus_{j=1}^{m} (H^{0,1}(\Sigma) \otimes (E_j)_y)^{\langle \mathbf{g} \rangle} \\
&= H^1(S^2, \phi_*(T_y\mathcal{G}^3_{(\mathbf{g})})^{\langle \mathbf{g} \rangle}) \oplus \bigoplus_{j=1}^{m} H^1(S^2, \phi_*((E_j)_y)^{\langle \mathbf{g} \rangle}),
\end{aligned}$$

where $\phi : \Sigma \to S^2$ is the branched covering and ϕ_* is the pushforward of constant sheaves. Let V be a $\langle \mathbf{g} \rangle$-vector space of (complex) rank v and let $m_{i,j} \in \mathbb{Z} \cap [0, r_i)$ be the weights of the action of g_i on V, where r_i is the order of g_i. Applying the index formula (Proposition 4.2.2 in [37]) to $(\phi_*(V))^{\langle \mathbf{g} \rangle}$, we have

$$\chi = v - \sum_{i=1}^{3} \sum_{j=1}^{v} \frac{m_{i,j}}{r_i}.$$

Here, we used the fact that $c_1(\phi_*(V)) = 0$ for a constant sheaf V. Note that if the $\langle \mathbf{g} \rangle$-action is trivial on V, then $\chi = v$. For $V = (E_j)_y$, we see that $v = 1$, and $m_{i,1}/r_i$ is just θ_{ij}.

From this setup, we draw the following two conclusions:

1. $(H^{0,1}(\Sigma) \otimes T_y\mathcal{G}^3_{(\mathbf{g})})^{\langle \mathbf{g} \rangle} = 0$, and
2. $(H^{0,1}(\Sigma) \otimes (E_j)_y)^{\langle \mathbf{g} \rangle}$ is non-trivial ($\Rightarrow$ rank one) $\iff$ $\sum_{i=1}^{3} \theta_{ij} = 2$. (Note that this sum is either 1 or 2.) Moreover, it is clear that

$$(H^{0,1}(\Sigma) \otimes (E_j)_y)^{\langle \mathbf{g} \rangle} \cong (E_j)_y.$$

It follows that

$$E_{(\mathbf{g})} = \bigoplus_{\sum_{i=1}^{3} \theta_{ij}=2} E_j = E'_{(g_1,g_2)}. \tag{5.4}$$

□

It remains to show that the two product structures on $H^*_{CR}(\mathcal{G};\mathbb{R})$ are one and the same.

Theorem 5.5 $(H^*_{CR}(\mathcal{G};\mathbb{R}), \cup) \cong (H^*_{CR}(\mathcal{G};\mathbb{R}), \wedge)$ *as rings.*

Proof Let α, β, and γ be classes from sectors $\mathcal{G}_{(g_1)}$, $\mathcal{G}_{(g_2)}$, and $\mathcal{G}_{(g_3)}$, respectively. We want to show that

$$\langle \alpha \cup \beta, \gamma \rangle = \int_{\mathcal{G}} i_{(g_1)}(\alpha) \wedge i_{(g_2)}(\beta) \wedge i_{(g_3)}(\gamma).$$

The right hand side is

$$\begin{aligned}
\int_{\mathcal{G}} i_{(g_1)}(\alpha) i_{(g_2)}(\beta) i_{(g_3)}(\gamma) &= \int_{\mathcal{G}} \alpha\beta\gamma \prod_{i=1}^{3} t(g_i) \\
&= \int_{\mathcal{G}} \alpha\beta\gamma \prod_{j=1}^{m} [\theta_j]^{\sum_{i=1}^{3} \mu_{ij}} \\
&= \int_{\mathcal{G}} \alpha\beta\gamma\, \Theta_{\mathcal{G}^3_{(\mathbf{g})}} \prod_{j=1}^{m} [\theta_j]^{\sum_{i=1}^{3} \mu_{ij}-1} \\
&= \int_{\mathcal{G}^3_{(\mathbf{g})}} e_1^*(\alpha) e_2^*(\beta) e_3^*(\gamma) \Theta_{(g_1,g_2)} \\
&= \int_{\mathcal{G}^3_{(\mathbf{g})}} e_1^*(\alpha) e_2^*(\beta) e_3^*(\gamma)\, \mathbf{e}(E_{(\mathbf{g})}).
\end{aligned}$$

Here $\Theta_{\mathcal{G}^3_{(\mathbf{g})}}$ is the Thom form of $\mathcal{G}^3_{(\mathbf{g})}$ in $\mathcal{G}$. Together with Poincaré duality, this calculation implies that the two products coincide. □

5.1.2 Examples

Now we will use the de Rham model to compute two examples. In both cases, the obstruction bundle contributes non-trivially. These examples were first computed by Jiang [74] and Park and Poddar [122] using much more complicated methods. The de Rham model, on the other hand, requires only a rather elementary computation.

Example 5.6 (Weighted projective space) Let $\mathcal{X} = \mathbb{WP}(a_0, \dots, a_n)$ be the weighted projective space of Example 1.15. This was defined as a quotient of $\mathbb{S}^{2n+1}$ by $\mathbb{S}^1$. When the positive integers a_i have no common factor, it forms an effective complex orbifold. Note that we can also present $\mathbb{WP}(a_0, \dots, a_n)$ as

$$\mathbb{WP}(V, \rho) = (V \setminus \{0\}) / \rho,$$

where $V = \mathbb{C}^{n+1}$ and $\rho : \mathbb{C}^* \to (\mathbb{C}^*)^{n+1} \subset GL(n+1, \mathbb{C})$ is the representation $\lambda \mapsto \mathrm{diag}(\lambda^{a_0}, \dots, \lambda^{a_n})$.

The twisted sectors of $\mathcal{X}$, which are themselves weighted projective spaces of smaller dimensions, are labeled by (torsion) elements of $\mathbb{S}^1$. More precisely, each $\lambda \in \mathbb{S}^1$ gives a decomposition $V = V^\lambda \oplus V^\lambda_\perp$, where V^λ is the subspace of V fixed by λ and $V^\lambda_\perp$ is the direct sum of the other eigenspaces. If $V^\lambda \neq \{0\}$, we let ρ^λ denote the restriction of the $\mathbb{C}^*$-action to V^λ. Then the pair $(\mathbb{WP}(V^\lambda, \rho^\lambda), \lambda)$ gives the twisted sector $\mathcal{X}_{(\lambda)}$. Thus, as a group, the Chen–Ruan cohomology is

$$H^*_{\mathrm{CR}}(\mathbb{WP}(V, \rho)) = \bigoplus_{\substack{\lambda \in \mathbb{S}^1 \\ V^\lambda \neq \{0\}}} H^*(\mathbb{WP}(V^\lambda, \rho^\lambda)).$$

To determine the degree shifting and the Chen–Ruan cup product, we only have to find the twisted factors.

Let $I \subset \{0, \dots, n\}$. Then $V_I = \{(z_0, \dots, z_n) \mid z_i = 0 \text{ for } i \in I\}$ are invariant subspaces of V; we denote the restricted action by ρ_I. We abbreviate $\{i\}$ as i in the subscripts. Let $\mathbb{WP}(V_i, \rho_i)$ be the corresponding codimension one subspace, and let $\xi_i \in H^2(\mathcal{X})$ denote its Thom class. Then ξ_i equals the Chern class of the line bundle defined by $\mathbb{WP}(V_i, \rho_i)$. The relations among the various ξ_i are $a_j \xi_i = a_i \xi_j$, given by meromorphic functions of the form $z_i^{a_j}/z_j^{a_i}$ for all pairs i, j. Set $y = \xi_i/a_i$. One sees that $H^*(\mathcal{X})$, the ordinary cohomology ring, is generated by the elements y, subject to the relations $y^{n+1} = 0$. Carrying out the same argument for each $\mathcal{X}_{(\lambda)} = (\mathbb{WP}(V^\lambda, \rho^\lambda), \lambda)$ of dimension at least one, we get generators

$$y_\lambda = \frac{\xi_i|_{\mathcal{X}_{(\lambda)}}}{a_i}$$

for the ordinary cohomology ring of $X_{(\lambda)}$, where the ith coordinate line is contained in V^λ. For point sectors $\mathcal{X}_{(\lambda)}$, we simply let $y_\lambda = 1$, the generator of $H^0(\mathcal{X}_{(\lambda)})$.

Suppose that $V^\lambda = V_I$. In this case, the twisted factor for $\mathcal{X}_{(\lambda)}$ can be written as

$$t(\lambda) = \prod_{i \in I} \xi_i^{\frac{1}{2\pi} \mathrm{Arg}(\lambda^{a_i})}.$$

Instead of using this twisted factor, we introduce a multiple of it which simplifies the notation:

$$t'(\lambda) = \prod_{i\in I}\left(\frac{\xi_i}{a_i}\right)^{\frac{1}{2\pi}\operatorname{Arg}(\lambda^{a_i})}. \tag{5.5}$$

When $V^\lambda = \{0\}$ we write $t'(\lambda) = (1)^{\frac{1}{2\pi}\operatorname{Arg}(\lambda)}$, and define it to be λ. Note that although the terms in the product (5.5) have the same base $y = (\xi_i/a_i)$, it would be inappropriate at this stage to simply add up the exponents. For one thing, we want to keep in mind the splitting of the normal bundle into line bundles; besides that, each factor is in fact a compactly supported form on a *different* line bundle. The formal product really means that we should pull back to the direct sum and then take the wedge product.

Now, the (scaled) twisted form corresponding to $(y_\lambda)^k$ is $(y_\lambda)^k t'(\lambda) = y^k t'(\lambda) \in H^*_{\mathrm{CR}}(\mathcal{X})$. Let λ_1 and $\lambda_2 \in \mathbb{S}^1$ with $\lambda_3 = \lambda_1\lambda_2$. Then

$$y^{k_1}t'(\lambda_1) \wedge y^{k_2}t'(\lambda_2) = y^{k_3}t'(\lambda_3),$$

where the terms in $t'(\lambda_1)$ and $t'(\lambda_2)$ combine by adding exponents with the same base (ξ_i/d_i), and in $t'(\lambda_3)$ we retain only the terms of the form $(\xi_i/d_i)^{\square}$, where $\square$ is the fractional part of the exponent. Of course, when $y^{k_3} = 0 \in H^*(\mathcal{X}_{(\lambda_3)})$, the product is zero.

To put it more combinatorially, we write the cohomology ring of $\mathrm{W}\mathbb{P}(a_0, \ldots, a_n)$ as

$$\mathbb{C}[Y_0, \ldots, Y_n]/(Y_i - Y_j, p \mid \deg p > n),$$

where $Y_i = \xi_i/a_i$ and p runs over all monomials in the Y_i. Then, representing the classes in $H^*_{\mathrm{CR}}(\mathcal{X})$ by twisted forms, we have

$$H^*_{\mathrm{CR}}(\mathcal{X}) = \left\{\prod_{i\notin I} Y_i \prod_{i\in I} Y_i^{\frac{1}{2\pi}\operatorname{Arg}(\lambda^{d_i})} \;\middle|\; V^\lambda = V_I \text{ as before, for } \lambda \in \mathbb{S}^1 \text{ and } I \subset \{0, \ldots, n\}\right\}\Big/\!\sim,$$

where the product is given by multiplication of monomials modulo the obvious relations for vanishing (given in the last sentence of the previous paragraph); besides these relations, we also mod out by the ideal generated by differences $Y_i - Y_j$.

Remark 5.7 If the weighted projective space is given by fans and so on, the computation above coincides with the formula given by Borisov, Chen, and Smith [26] for general toric Deligne–Mumford stacks.

Example 5.8 (Mirror quintic orbifolds) We next consider the mirror quintic orbifold $\mathcal{Y}$, which is defined as a generic member of the anti-canonical linear system in the following quotient of $\mathbb{C}P^4$ by $(\mathbb{Z}/5\mathbb{Z})^3$:

$$[z_1 : z_2 : z_3 : z_4 : z_5] \sim [\xi^{a_1} z_1 : \xi^{a_2} z_2 : \xi^{a_3} z_3 : \xi^{a_4} z_4 : \xi^{a_5} z_5],$$

where $\sum a_i = 0 \mod 5$ and $\xi = e^{\frac{2\pi i}{5}}$. Concretely, we obtain $\mathcal{Y}$ as the quotient of a quintic of the form

$$Q = \{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \psi z_1 z_2 z_3 z_4 z_5 = 0\}$$

under the $(\mathbb{Z}/5\mathbb{Z})^3$-action, where $\psi^5 \neq -5^5$ (cf. Example 1.12).

The computation for the mirror quintic was first done in [122]. The ordinary cup product on $\mathcal{Y}$ is computed in [122, §6], and we refer the reader there for details. We also consult [122, §5] for the description of the twisted sectors of $\mathcal{Y}$. These are either points or curves. The main simplification in applying the de Rham method lies in computing the contributions from the twisted sectors that are curves. Let $\mathcal{Y}_{(\mathbf{g})}$ be a 3-sector which is an orbifold curve, where as usual $(\mathbf{g}) = (g_1, g_2, g_3)$. Such a curve only occurs as the intersection of $\mathcal{Y}$ with some two-dimensional subvariety of $\mathcal{X} = \mathbb{C}P^4/(\mathbb{Z}/5\mathbb{Z})^3$ invariant under the Hamiltonian torus action. It follows that the isotropy group for a generic point in $\mathcal{Y}_{(\mathbf{g})}$ must be $G \cong \mathbb{Z}/5\mathbb{Z}$, and we have $g_i \in G$. Furthermore, under the evaluation maps to $\mathcal{Y}$, the sectors $\mathcal{Y}_{(g_i)}$ and $\mathcal{Y}_{(\mathbf{g})}$ have the same image, which we denote by $\mathcal{Y}_{(G)}$.

Using the de Rham model, we note that the formal maps

$$i_{(g_i)} : H^*(\mathcal{Y}_{(g_i)}) \to H_{\mathrm{CR}}^{*+\iota_{(g_i)}}(\mathcal{Y}),$$

all factor through a tubular neighborhood of $\mathcal{Y}_{(G)}$ in $\mathcal{Y}$. Since $\mathcal{Y}$ is a Calabi–Yau orbifold, the degree shift $\iota_{(g_i)}$ is always a non-negative integer. In particular, if $g_i \neq \mathrm{id} \in G$, we must have $\iota_{(g_i)} = 1$. Let $\alpha_i \in H^*(\mathcal{Y}_{(g_i)})$. We consider the Chen–Ruan cup product $\alpha_1 \cup \alpha_2$. It suffices to evaluate the non-zero pairings

$$\langle \alpha_1 \cup \alpha_2, \alpha_3 \rangle = \int_{\mathcal{Y}} \bigwedge_{i=1}^{3} i_{(g_i)}(\alpha_i) \neq 0.$$

When $g_3 = \mathrm{id}$, we see that the Chen–Ruan cup product reduces to (ordinary) Poincaré duality. When $g_i \neq \mathrm{id}$ for $i = 1, 2, 3$, then by directly checking degrees we find that $\alpha_i \in H^0(\mathcal{Y}_{(g_i)})$ for all i, and the wedge product is a multiple of the product of the twist factors $t(g_i) = [\theta_1]^{\mu_{i1}}[\theta_2]^{\mu_{i2}}$. Here the $[\theta_j]$ are the Thom classes of the line bundle factors of the normal bundle. Without loss of generality, suppose $\alpha_i = 1_{(g_i)}$. Since $g_i \neq \mathrm{id}$ by assumption, we have $\mu_{ij} > 0$

for all i, j. Thus,

$$\int_{\mathcal{Y}} \bigwedge_{i=1}^{3} i_{(g_i)}(1_{(g_i)}) = \int_{\mathcal{Y}_{(G)}} c_\diamond,$$

where $c_\diamond$ stands for the Chern class corresponding to either $[\theta_1]$ or $[\theta_2]$ (they are equal). Let X_2 be the two-dimensional invariant subvariety of $\mathcal{X} = \mathbb{C}P^4/(\mathbb{Z}/5\mathbb{Z})^3$ such that $\mathcal{Y} \cap X_2 = \mathcal{Y}_{(G)}$. Then there are two invariant three-dimensional subvarieties $X_{3,1}$ and $X_{3,2}$ containing X_2. Let $\mathcal{Y}_j = \mathcal{Y} \cap X_{3,j}$ for $j = 1, 2$. Then $c_\diamond$ above is simply the Chern class c_j of the normal bundle of $\mathcal{Y}_{(G)}$ in $\mathcal{Y}_j$ (for either value of j). To finish the computation, we note that the whole local picture can be lifted to $Q \subset \mathbb{C}P^4$, where the Chern classes corresponding to c_j obviously integrate to 5. Quotienting by $(\mathbb{Z}/5\mathbb{Z})^3$, we obtain

$$\int_{\mathcal{Y}_{(G)}} c_\diamond = \frac{5}{125} = \frac{1}{25}.$$

5.2 Symmetric products

Let S_n be the symmetric group on n letters, and let X be a manifold. Then S_n acts on X^n by permuting factors. The global quotient X^n/S_n is called the nth *symmetric product* of X. We first considered this orbifold in Example 1.13. When X is a complex manifold, X^n/S_n is complex as well. An important particular case occurs when X is an algebraic surface. In this case, there is a famous crepant resolution given by the Hilbert scheme $X^{[n]}$ of points of the algebraic surface:

$$X^{[n]} \to X^n/S_n.$$

The topology of $X^{[n]}$ and X^n/S_n and their relation to each other have been interesting questions in algebraic geometry that have undergone intensive study since the 1990s. The result is a beautiful story involving algebraic geometry, topology, algebra, and representation theory. Although a discussion of the Hilbert scheme of points is beyond the scope of this book, we will present the symmetric product side of the story in this chapter, along with extensive references at the end of the book for the interested reader. The central theme is that the direct sum of the cohomologies of all the symmetric products of X is an irreducible representation of a *super Heisenberg algebra*. Throughout the remainder of this chapter, we will understand all coefficients to be complex unless stated otherwise.

5.2.1 The Heisenberg algebra action

Let H be a finite-dimensional complex *super vector space*. That is, $H = H_{\text{even}} \oplus H_{\text{odd}}$ is a complex vector space together with a chosen $\mathbb{Z}/2\mathbb{Z}$-grading. Assume also that H comes equipped with a *super inner product* $\langle\ ,\ \rangle$. For instance, H could be the cohomology of a manifold, and the inner product could be the Poincaré pairing. For any homogeneous element $\alpha \in H$, we denote its degree by $|\alpha|$, so $|\alpha| = 0$ if $\alpha \in H_{\text{even}}$ and $|\alpha| = 1$ for $\alpha \in H_{\text{odd}}$.

Definition 5.9 The *super Heisenberg algebra* associated to H is the super Lie algebra $\mathcal{A}(H)$ with generators $\mathfrak{p}_l(\alpha)$ for each non-zero integer l and each $\alpha \in H$, along with a central element $\mathfrak{c}$. These are subject to the following relations. First, the generators $\mathfrak{p}_l(\alpha)$ are linear in α, and for homogeneous α we let $\mathfrak{p}_l(\alpha)$ have degree $|\alpha|$. Second, the super Lie bracket must satisfy

$$[\mathfrak{p}_l(\alpha), \mathfrak{p}_m(\beta)] = l\delta_{l+m,0}\langle\alpha, \beta\rangle\mathfrak{c}. \tag{5.6}$$

The $\mathfrak{p}_l(\alpha)$ are called *annihilation operators* when $l > 0$, and *creation operators* when $l < 0$. If H^{odd} is trivial, then H is just a vector space, and we obtain an ordinary Lie algebra instead of a super Lie algebra. The *classical Heisenberg algebra* is the algebra obtained when H is the trivial super vector space.

We digress briefly to discuss some representations of these Heisenberg algebras (see [78]). Let $F = \bigoplus_{n\in\mathbb{Z}_{>0}} H_n$ with each $H_n = H$, and let $\mathrm{Sym}(F)$ be the *supersymmetric algebra* on F. That is, $\mathrm{Sym}(F)$ is the quotient of the tensor algebra on F by the ideal generated by elements of the form $a \otimes b - (-1)^{|a||b|} b \otimes a$. This is naturally a supercommutative superalgebra. If we choose bases $\{\alpha_i\}$ and $\{\beta_i\}$ $(i = 1, \ldots, k)$ of H that are dual with respect to the pairing:

$$\langle\beta_i, \alpha_j\rangle = \delta_{i,j},$$

then $\mathrm{Sym}(F)$ may be identified with a polynomial algebra in the variables $x_n^{\alpha_i}$, where $n \in \mathbb{Z}_{>0}$ and the variables indexed by odd basis elements anticommute with each other. We define a representation of the super Heisenberg algebra $\mathcal{A}(H)$ on $\mathrm{Sym}(F)$ as follows. Let the central element $\mathfrak{c}$ act as the identity endomorphism. For $l > 0$ and $p \in \mathrm{Sym}(F)$ a polynomial, let

$$\mathfrak{p}_l(\alpha)p = l\sum_j \langle\alpha, \alpha_j\rangle \frac{\partial p}{\partial x_l^{\alpha_j}},$$

and

$$\mathfrak{p}_{-l}(\alpha)p = \sum_j \langle\beta_j, \alpha\rangle x_l^{\alpha_j} p.$$

The reader may check that this defines a homomorphism of super Lie algebras from $\mathcal{A}(H)$ to the super Lie algebra End(Sym(F)). The constant polynomial 1 has the property $\mathfrak{p}_l(\alpha)(1) = 0$ for any $l > 0$; in other words, it is annihilated by all the annihilation operators. Any vector with this property is called a *highest weight vector*. A highest weight vector is also often referred to as a *vacuum vector* (or simply a *vacuum*). A representation of the Heisenberg algebra is called a *highest weight representation* if it contains a highest weight vector. A well-known result states that an irreducible highest weight representation of $\mathcal{A}(H)$ is unique up to isomorphism, the essential idea being that any such representation is generated by a unique highest weight vector. In such cases, we use $|0\rangle$ to denote the highest weight vector.

Let us recall how the Chen–Ruan cohomology ring from Chapter 4 works in the case of a global quotient. Let M be a complex manifold with a smooth action of the finite group G. We consider the space

$$\bigsqcup_{g \in G} M^g = \{(g, x) \in G \times M \mid gx = x\}.$$

G acts on this space by $h \cdot (g, x) = (hgh^{-1}, hx)$. The inertia orbifold is then $\wedge(M/G) = (\bigsqcup_{g \in G} M^g)/G$. As a vector space, $H^*(M, G)$ is the cohomology group of $\bigsqcup_{g \in G} M^g$ with complex coefficients (see the remarks on page 91 for more details). The vector space $H^*(M, G)$ has a natural induced G action, denoted by $\mathrm{ad}_h : H^*(M^g) \to H^*(M^{h^{-1}gh})$ for each $h \in G$. As a vector space, the Chen–Ruan cohomology group $H^*_{\mathrm{CR}}(M/G)$ is the G-invariant part of $H^*(M, G)$, and is isomorphic to

$$\bigoplus_{(g) \in G_*} H^*(M^g / \mathbf{C}(g)),$$

where G_* denotes the set of conjugacy classes of G, and $\mathbf{C}(g) = \mathbf{C}_G(g)$ denotes the centralizer of g in G.

For the identity element $1 \in G$, we have $H^*(M^1/Z(1)) \cong H^*(M/G)$. Thus we can regard any $\alpha \in H^*(M/G)$ as an element of $H^*_{\mathrm{CR}}(M/G)$. Also, given $a = \sum_{g \in G} a_g g$ in the group algebra $\mathbb{C}G$ (resp. $(\mathbb{C}G)^G$), we may regard a as an element in $H^*(M, G)$ (resp. $H^*_{\mathrm{CR}}(M/G)$), whose component in each $H^*(M^g)$ is $a_g \cdot 1_{M^g} \in H^0(M^g)$ (see Section 4.4).

If K is a subgroup of G, then we can define a *restriction map* from $H^*(M, G)$ to $H^*(M, K)$ by projecting to the subspace $\bigoplus_{g \in K} H^*(M^g)$. Restricted to the G-invariant part of $H^*(M, G)$, this naturally induces a *degree-preserving* linear map

$$\mathrm{Res}^G_K : H^*_{\mathrm{CR}}(M/G) \to H^*_{\mathrm{CR}}(M/K).$$

Dually, we define the *induction map*

$$\operatorname{Ind}_K^G : H^*(M, K) \to H^*_{\mathrm{CR}}(M/G)$$

by sending $\alpha \in H^*(M^h)$ for $h \in K$ to

$$\operatorname{Ind}_K^G(\alpha) = \frac{1}{|K|} \sum_{g \in G} \operatorname{ad}_g(\alpha).$$

Note that $\operatorname{Ind}_K^G(\alpha)$ is automatically G-invariant. Again, by restricting to the invariant part of the domain, we obtain a *degree-preserving* linear map

$$\operatorname{Ind}_K^G : H^*_{\mathrm{CR}}(M/K) \to H^*_{\mathrm{CR}}(M/G).$$

We often write the restriction (induction) maps as Res_K or Res (Ind^G or Ind) when the groups involved are clear from the context. Suppose that we have a chain of subgroups $H \subseteq K \subseteq L$. Then on the Chen–Ruan cohomology, we have

$$\operatorname{Ind}_K^L \operatorname{Ind}_H^K = \operatorname{Ind}_H^L, \quad \text{and } \operatorname{Res}_H^K \operatorname{Res}_K^L = \operatorname{Res}_H^L.$$

When dealing with restrictions and inductions of modules, Mackey's Decomposition Theorem provides a useful tool, see Theorem 2.9 on page 85 in [53]. Although our restrictions and inductions are not the usual ones, we can still prove a similar decomposition result.

Lemma 5.10 *Suppose we have two subgroups H and L of a finite group Γ. Fix a set S of representative elements in the double cosets $H\backslash\Gamma/L$. Let $L_s = s^{-1}Hs \cap L$ and $H_s = sL_s s^{-1} \subseteq H$. Then, on the Chen–Ruan cohomology,*

$$\operatorname{Res}_L \operatorname{Ind}_H^\Gamma = \sum_{s \in S} \operatorname{Ind}_{L_s}^L \operatorname{ad}_s \operatorname{Res}_{H_s}^H,$$

where $\operatorname{ad}_s : H^*_{\mathrm{CR}}(M/H_s) \xrightarrow{\cong} H^*_{\mathrm{CR}}(M/L_s)$ *is the isomorphism induced by* $\operatorname{ad}_s : H^*(M, \Gamma) \to H^*(M, \Gamma)$.

Proof First, fix $\alpha \in H^*(M^g)$. Then

$$\operatorname{Res}_L \operatorname{Ind}_H^\Gamma(\alpha) = \frac{1}{|H|} \sum_{s^{-1}gs \in L} \operatorname{ad}_s(\alpha).$$

This can be rewritten as

$$\operatorname{Res}_L \operatorname{Ind}_H^\Gamma = \frac{1}{|H|} \sum_{s \in \Gamma} \operatorname{ad}_s \operatorname{Res}_{H_s}^H.$$

Since the kernel of the $H \times L$ action on the double cosets $L\backslash\Gamma/H$ is given by the equation $hs = sl$, i.e., $s^{-1}hs = l$, we see that this kernel can be identified

with the group L_s. This means that

$$\mathrm{Res}_L \, \mathrm{Ind}_H^\Gamma = \frac{1}{|H|} \frac{1}{|L_s|} \sum_{s \in S} \sum_{h \in H} \sum_{l \in L} \mathrm{ad}_{hsl} \, \mathrm{Res}_{H_{hsl}}^H .$$

Now for each $\alpha \in H^*(M, \Gamma)$, let

$$\alpha(H, s, L) = \mathrm{Ind}_{L_s}^L \, \mathrm{Res}_{L_s}^{s^{-1}Hs} \, \mathrm{ad}_s \, \alpha.$$

Then if we replace s by another double coset representative $hsl \in HsL$, a calculation from definitions shows that $\alpha(H, hsl, L) = \mathrm{ad}_h(\alpha)(H, s, L)$. Consequently, one finds that

$$\begin{aligned}
\mathrm{Res}_L \, \mathrm{Ind}_H^\Gamma &= \sum_{s \in S} \frac{1}{|L_s|} \sum_{l \in L} \mathrm{ad}_l \, \frac{1}{|H|} \sum_{h \in H} \mathrm{ad}_{hs} \, \mathrm{Res}_{H_{hH_s h^{-1}}}^H \\
&= \sum_{s \in S} \mathrm{Ind}_{L_s}^L \, \frac{1}{|H|} \sum_{h \in H} \mathrm{ad}_s \, \mathrm{ad}_h \, \mathrm{Res}_{H_{hs}}^H \\
&= \sum_{s \in S} \mathrm{Ind}_{L_s}^L \, \mathrm{ad}_s \, \mathrm{Res}_{H_s}^H \, \frac{1}{|H|} \sum_{h \in H} \mathrm{ad}_h .
\end{aligned}$$

But if we apply this to $\alpha \in H_{CR}^*(M/\Gamma)$, the last operator $(1/|H|) \sum_{h \in H} \mathrm{ad}_h = \mathrm{Ind}_H^H$ is the identity, and the lemma is proved. □

We are now ready to consider symmetric products. Fix a closed complex manifold X of *even* complex dimension d. Our main objects of study are the Chen–Ruan cohomology rings $H_{\mathrm{CR}}^*(X^n/S_n)$. We write

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} H_{\mathrm{CR}}^*(X^n/S_n; \mathbb{C}),$$

where we use the convention that X^0/S_0 is a point.

For each n, we introduce a linear map

$$\omega_n : H^*(X) \to H_{\mathrm{CR}}^*(X^n/S_n),$$

defined as follows. First, recall that the n-cycles make up a conjugacy class in S_n; there are $(n-1)!$ such cycles. For each n-cycle σ_n, the fixed-point set $(X^n)^{\sigma_n}$ is the diagonal copy of X, so we have an isomorphism $H^*((X^n)^{\sigma_n}) \cong H^*(X)$. Given $\alpha \in H^r(X)$, we let $\omega_n(\alpha) \in H_{\mathrm{CR}}^{r+d(n-1)}(X^n/S_n)$ be the sum of the $(n-1)!$ elements associated to $n\alpha$ under these isomorphisms as σ_n runs over the set of n-cycles. The reader should check that the degree shift is indeed $d(n-1)$ (see also the discussion on page 124). We define a second linear map,

$$\mathrm{ch}_n : H_{\mathrm{CR}}^*(X^n/S_n) \to H^*(X),$$

to be $1/(n-1)!$ times the sum of the compositions

$$H^*_{\mathrm{CR}}(X^n/S_n) \to H^*((X^n)^{\sigma_n}) \stackrel{\cong}{\to} H^*(X),$$

as σ_n runs over the n-cycles, where the first map is the projection. In particular,

$$\mathrm{ch}_n(\omega_n(\alpha)) = n\alpha.$$

Let $\mathcal{A}(H^*(X))$ be the super Heisenberg algebra associated to the cohomology of X and its Poincaré pairing. We wish to define a representation of $\mathcal{A}(H^*(X))$ on $\mathcal{H}$. As usual, we let the central element $\mathfrak{c}$ act as the identity endomorphism $\mathrm{Id}_{\mathcal{H}}$. Let $\alpha \in H^*(X)$, and let $n > 0$. We let the creation operator $\mathfrak{p}_{-n}(\alpha)$ act as the endomorphism given by the composition

$$\begin{aligned} H^*_{\mathrm{CR}}(X^k/S_k) &\stackrel{\omega_n(\alpha)\otimes\,\cdot}{\longrightarrow} H^*_{\mathrm{CR}}(X^n/S_n) \bigotimes H^*_{\mathrm{CR}}(X^k/S_k) \\ &\stackrel{\cong}{\longrightarrow} H^*_{\mathrm{CR}}(X^{n+k}/(S_n \times S_k)) \stackrel{\mathrm{Ind}}{\longrightarrow} H^*_{\mathrm{CR}}(X^{n+k}/S_{n+k}), \end{aligned}$$

for each $k \geq 0$, where the second map is the Künneth isomorphism. Similarly, we let the annihilation operator $\mathfrak{p}_n(\alpha)$ act as the endomorphism given by

$$\begin{aligned} H^*_{\mathrm{CR}}(X^{n+k}/S_{n+k}) &\stackrel{\mathrm{Res}}{\longrightarrow} H^*_{\mathrm{CR}}(X^{n+k}/(S_n \times S_k)) \\ &\stackrel{\cong}{\longrightarrow} H^*_{\mathrm{CR}}(X^n/S_n) \bigotimes H^*_{\mathrm{CR}}(X^k/S_k) \\ &\stackrel{\mathrm{ch}_n}{\longrightarrow} H^*(X) \bigotimes H^*_{\mathrm{CR}}(X^k/S_k) \stackrel{\langle\alpha,\cdot\rangle\otimes\mathrm{id}}{\longrightarrow} H^*_{\mathrm{CR}}(X^k/S_k) \end{aligned}$$

for each $k \geq 0$; we let $\mathfrak{p}_n(\alpha)$ act as the zero operator on $H^*(X^i/S_i)$ for $i < n$. In particular,

$$\mathfrak{p}_{-1}(\alpha)(y) = \frac{1}{(n-1)!} \sum_{g \in S_n} \mathrm{ad}_g(\alpha \otimes y) \tag{5.7}$$

for $y \in H^*_{\mathrm{CR}}(X^{n-1}/S_{n-1})$.

Theorem 5.11 *Under the associations given above, $\mathcal{H}$ is an irreducible highest weight representation of the super Heisenberg algebra $\mathcal{A}(H^*(X))$ with vacuum vector $|0\rangle = 1 \in H^*_{CR}(X^0/S_0) \cong \mathbb{C}$.*

Proof It is easy to check that

$$[\mathfrak{p}_n(\alpha), \mathfrak{p}_m(\beta)] = 0$$

for $n, m > 0$ or $n, m < 0$; we leave it to the reader. Consider instead the case $[\mathfrak{p}_m(\beta), \mathfrak{p}_{-n}(\alpha)]$ for $n, m > 0$. To simplify signs, we assume that all cohomology classes involved have even degrees. By Lemma 5.10, for $\kappa \in H^*_{\mathrm{CR}}(X^k/S_k)$ we

have

$$\operatorname{Res}_{S_m\times S_l}\operatorname{Ind}_{S_n\times S_k}^{S_{n+k}}(\omega_n(\alpha)\otimes\kappa)=\sum_{s\in S}\operatorname{Ind}_{L_s}^{S_l\times S_m}\operatorname{ad}_s\operatorname{Res}_{H_s}^{S_n\times S_k}(\omega_n(\alpha)\otimes\kappa),$$

where S is again a set of double coset representatives and $n+k=l+m$.

It is well known that the set of double cosets $S=(S_l\times S_m)\backslash S_{n+k}/(S_n\times S_k)$ is parameterized by the set $\mathcal{M}$ of 2×2 matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \qquad a_{ij}\in\mathbb{Z},$$

satisfying

$$a_{11}+a_{12}=n, \qquad a_{21}+a_{22}=k,$$
$$a_{11}+a_{21}=m, \qquad a_{12}+a_{22}=l.$$

Then,

$$\begin{aligned}
&\operatorname{Res}_{S_m\times S_l}\operatorname{Ind}_{S_n\times S_k}^{S_{n+k}}(\omega_n(\alpha)\otimes\kappa)\\
&\quad=\sum_{A\in\mathcal{M}}\operatorname{Ind}_{S_{a_{11}}\times S_{a_{21}}\times S_{a_{12}}\times S_{a_{22}}}^{S_l\times S_m}\operatorname{Res}_{S_{a_{11}}\times S_{a_{12}}\times S_{a_{21}}\times S_{a_{22}}}^{S_n\times S_k}(\omega_n(\alpha)\otimes\kappa)\\
&\quad=\sum_{A\in\mathcal{M}}\operatorname{Ind}_{S_{a_{11}}\times S_{a_{21}}\times S_{a_{12}}\times S_{a_{22}}}^{S_l\times S_m}\big(\operatorname{Res}_{S_{a_{11}}\times S_{a_{12}}}^{S_n}(\omega_n(\alpha))\otimes\operatorname{Res}_{S_{a_{21}}\times S_{a_{22}}}^{S_k}(\kappa)\big).
\end{aligned}$$

Clearly,

$$\operatorname{Res}_{S_{a_{11}}\times S_{a_{12}}}^{S_n}(\omega_n(\alpha))=0$$

unless $a_{11}=n, a_{12}=0$ or $a_{11}=0, a_{12}=n$. Moreover,

$$\operatorname{ch}_m\left(\operatorname{Ind}_{S_{a_{11}}\times S_{a_{21}}\times S_{a_{12}}\times S_{a_{22}}}^{S_l\times S_m}\big(\omega_n(\alpha)\otimes\operatorname{Res}_{S_{a_{21}}\times S_{a_{22}}}^{S_k}(\kappa)\big)\right)=0$$

unless $a_{11}=m, a_{21}=0$ or $a_{11}=0, a_{21}=m$. In that case, either $m=n, l=k$ or $m+a_{22}=k,\ n+a_{22}=l$. When $m=n, l=k$, we obtain $n\langle\alpha,\beta\rangle\operatorname{Id}$. In the second case, we obtain $(-1)^{|\alpha||\beta|}\mathfrak{p}_{-m}(\beta)\mathfrak{p}_n(\alpha)$. Hence, $[\mathfrak{p}_n(\alpha),\mathfrak{p}_{-m}(\beta)]=n\delta_{n-m,0}\langle\alpha,\beta\rangle\operatorname{Id}$, as desired. □

We can compute $\mathcal{H}$ explicitly using ideas of Vafa and Witten [155]. First, we compute the cohomology of the non-twisted sector. With complex coefficients, this is isomorphic to $H^*(X^n/S_n;\mathbb{C})$, the cohomology of the quotient space. It is easy to see that $H^*(X^n/S_n;\mathbb{C})\cong H^*(X^n;\mathbb{C})^{S_n}$. Let $\alpha_i\in H^*(X;\mathbb{C})$ for $i=1,\ldots,n$. Then $\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n\in H^*(X^n;\mathbb{C})$, and every class in $H^*(X^n;\mathbb{C})^{S_n}$ is of the form $\sum_{g\in S_n}\operatorname{ad}_g(\alpha_1\otimes\cdots\otimes\alpha_n)$ for some such set $\{\alpha_i\}$.

We observe that

$$\mathrm{Ind}_{S_l\times S_{n+k}}^{S_{l+n+k}}\left(\omega_l(\alpha)\otimes \mathrm{Ind}_{S_n\times S_k}^{S_{n+k}}(\omega_n(\beta)\otimes\kappa)\right) = \mathrm{Ind}_{S_l\times S_n\times S_k}^{S_{l+n+k}}(\omega_l(\alpha)\otimes\omega_n(\beta)\otimes\kappa). \tag{5.8}$$

Using this formula repeatedly, one can show that

$$\mathfrak{p}_{-1}(\alpha_1)\ldots\mathfrak{p}_{-1}(\alpha_n)|0\rangle = \sum_{g\in S_n}\mathrm{ad}_g(\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n).$$

The twisted sectors are represented by the connected components of $(X^n)^g/\mathbf{C}(g)$ as g varies over representatives of each conjugacy class $(g)\in(S_n)_*$. It is well known that the conjugacy class of an element $g\in S_n$ is determined by its *cycle type*. Suppose that g has cycle type $1^{n_1}2^{n_2}\cdots k^{n_k}$, where i^{n_i} indicates that g has n_i cycles of length i. There is an associated partition $n=\sum_i in_i$. One sees that the fixed-point locus is $(X^n)^g = X^{n_1}\times\cdots\times X^{n_k}$, while the centralizer is

$$\mathbf{C}(g) = \left(S_{n_1}\ltimes(\mathbb{Z}/1\mathbb{Z})^{n_1}\right)\times\cdots\times\left(S_{n_k}\ltimes(\mathbb{Z}/k\mathbb{Z})^{n_k}\right).$$

Hence, as a topological space, the twisted sector is

$$(X^n)^g/\mathbf{C}(g) = X^{n_1}/S_{n_1}\times\cdots\times X^{n_k}/S_{n_k},$$

although it has a different orbifold structure involving the extra isotropy groups $(\mathbb{Z}/i\mathbb{Z})^{n_i}$ for $i=1,\ldots,k$. By choosing appropriate classes $\alpha^i_j\in H^*(X)$ as j runs from 1 to n_j and i from 1 to k, we can represent any cohomology element of the twisted sector in the form

$$\sum_{h\in S_n}\mathrm{ad}_h\left(\bigotimes_i\bigotimes_j^{n_i}\alpha^i_j\right).$$

Again, by repeated use of formula (5.8), this is precisely

$$\mathfrak{p}_{-1}(\alpha^1_1)\cdots\mathfrak{p}_{-1}(\alpha^1_{n_1})\mathfrak{p}_{-2}(\alpha^2_1)\cdots\mathfrak{p}_{-2}(\alpha^2_{n_2})\cdots\mathfrak{p}_{-k}(\alpha^k_1)\cdots\mathfrak{p}_{-k}(\alpha^k_{n_k})|0\rangle. \tag{5.9}$$

Let us introduce some notation to simplify this expression. We will also assume again that X has all cohomology concentrated in even degrees to simplify signs. Choose a basis $\{\alpha_i\}_{i=1}^N$ of $H^*(X)$. Let $\lambda=(\lambda^1,\ldots,\lambda^N)$ be a multipartition. That is, each $\lambda^i=(\lambda^i_1,\ldots,\lambda^i_{\ell(\lambda^i)})$ is a partition of length $\ell(\lambda^i)$. Write

$$\mathfrak{p}_\lambda = \prod_{i=1}^N\mathfrak{p}_{\lambda^i}(\alpha_i),$$

where

$$\mathfrak{p}_{\lambda^i} = \mathfrak{p}_{-\ell(\lambda^i)}(\alpha_{\lambda^i_1})\mathfrak{p}_{-\ell(\lambda^i)}(\alpha_{\lambda^i_2})\ldots\mathfrak{p}_{-\ell(\lambda^i)}(\alpha_{\lambda^i_{\ell(\lambda_i)}}).$$

Putting this notation together with equation (5.9), we have now proved that $H^*_{\mathrm{CR}}(X^n/S_n)$ has the basis

$$\left\{ \mathfrak{p}_\lambda|0\rangle \;\middle|\; \sum_i \ell(\lambda^i) = n \right\}.$$

$\mathcal{H}$ has a natural pairing induced by the Poincaré pairing on Chen–Ruan cohomology. We compute the pairing on the basis elements $\mathfrak{p}_\lambda|0\rangle$. If λ is a multipartition, let $|\lambda| = (\ell(\lambda^1), \ell(\lambda^2), \ldots, \ell(\lambda^k))$. Suppose μ is another multipartition. If $|\lambda| \neq |\mu|$ as partitions, then $\mathfrak{p}_\lambda|0\rangle$ and $\mathfrak{p}_\mu|0\rangle$ belong to different sectors, and so they are orthogonal to each other. Here, one should note that g^{-1} is conjugate to g in the symmetric group S_n, so that the isomorphic orbifolds $X^n_g/\mathbf{C}(g) \cong X^n_{g^{-1}}/\mathbf{C}(g^{-1})$ are viewed as one and the same sector. Suppose that $\mathfrak{p}_\lambda|0\rangle$ and $\mathfrak{p}_\mu|0\rangle$ are both in the sector $(X^n)^g/\mathbf{C}(g)$, where g has cycle type $1^{n_1}2^{n_2}\ldots l^{n_l}$. We calculate:

$$\begin{aligned}
\big\langle \mathfrak{p}_\lambda|0\rangle, \mathfrak{p}_\mu|0\rangle \big\rangle &= \frac{1}{|S_n|}\sum_{f,h}\left\langle \mathrm{ad}_f\left(\bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\lambda^i_j}\right), \ \mathrm{ad}_h\left(\bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\mu^i_j}\right)\right\rangle \\
&= \sum_h \left\langle \bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\lambda^i_j}, \ \mathrm{ad}_h\left(\bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\mu^i_j}\right)\right\rangle \\
&= \sum_{h^{-1}gh=g^{-1}} \left\langle \bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\lambda^i_j}, \ \mathrm{ad}_h\left(\bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\mu^i_j}\right)\right\rangle \\
&= \sum_{h\in\mathbf{C}(g)} \left\langle \bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\lambda^i_j}, \ \mathrm{ad}_h\left(\bigotimes_{i=1}^N \bigotimes_{j=1}^{n_i} \alpha_{\mu^i_j}\right)\right\rangle \\
&= 1^{n_1}2^{n_2}\cdots l^{n_l}\sum_{j_1,j_2}\left\langle \alpha_{\lambda^i_{j_1}}, \alpha_{\mu^i_{j_2}}\right\rangle.
\end{aligned}$$

Here, we again use the fact that g^{-1} is conjugate to g, as well as the description of $\mathbf{C}(g)$ given earlier.

Lemma 5.12 $\mathfrak{p}_n(\beta)^\dagger = \mathfrak{p}_{-n}(\beta)$, *where* $\mathfrak{p}_n(\beta)^\dagger$ *is the adjoint with respect to the Poincaré pairing.*

Proof For simplicity, we assume again that all cohomology classes are of even degree. Suppose that $n > 0$. By definition,

$$\begin{aligned}
\big\langle \mathfrak{p}_n(\beta)^\dagger \mathfrak{p}_\lambda|0\rangle, \mathfrak{p}_\mu|0\rangle \big\rangle &= \big\langle \mathfrak{p}_\lambda|0\rangle, \mathfrak{p}_n(\beta)\mathfrak{p}_\mu|0\rangle \big\rangle \\
&= \sum_{i,j} \delta_{n,\ell(\mu^i)} n\langle \beta, \alpha_{\mu^i_j}\rangle \mathfrak{p}_{\tilde{\mu}_{ij}}|0\rangle,
\end{aligned}$$

where $\tilde{\mu}_{ij}$ is the multipartition obtained from μ by deleting μ^i_j. By comparing this expression with the lemma, we conclude that $\mathfrak{p}_n(\beta)^\dagger = \mathfrak{p}_{-n}(\beta)$. □

Before we can compute the Chen–Ruan ring structure for the symmetric product, we need to find the degree-shifting numbers. We will see that the theory is slightly different according to whether $d = \dim_{\mathbb{C}} X$ is even or odd. Let $\sigma \in S_j$ be a j-cycle. Then its action on a fiber $(\mathbb{C}^d)^j$ of $T(X^j)|_{(X^j)^\sigma}$ has eigenvalues $e^{\frac{2\pi i p}{j}}$, each with multiplicity d, for $p = 0, \ldots, j$. Therefore, the degree shifting number is $\iota_{(\sigma)} = \frac{1}{2}(j-1)d$. Now let $g \in S_n$ be a general permutation, with cycle type $1^{n_1} \cdots k^{n_k}$. Let $\ell(g)$ be the *length*[1] of g, i.e., the minimum number m of transpositions $\tau_1, \ldots, \tau_m$ such that $g = \tau_1 \cdots \tau_m$. In our case, $\ell(g) = \sum_i n_i(i-1)$, and we see that the degree-shifting number is $\iota_{(g)} = \frac{1}{2} d\ell(g)$. Note that when d is even, $\iota_{(g)}$ is an integer; otherwise, $\iota_{(g)}$ may be fractional. In particular, when $d = 2$, $\iota_{(g)} = \ell(g)$. Throughout the rest of this chapter, we will assume that d is even, and hence that the degree-shifting numbers are all integral. Of course, the actual shifts are by $2\iota_{(g)}$, which is always an integer, so the Chen–Ruan cohomology is integrally graded in either case.

At this point we can already provide a computation of the Euler characteristic $\chi_{\mathcal{H}}$. By convention, the operator $\mathfrak{p}_n(\alpha)$ is even or odd if α is even or odd, respectively. Furthermore, when the dimension of X is even, the degree shifts do not change the parity of Chen–Ruan cohomology classes. Hence, the class $\mathfrak{p}_{-l_1}(\alpha_1) \cdots \mathfrak{p}_{-l_k}(\alpha_k)|0\rangle$ is even (odd) if it is even (odd) as a cohomology class in $H^*_{\mathrm{CR}}(X^n/S_n)$. Therefore,

$$\chi_{\mathcal{H}} = \sum_n q^n \chi(H^*_{\mathrm{CR}}(X^n/S_n)).$$

A routine calculation now shows that

$$\chi_{\mathcal{H}} = \prod_n \frac{1}{(1-q^n)^{\chi(X)}}.$$

The irreducible highest weight representation of the classical Heisenberg algebra is naturally a representation of the Virasoro algebra. This classical theorem can be generalized to our situation as well. Those readers solely interested in the computation of the Chen–Ruan product may skip to the next section after reading the definition of τ_* below; the Virasoro action is not otherwise used in the last section.

For $k \geq 1$, let

$$\tau_{k*}\alpha : H^*(X) \to H^*(X^k) \cong H^*(X)^{\otimes k}, \tag{5.10}$$

[1] Despite the similar notation, this length should not be confused with the length of partitions discussed just previously.

be the linear pushforward map induced by the diagonal embedding $\tau_k : X \to X^k$. Let $\mathfrak{p}_{m_1} \ldots \mathfrak{p}_{m_k}(\tau_{k*}\alpha)$ denote $\sum_j \mathfrak{p}_{m_1}(\alpha_{j,1}) \ldots \mathfrak{p}_{m_k}(\alpha_{j,k})$, where we write $\tau_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$. We will write $\tau_*\alpha$ for $\tau_{k*}\alpha$ when there is no cause for confusion.

Lemma 5.13 *Let $k, u \geq 1$ and $\alpha, \beta \in H^*(X)$. Assume that $\tau_{k*}(\alpha) = \sum_i \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k}$ under the Künneth decomposition of $H^*(X^k)$. Then for $0 \leq j \leq k$, we have*

$$\tau_{k*}(\alpha\beta) = \sum_i (-1)^{|\beta| \cdot \sum_{l=j+1}^{k} |\alpha_{i,l}|} \cdot \left(\bigotimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\alpha_{i,j}\beta) \otimes \left(\bigotimes_{t=j+1}^{k} \alpha_{i,t} \right),$$

$$\tau_{(k-1)*}(\alpha\beta) = \sum_i (-1)^{|\beta| \sum_{l=j+1}^{k} |\alpha_{i,l}|} \int_X \alpha_{i,j}\beta \cdot \bigotimes_{\substack{1 \leq s \leq k \\ s \neq j}} \alpha_{i,s},$$

$$\tau_{(k+u-1)*}(\alpha) = \sum_i \left(\bigotimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\tau_{u*}\alpha_{i,j}) \otimes \left(\bigotimes_{t=j+1}^{k} \alpha_{i,t} \right).$$

Proof Recall the projection formula $f_*(\alpha f^*(\beta)) = f_*(\alpha)\beta$ for $f : X \to Y$. We have

$$\begin{aligned}
&\sum_i (-1)^{|\beta| \cdot \sum_{l=j+1}^{k} |\alpha_{i,l}|} \cdot \left(\bigotimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\alpha_{i,j}\beta) \otimes \left(\bigotimes_{t=j+1}^{k} \alpha_{i,t} \right) \\
&\quad = \left(\sum_i \alpha_{i,1} \otimes \cdots \otimes \alpha_{i,k} \right) \cdot p_j^*(\beta) = \tau_{k*}(\alpha) \cdot p_j^*(\beta) \\
&\quad = \tau_{k*}\left(\alpha \cdot (p_j \circ \tau_k)^*(\beta) \right) = \tau_{k*}(\alpha\beta),
\end{aligned}$$

where p_j is the projection of X^k to the jth factor. This proves the first formula. The proofs of the other two are similar. □

Definition 5.14 Define operators $\mathfrak{L}_n(\alpha)$ on $\mathcal{H}$ for $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$ by

$$\mathfrak{L}_n(\alpha) = \frac{1}{2} \sum_{\nu \in \mathbb{Z}} \mathfrak{p}_{n-\nu}\mathfrak{p}_\nu(\tau_*\alpha), \text{ if } n \neq 0$$

and

$$\mathfrak{L}_0(\alpha) = \sum_{\nu > 0} \mathfrak{p}_{-\nu}\mathfrak{p}_\nu(\tau_*\alpha),$$

where we let $\mathfrak{p}_0(\alpha)$ be the zero operator on $\mathcal{H}$.

Remark 5.15 The sums that appear in the definition are formally infinite. However, as operators on any fixed vector in $\mathcal{H}$, only finitely many summands are non-zero. Hence, the sums are locally finite and the operators $\mathcal{L}_n$ are well defined.

Remark 5.16 Using the physicists' *normal ordering convention*

$$:\mathfrak{p}_n\mathfrak{p}_m := \begin{cases} \mathfrak{p}_n\mathfrak{p}_m & \text{if } n \leq m, \\ \mathfrak{p}_m\mathfrak{p}_n & \text{if } n \geq m, \end{cases}$$

the operators $\mathfrak{L}_n$ can be uniformly expressed as

$$\mathfrak{L}_n(\alpha) = \frac{1}{2}\sum_{\nu\in\mathbb{Z}} : \mathfrak{p}_{n-\nu}\mathfrak{p}_\nu : (\tau_*\alpha).$$

Theorem 5.17 *The operators $\mathfrak{L}_n$ and $\mathfrak{p}_n$ on $\mathcal{H}$ satisfy the following supercommutation relations:*

1. $[\mathfrak{L}_n(\alpha), \mathfrak{p}_m(\beta)] = -m\mathfrak{p}_{n+m}(\alpha\beta)$, *and*
2. $[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m)\mathfrak{L}_{n+m}(\alpha\beta) - \frac{1}{12}(n^3-n)\delta_{n+m,0}(\int_X \mathbf{e}(X)\alpha\beta)\,\mathrm{Id}_{\mathcal{H}}$.

Here, $\mathbf{e}(X)$ is the Euler class of X. Taking only the operators $\mathfrak{L}_n(1)$, $n \in \mathbb{Z}$, we see that the classical Virasoro algebra [78] acts on $\mathcal{H}$ with central charge equal to the Euler number of X.

Proof Assume first that $n \neq 0$. For any classes α and β with

$$\tau_*\alpha = \sum_i \alpha_i' \otimes \alpha_i'',$$

we have

$$\begin{aligned}
&[\mathfrak{p}_{n-\nu}(\alpha_i')\mathfrak{p}_\nu(\alpha_i''), \mathfrak{p}_m(\beta)] \\
&\quad = \mathfrak{p}_{n-\nu}(\alpha_i')[\mathfrak{p}_\nu(\alpha_i''), \mathfrak{p}_m(\beta)] + (-1)^{|\beta||\alpha_i''|}[\mathfrak{p}_{n-\nu}(\alpha_i'), \mathfrak{p}_m(\beta)]\mathfrak{p}_\nu(\alpha_i'') \\
&\quad = (-m)\delta_{m+\nu,0}\cdot \mathfrak{p}_{n+m}(\alpha_i')\cdot \int_X \alpha_i''\beta \\
&\qquad + (-1)^{|\beta||\alpha|}(-m)\delta_{n+m-\nu,0}\cdot \int_X \beta\alpha_i' \cdot \mathfrak{p}_{n+m}(\alpha_i'').
\end{aligned}$$

If we sum over all ν and i, we get

$$2[\mathfrak{L}_n(\alpha), \mathfrak{p}_m(\beta)] = \sum_\nu [\mathfrak{p}_{n-\nu}\mathfrak{p}_\nu\tau_*(\alpha), \mathfrak{p}_m(\beta)] = (-m)\cdot\mathfrak{p}_{n+m}(\gamma)$$

with

$$\gamma = pr_{1*}(\tau_*(\alpha)\cdot pr_2^*(\beta)) + (-1)^{|\beta|\cdot|\alpha|}\cdot pr_{2*}(pr_1^*(\beta)\cdot\tau_*(\alpha)) = 2\alpha\beta.$$

Now suppose that $n = 0$. Then for $\nu > 0$, we have

$$[\mathfrak{p}_{-\nu}\mathfrak{p}_\nu(\tau_*(\alpha)), \mathfrak{p}_m(\beta)] = -m \cdot \mathfrak{p}_m(\alpha\beta) \cdot (\delta_{m-\nu} + \delta_{m+\nu}).$$

Thus, summing over all $\nu > 0$, we find again

$$[\mathfrak{L}_0(\alpha), \mathfrak{p}_m(\beta)] = -m \cdot \mathfrak{p}_m(\alpha\beta).$$

This proves the first part of the theorem.

As for the second part, assume first that $n \geq 0$. In order to avoid case considerations, let us agree that $\mathfrak{p}_{k/2}$ is the zero operator if k is odd. Then we may write

$$\mathfrak{L}_m(\alpha) = \frac{1}{2}\mathfrak{p}_{m/2}^2(\tau_*\alpha) + \sum_{\mu > \frac{m}{2}} \mathfrak{p}_{m-\mu}\mathfrak{p}_\mu(\tau_*\alpha).$$

By the first part of the theorem, we have

$$[\mathfrak{L}_n(\alpha), \mathfrak{p}_{m-\mu}\mathfrak{p}_\mu(\tau_*(\beta))] = (-\mu\mathfrak{p}_{n+\mu}\mathfrak{p}_{m-\mu} + (\mu - m)\mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu})\tau_*(\alpha\beta).$$

In the following calculation, we suppress the cohomology classes α and β (as well as various Kronecker δs) until the very end. Summing over all $\mu \geq 0$, we get

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_m] &= -\frac{m}{4}(\mathfrak{p}_{n+m/2}\mathfrak{p}_{m/2} + \mathfrak{p}_{m/2}\mathfrak{p}_{n+m/2}) \\
&\quad + \sum_{\mu > \frac{m}{2}} (\mu - m)\mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu} + \sum_{\mu > \frac{m}{2}} (-\mu)\mathfrak{p}_{n+\mu}\mathfrak{p}_{m-\mu} \\
&= -\frac{m}{4}(\mathfrak{p}_{n+m/2}\mathfrak{p}_{m/2} + \mathfrak{p}_{m/2}\mathfrak{p}_{n+m/2}) \\
&\quad + \sum_{\mu > \frac{m}{2}} (\mu - m)\mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu} + \sum_{\mu > n + \frac{m}{2}} (n - \mu)\mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu}.
\end{aligned}$$

Hence

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_m] - (n - m)\sum_{\mu > \frac{n+m}{2}} \mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu} &= -\frac{m}{4}(\mathfrak{p}_{n+m/2}\mathfrak{p}_{m/2} + \mathfrak{p}_{m/2}\mathfrak{p}_{n+m/2}) \\
&\quad + \sum_{\frac{m}{2} < \mu \leq \frac{m+n}{2}} (\mu - m)\mathfrak{p}_\mu\mathfrak{p}_{m+n-\mu} \\
&\quad - \sum_{\frac{n+m}{2} < \mu \leq n + \frac{m}{2}} (n - \mu)\mathfrak{p}_\mu\mathfrak{p}_{n+m-\mu}.
\end{aligned}$$

Now split off the summands corresponding to the indices $\mu = \frac{1}{2}(m + n)$ and $\mu = n + \frac{1}{2}m$ from the sums. Substituting $n + m - \mu$ for μ in the second sum

on the right hand side, we are left with the expression

$$[\mathfrak{L}_n, \mathfrak{L}_m] - (n-m)\mathfrak{L}_{n+m} = -\frac{m}{4}[\mathfrak{p}_{m/2}, \mathfrak{p}_{n+m/2}] + \sum_{\frac{m}{2}<\mu<\frac{n+m}{2}} (\mu - m)[\mathfrak{p}_\mu, \mathfrak{p}_{n+m-\mu}].$$

The right hand side is zero unless $n + m = 0$. In this case, observe that the composition

$$H^*(X) \xrightarrow{\tau_*} H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)$$

is multiplication with $\mathbf{e}(X)$. Hence, we see that

$$[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m)\mathfrak{L}_{n+m}(\alpha\beta) + \delta_{n+m} \cdot \int_X \mathbf{e}(X)\alpha\beta \cdot N,$$

where N is the number

$$N = \begin{cases} \sum_{0<\nu<\frac{n}{2}} \nu(\nu - n) & \text{if } n \text{ is odd,} \\ \sum_{0<\nu<\frac{n}{2}} \nu(\nu - n) - \frac{1}{8}n^2 & \text{if } n \text{ is even.} \end{cases}$$

An easy computation shows that in both cases N equals $(n - n^3)/12$. □

5.2.2 The obstruction bundle

In this section, we compute the ring structure of $H^*_{\mathrm{CR}}(X^n/S_n)$. The first such computations were done by Fantechi and Gottsche [52] and Uribe [153]. In combination with results of Lehn and Sorger [93], they proved the Cohomological Hyperkähler Resolution Conjecture 4.24 for symmetric products of $K3$ and $\mathbb{T}^4$, with resolutions the corresponding Hilbert schemes of points. They achieved this via direct computations.

From the definition, it is clear that the cup product is determined once we understand the relevant obstruction bundles. To do so, we introduce some additional notation. For $\sigma, \rho \in S_n$, let $T \subset [n] = \{1, 2, \ldots, n\}$ be a set stable under the action of σ; we will denote by $\mathcal{O}(\sigma; T)$ the set of orbits under the action of σ on T. If T is both σ-stable and ρ-stable, $\mathcal{O}(\sigma, \rho; T)$ will be the set of orbits under the action of the subgroup $\langle\sigma, \rho\rangle$ generated by σ and ρ. When $T = [n]$, we drop it from the notation, so $\mathcal{O}(\sigma, [n])$ will be denoted by $\mathcal{O}(\sigma)$, and so on. For instance, if $\ell(\sigma)$ once again denotes the length of the permutation σ, then

$$\ell(\sigma) + |\mathcal{O}(\sigma)| = n.$$

Superscripts on X will count the number of copies in the Cartesian product, and, in this section only, subscripts will be elements of the group and will

determine fixed-point sets. Hence, X^n_σ will denote those points fixed under the action of σ on X^n.

Let $\mathcal{Y} = X^n/S_n$. For $h_1, h_2 \in S_n$, the obstruction bundle $E_{(h_1,h_2)}$ over $\mathcal{Y}_{(h_1,h_2)}$ is defined by

$$E_{(\mathbf{h})} = \left(H^1(\Sigma) \otimes e^* T\mathcal{Y}\right)^G ,$$

where $G = \langle h_1, h_2 \rangle$ and Σ is an orbifold Riemann surface provided with a G action such that $\Sigma/G = (S^2, (x_1, x_2, x_3), (k_1, k_2, k_3))$ is an orbifold sphere with three marked points.

Let E_{h_1,h_2} be the pullback of $E_{(h_1,h_2)}$ under $\pi : X^n_{h_1,h_2} \to \mathcal{Y}_{(h_1,h_2)}$. Because $H^1(\Sigma)$ is a trivial bundle,

$$E_{h_1,h_2} = \pi^* E_{(h_1,h_2)} = \left(H^1(\Sigma) \otimes \Delta^* T X^n\right)^G ,$$

where $\Delta : X^n_{h_1,h_2} \hookrightarrow X^n$ is the inclusion (if $q : X^n \to \mathcal{Y}$ is the quotient map, then $q \circ \Delta = e \circ \pi$).

Without loss of generality, we can assume that $|\mathcal{O}(h_1, h_2)| = k$, and that $n_1 + \cdots + n_k = n$ is a partition of n such that

$$T_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}$$

and $\{T_1, T_2, \ldots, T_k\} = \mathcal{O}(h_1, h_2)$. We will show that the obstruction bundle $E_{h_1,h_2} = \prod_i E^i_{h_1,h_2}$ is the product of k bundles over X, where the factor $E^i_{h_1,h_2}$ corresponds to the orbit T_i.

Let $\Delta_i : X \to X^{n_i}$, $i = 1, \ldots, k$ be the diagonal inclusions. Then the bundles $\Delta_i^* T X^{n_i}$ become G-bundles via the restriction of the action of G on the orbit T_i, and

$$\Delta^* T X^n \cong \Delta_1^* T X^{n_1} \times \cdots \times \Delta_k^* T X^{n_k}$$

as G-vector bundles. This stems from the fact that the orbits T_i are G-stable, hence G induces an action on each X^{n_i}. Therefore, the obstruction bundle splits as

$$E_{h_1,h_2} = \prod_{i=1}^{k} \left(H^1(\Sigma) \otimes \Delta_i^* T X^{n_i}\right)^G . \tag{5.11}$$

We can simplify the previous expression a bit further. Let G_i be the subgroup of S_{n_i} obtained from G when its action is restricted to the elements in T_i; then we have a surjective homomorphism

$$\lambda_i : G \to G_i ,$$

where the action of G on $\Delta_i^* T X^{n_i}$ factors through G_i. So we have

$$\left(H^1(\Sigma) \otimes \Delta_i^* T X^{n_i}\right)^G \cong \left(H^1(\Sigma)^{\ker(\lambda_i)} \otimes \Delta_i^* T X^{n_i}\right)^{G_i} .$$

Now let $\Sigma_i = \Sigma / \ker(\lambda_i)$; it is an orbifold Riemann surface with a G_i action such that Σ_i / G_i becomes an orbifold sphere with three marked points (the markings are with respect to the generators $\lambda_i(h_1)$, $\lambda_i(h_2)$, and $\lambda_i((h_1 h_2)^{-1})$ of G_i). So, just as in the definition of the obstruction bundle $E_{(\mathbf{h})}$, we define

$$E^i_{h_1,h_2} = \left(H^1(\Sigma_i) \otimes \Delta_i^* T X^{n_i}\right)^{G_i}.$$

Then the obstruction bundle splits as

$$E_{h_1,h_2} = \prod_{i=1}^{k} E^i_{h_1,h_2},$$

as desired.

As the action of G_i in $\Delta_i^* T X^{n_i}$ is independent of the structure of X, we have

$$\Delta_i^* T X^{n_i} \cong T X \otimes \mathbb{C}^{n_i}$$

as G_i-vector bundles, where $G_i \subseteq S_{n_i}$ acts on $\mathbb{C}^{n_i}$ in the natural way via the regular representation. Then

$$E^i_{h_1,h_2} \cong T X \otimes (H^1(\Sigma_i) \otimes \mathbb{C}^{n_i})^{G_i}. \tag{5.12}$$

Defining $r(h_1, h_2)(i) = \dim_{\mathbb{C}}(H^1(\Sigma_i) \otimes \mathbb{C}^{n_i})^{G_i}$, it follows that the Euler class of $E^i_{h_1,h_2}$ equals the Euler class of X raised to this exponent: $\mathbf{e}(E^i_{h_1,h_2}) = \mathbf{e}(X)^{r(h_1,h_2)(i)}$. However, the underlying space is only one copy of X. We conclude that

$$\mathbf{e}(E^i_{h_1,h_2}) = \begin{cases} 1 & \text{if } r(h_1, h_2)(i) = 0, \\ \mathbf{e}(X) & \text{if } r(h_1, h_2)(i) = 1, \\ 0 & \text{if } r(h_1, h_2)(i) \geq 2. \end{cases} \tag{5.13}$$

We have proved the following theorem.

Theorem 5.18

$$\mathbf{e}(E_{h_1,h_2}) = \prod_{i=1}^{k} \mathbf{e}(E^i_{h_1,h_2}),$$

where

$$\mathbf{e}(E^i_{h_1,h_2}) = \begin{cases} 1 & \textit{if } r(h_1, h_2)(i) = 0, \\ \mathbf{e}(X) & \textit{if } r(h_1, h_2)(i) = 1, \\ 0 & \textit{if } r(h_1, h_2)(i) \geq 2. \end{cases}$$

5.2.3 LLQW axioms

The computation above, while interesting and correct, exhibits relatively little of the deeper structure of Chen–Ruan cohomology. To rectify this shortcoming, Qin and Wang [127] devised a very different approach to the Chen–Ruan cohomology of symmetric products motivated by the study of the cohomology of the Hilbert scheme of points. Building on early work of Lehn [91] and Li, Qin, and Wang [98, 99] on the Hilbert scheme, their approach is to axiomatize the cohomology rings: the results are the *LLQW axioms* referred to in the title of this section. Once the cohomology is axiomatized, one need only check the axioms for both rings in order to verify the Hyperkähler Resolution Conjecture 4.24.

Using this method, Qin and Wang were able to prove the Hyperkähler Resolution Conjecture for the Hilbert schemes of points of both the cotangent bundle $T^*\Sigma$ of a Riemann surface and also the minimal resolution of $\mathbb{C}^2/\Gamma$ [100, 126]. Throughout this section, we assume that the complex manifold X is of even complex dimension $2d$. As before, $\cup$ will denote the Chen–Ruan product, while juxtaposition will be the Heisenberg action. Instead of introducing the LLQW axioms immediately, we start by establishing key properties of the ring structure from a representation theoretic point of view. In the process, the LLQW axioms will naturally arise.

The construction starts with a set of special classes in $H^*_{\mathrm{CR}}(X^n/S_n;\mathbb{C})$. On the Hilbert scheme side, this was motivated by the Chern character of a certain universal sheaf. As in the last section, however, the symmetric product side of the story is purely combinatorial. Recall [77, 118] that the *Jucys–Murphy elements* $\xi_{j;n}$ associated to the symmetric group S_n are defined to be the following sums of transpositions:

$$\xi_{j;n} = \sum_{i<j} (i, j) \in \mathbb{C}S_n, \quad j = 1, \dots, n.$$

When it is clear from the context, we may simply write ξ_j instead of $\xi_{j;n}$. Let Ξ_n be the set $\{\xi_1, \dots, \xi_n\}$. According to Jucys [78], the kth elementary symmetric function $e_k(\Xi_n)$ in the variables Ξ_n is equal to the sum of all permutations in S_n having exactly $(n-k)$ cycles.

Given $\gamma \in H^*(X)$, we write

$$\gamma^{(i)} = 1^{\otimes i-1} \otimes \gamma \otimes 1^{\otimes n-i} \in H^*(X^n),$$

and regard it as a cohomology class in $H^*(X^n, S_n)$ associated to the identity conjugacy class. We define $\xi_i(\gamma) = \xi_i + \gamma^{(i)} \in H^*(X^n, S_n)$.

Regarding $\xi_i = \xi_i(0) \in H^*(X^n, S_n)$, we let

$$\xi_i^{\cup k} = \overbrace{\xi_i \cup \cdots \cup \xi_i}^{k \text{ times}} \in H^*(X^n, S_n),$$

and define

$$e^{-\xi_i} = \sum_{k \geq 0} \frac{1}{k!}(-\xi_i)^{\cup k} \in H^*(X^n, S_n).$$

Definition 5.19 For homogeneous elements $\alpha \in H^{|\alpha|}(X)$, we define the class $O^k(\alpha, n) \in H^*_{\mathrm{CR}}(X^n/S_n)$ to be

$$O^k(\alpha, n) = \sum_{i=1}^{n} (-\xi_i)^{\cup k} \cup \alpha^{(i)} \in H_{\mathrm{CR}}^{dk+|\alpha|}(X^n/S_n),$$

and extend this linearly to all $\alpha \in H^*(X)$. We put

$$O(\alpha, n) = \sum_{k \geq 0} \frac{1}{k!} O^k(\alpha, n) = \sum_{i=1}^{n} e^{-\xi_i} \cup \alpha^{(i)}.$$

We obtain operators $\mathfrak{O}^k(\alpha) \in \mathrm{End}(\mathcal{H})$ (resp. $\mathfrak{O}(\alpha)$) by cupping with $O^k(\alpha, n)$ (resp. $O(\alpha, n)$) in $H^*_{\mathrm{CR}}(X^n/S_n)$ for each $n \geq 0$.

Remark 5.20 We can see that $O^k(\alpha, n) \in H^*(X^n, S_n)$ is S_n-invariant as follows. For $\gamma \in H^*(X)$, note that $e_j(\xi_1(\gamma), \ldots, \xi_n(\gamma))$ lies in $\mathcal{H}$, where $e_j(\xi_1(\gamma), \ldots, \xi_n(\gamma))$ is the jth elementary symmetric function for $(1 \leq j \leq n)$. So $\mathcal{H}$ contains all symmetric functions in the classes $\xi_i(\gamma)$. In particular, $O(e^{-\gamma}, n) = \sum_i (e^{-\xi_i} \cup (e^{-\gamma})^{(i)}) = \sum_i e^{-\xi_i(\gamma)} \in \mathcal{H}$. Letting γ vary, we see that $O(\alpha, n)$ and similarly $O^k(\alpha, n)$ lie in $\mathcal{H}$.

The operator $\mathfrak{O}^1(1_X) \in \mathrm{End}(\mathcal{H})$ plays a special role in the theory. Given an operator $\mathfrak{f} \in \mathrm{End}(\mathcal{H})$, we write $\mathfrak{f}' = [\mathfrak{O}^1(1_X), f]$, and recursively define $\mathfrak{f}^{(k+1)} = (\mathfrak{f}^{(k)})'$. It follows directly from the Jacobi identity that $\mathfrak{f} \to \mathfrak{f}'$ is a derivation – i.e., for any two operators $\mathfrak{a}$ and $\mathfrak{b} \in \mathrm{End}(\mathcal{H})$, the "Leibniz rule" holds:

$$(\mathfrak{a}\mathfrak{b})' = \mathfrak{a}'\mathfrak{b} + \mathfrak{a}\mathfrak{b}' \quad \text{and} \quad [\mathfrak{a}, \mathfrak{b}]' = [\mathfrak{a}', \mathfrak{b}] + [\mathfrak{a}, \mathfrak{b}'].$$

We start our calculation from this simplest operator $\mathfrak{O}^1(1_X)$. Indeed, we can determine it explicitly.

Our convention for vertex operators or fields is to write them in the form

$$\phi(z) = \sum_n \phi_n z^{-n-\Delta},$$

where Δ is the conformal weight of the field $\phi(z)$. We define the normally ordered product $:\phi_1(z) \cdots \phi_k(z):$ as usual (see [78], for example, for more details).

For $\alpha \in H^*(X)$, we define a vertex operator $\mathfrak{p}(\alpha)(z)$ by putting

$$\mathfrak{p}(\alpha)(z) = \sum_{n\in\mathbb{Z}} \mathfrak{p}_n(\alpha) z^{-n-1}.$$

Recall the pushforward τ_{p*} defined in equation (5.10). The field $: \mathfrak{p}(z)^p : (\tau_{p*}\alpha)$ (most often written as $: \mathfrak{p}(z)^p : (\tau_*\alpha)$ below) is defined to be

$$\sum_i : \mathfrak{p}(\alpha_{i,1})(z)\mathfrak{p}(\alpha_{i,2})(z)\cdots\mathfrak{p}(\alpha_{i,p})(z) :$$

where $\tau_{p*}\alpha = \sum_i \alpha_{i,1} \otimes \alpha_{i,2} \otimes \cdots \otimes \alpha_{i,p} \in H^*(X)^{\otimes p}$. We rewrite $: \mathfrak{p}(z)^p :$ $(\tau_*\alpha)$ componentwise as

$$: \mathfrak{p}(z)^p : (\tau_*\alpha) = \sum_m : \mathfrak{p}^p :_m (\tau_*\alpha)\, z^{-m-p}.$$

Here, the coefficient $: \mathfrak{p}^p :_m (\tau_*\alpha) \in \mathrm{End}(\mathcal{H})$ of z^{-m-p} is the mth Fourier component of the field $: \mathfrak{p}(z)^p : (\tau_*\alpha)$; it maps $H^*_{\mathrm{CR}}(X^n/S_n)$ to $H^*_{\mathrm{CR}}(X^{n+m}/S_{n+m})$.

Theorem 5.21 *We have* $\mathfrak{O}^1(1_X) = -\frac{1}{6} : \mathfrak{p}^3 :_0 (\tau_* 1_X)$.

Proof It is clear that

$$: \mathfrak{p}^3 :_0 = \sum_{l_1+l_2+l_3=0} : \mathfrak{p}_{l_1}\mathfrak{p}_{l_2}\mathfrak{p}_{l_3}(\tau_{3*}1_X) :,$$

and so

$$\frac{1}{6} : \mathfrak{p}^3 :_0 = \sum_{\substack{l_1+l_2+l_3=0,\\ l_1\le l_2\le l_3}} \mathfrak{p}_{l_1}\mathfrak{p}_{l_2}\mathfrak{p}_{l_3}(\tau_{3*}1_X).$$

Since $l_1 + l_2 + l_3 = 0$, they cannot all be positive. There are two cases: either $l_1 < 0 < l_2, l_3$ or $l_1, l_2 < 0 < l_3$. Suppose we have the former case. Then $l_1 = -(l_2 + l_3)$. Consider the action of the operator $\mathfrak{p}_{-(l_2+l_3)}\mathfrak{p}_{l_2}\mathfrak{p}_{l_3}(\tau_{3^*1_X})$ on a basis element

$$\mathfrak{p}_{-1}(\alpha_1^1)\cdots\mathfrak{p}_{-1}(\alpha_{n_1}^1)\mathfrak{p}_{-2}(\alpha_1^2)\cdots\mathfrak{p}_{-2}(\alpha_{n_2}^2)\cdots\mathfrak{p}_{-k}(\alpha_1^k)\cdots\mathfrak{p}_{-k}(\alpha_{n_k}^k)|0\rangle.$$

The result is zero unless $l_2, l_3 \le k$. Using the supercommutation relations (5.6), we find that when $l_2 \ne l_3$,

$$\begin{aligned}
&\mathfrak{p}_{-(l_2+l_3)}\mathfrak{p}_{l_2}\mathfrak{p}_{l_3}(\tau_{3^*1_X})\big(\mathfrak{p}_{-1}(\alpha_1^1)\cdots\mathfrak{p}_{-1}(\alpha_{n_1}^1)\mathfrak{p}_{-2}(\alpha_1^2)\cdots\mathfrak{p}_{-2}(\alpha_{n_2}^2)\cdots\mathfrak{p}_{-k}(\alpha_1^k)\cdots\mathfrak{p}_{-k}(\alpha_{n_k}^k)|0\rangle\big)\\
&\quad = l_2 l_3 \sum_{\substack{1\le i\le n_{l_2},\\ 1\le j\le n_{l_3}}} \mathfrak{p}_{-l_2-l_3}(\alpha_i^{l_2} \cup \alpha_j^{l_3})\mathfrak{p}_{-1}(\alpha_1^1)\cdots\widehat{\mathfrak{p}_{-l_2}(\alpha_i^{l_2})}\cdots\widehat{\mathfrak{p}_{-l_3}(\alpha_j^{l_3})}\cdots|0\rangle.
\end{aligned}$$

When, on the other hand, $l_2 = l_3 \leq k$, we get

$$\mathfrak{p}_{-(2l_2)}\mathfrak{p}_{l_2}\mathfrak{p}_{l_2}(\tau_{3*}1_X)\left(\mathfrak{p}_{-1}(\alpha_1^1)\cdots\mathfrak{p}_{-1}(\alpha_{n_1}^1)\mathfrak{p}_{-2}(\alpha_1^2)\cdots\mathfrak{p}_{-2}(\alpha_{n_2}^2)\cdots\mathfrak{p}_{-k}(\alpha_1^k)\cdots\mathfrak{p}_{-k}(\alpha_{n_k}^k)|0\rangle\right)$$
$$= l_2^2 \sum_{1\leq i\leq j\leq n_{l_2}} \mathfrak{p}_{-2l_2}(\alpha_i^{l_2} \cup \alpha_j^{l_2})\mathfrak{p}_{-1}(\alpha_1^1)\cdots\widehat{\mathfrak{p}_{-l_2}(\alpha_i^{l_2})}\cdots\widehat{\mathfrak{p}_{-l_2}(\alpha_j^{l_2})}\cdots|0\rangle.$$

In either case, we join two cycles of length n_2, n_3 to form a cycle of length $n_2 + n_3$. Proceeding similarly in the case $l_1, l_2 < 0$, where $l_3 = -(l_1 + l_2)$, we get

$$\mathfrak{p}_{l_1}\mathfrak{p}_{l_2}\mathfrak{p}_{-(l_1+l_2)}(\tau_{3*}1_X)\left(\mathfrak{p}_{-1}(\alpha_1^1)\cdots\mathfrak{p}_{-1}(\alpha_{n_1}^1)\mathfrak{p}_{-2}(\alpha_1^2)\cdots\mathfrak{p}_{-2}(\alpha_{n_2}^2)\cdots\mathfrak{p}_{-k}(\alpha_1^k)\cdots\mathfrak{p}_{-k}(\alpha_{n_k}^k)|0\rangle\right)$$
$$= -(l_1 + l_2) \sum_{1\leq i\leq n_{-l_1-l_2}} \mathfrak{p}_{l_1}\mathfrak{p}_{l_2}(\tau_{2*}\alpha_i)\mathfrak{p}_{-l_1}(\alpha_1^1)\cdots\widehat{\mathfrak{p}_{l_1+l_2}(\alpha_i^{n(-l_1-l_2)})}\cdots|0\rangle.$$

Next, we compute $\mathfrak{O}^1(1_X)$. By definition, $O^1(1_X) = -\sum_{i<j}(i, j)$. The cohomology from the sector $X_g^n/\mathbf{C}(g)$ is of the form

$$\mathfrak{p}_\lambda|0\rangle = \mathfrak{p}_{-1}(\alpha_1^1)\cdots\mathfrak{p}_{-1}(\alpha_{n_1}^1)\mathfrak{p}_{-2}(\alpha_1^2)\cdots\mathfrak{p}_{-2}(\alpha_{n_2}^2)\cdots\mathfrak{p}_{-k}(\alpha_1^k)\cdots\mathfrak{p}_{-k}(\alpha_{n_k}^k)|0\rangle$$
$$= \sum_{h\in S_n} \mathrm{ad}_h\left(\bigotimes_i\bigotimes_j \alpha_j^i\right),$$

for an appropriate multipartition λ. On such a class, we calculate

$$\begin{aligned}
\mathfrak{O}^1(1_X)(\mathfrak{p}_\lambda|0\rangle) &= O^1(1_X) \cup \mathfrak{p}_\lambda|0\rangle \\
&= -\sum_{a<b}\sum_{h\in S_n}(a, b) \cup \mathrm{ad}_h\left(\bigotimes_i\bigotimes_j \alpha_j^i\right) \\
&= -\sum_{h\in S_n}\sum_{a<b}(a, b) \cup \mathrm{ad}_h\left(\bigotimes_i\bigotimes_j \alpha_j^i\right).
\end{aligned}$$

Suppose g has an i-cycle and a j-cycle such that a is in the i-cycle and b is in the j-cycle. Then the transposition (a, b) will join the two cycles into a single cycle of length $i + j$. Moreover, as a varies within the cycle of length i, and b within the cycle of length j, the resulting permutation $(a, b)g$ has the same cycle type, and hence gives ij copies of the same Chen–Ruan cohomology class.

Next, we consider the obstruction bundles. Suppose that the cohomology classes corresponding to our i- and j-cycles are α_l^i and α_k^j, respectively. The relevant part of the two-sector $X_{(a,b),g}$ is X. There is no obstruction bundle in this case. The corresponding operation on cohomology is the pullback of $\alpha_l^i \otimes \alpha_k^j$ by the diagonal embedding $X \to X \times X$, followed by the pushforward

through the identity map $X \to X$. Thus, we simply obtain $\alpha_l^i \cup \alpha_k^j$, precisely matching the first two cases above.

If both a and b are inside an m-cycle of g, the product $(a, b)g$ breaks the m-cycle into two cycles of length $b - a$ and $m - (b - a)$. Fix $i = b - a$ and $j = m - i$. We still have freedom to move a inside the m cycle, with the resulting products having the same cycle types. Therefore, we obtain $m = i + j$ copies of the same class. Suppose that the cohomology class corresponding to the $i + j$ cycles is α_l^{i+j}. There is no obstruction bundle in this case either. The corresponding operation on cohomology is the pullback of α_l^{i+j} by the identity map $X \to X$, followed by the pushforward through the diagonal embedding $X \to X \times X$, which is just $\tau_* \alpha_l^{i+j}$. This matches the third case above, and the theorem is proved. $\square$

The other key property is formulated in terms of the interaction between the cup product operator $\mathfrak{O}(\gamma)$ and the Heisenberg operator $\mathfrak{p}_{-1}(\alpha)$.

Theorem 5.22 *Let $\gamma, \alpha \in H^*(X)$. Then for each $k \geq 0$, we have*

$$\left[\mathfrak{O}^k(\gamma), \mathfrak{p}_{-1}(\alpha)\right] = \mathfrak{p}_{-1}^{(k)}(\gamma\alpha).$$

Proof To simplify signs, we assume that the cohomology classes γ and α are of even degree. Recall that

$$\mathfrak{p}_{-1}(\alpha)(y) = \frac{1}{(n-1)!} \sum_{g \in S_n} \mathrm{ad}_g(\alpha \otimes y)$$

for $y \in H^*_{CR}(X^{n-1}/S_{n-1})$. Regarding S_{n-1} as the subgroup $S_{n-1} \times 1$ of S_n, we introduce an injective ring homomorphism

$$\iota : H^*(X^{n-1}, S_{n-1}) \to H^*(X^n, S_n)$$

by sending κ_σ to $\kappa_\sigma \otimes 1_X$, where $\sigma \in S_{n-1}$, and κ_σ is a class coming from the σ-fixed locus. Thus

$$\begin{aligned}
(n-1)! \left[\mathfrak{O}^k(\gamma), \mathfrak{p}_{-1}(\alpha)\right](y) &= (n-1)! \big(\mathfrak{O}^k(\gamma) \cdot \mathfrak{p}_{-1}(\alpha)(y) - \mathfrak{p}_{-1}(\alpha) \cdot \mathfrak{O}^k(\gamma)(y)\big) \\
&= O^k(\gamma, n) \cup \sum_{g \in S_n} \mathrm{ad}_g(\alpha \otimes y) \\
&\quad - \sum_{g \in S_n} \mathrm{ad}_g(\alpha \otimes (O^k(\gamma, n-1) \cup y)) \\
&= \sum_{g \in S_n} \mathrm{ad}_g\big((O^k(\gamma, n) - \iota(O^k(\gamma, n-1))) \cup (\alpha \otimes y)\big),
\end{aligned}$$

where we use the fact that $O^k(\gamma, n)$ is S_n-invariant. By definition, we have $O^k(\gamma, n) - \iota(O^k(\gamma, n-1)) = (-\xi_{n;n})^{\cup k} \cup \gamma^{(n)}$. Thus, we obtain

$$\begin{aligned}(n-1)!\,\big[\mathfrak{O}^k(\gamma), \mathfrak{p}_{-1}(\alpha)\big](y) &= \sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup k} \cup \gamma^{(n)} \cup (\alpha \otimes y)\right)\\ &= \sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes y)\right).\end{aligned}$$

It remains to prove that

$$\sum_{g\in S_n} \mathrm{ad}_g\left((-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes y)\right) = (n-1)!\ \mathfrak{p}_{-1}^{(k)}(\gamma\alpha)(y). \tag{5.14}$$

We prove this by induction. It is clearly true for $k = 0$. Note that

$$O^1(1_X, n) - \iota(O^1(1_X, n-1)) = -\xi_{n;n}.$$

Under the assumption that (5.14) holds for k, we have

$$\begin{aligned}&\sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup(k+1)} \cup (\gamma\alpha \otimes y)\right)\\ &\quad= \sum_g \mathrm{ad}_g\left((O^1(1_X, n) - \iota(O^1(1_X, n-1))) \cup (-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes y)\right)\\ &\quad= O^1(1_X, n) \cup \sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes y)\right)\\ &\qquad - \sum_g \mathrm{ad}_g\left(\iota(O^1(1_X, n-1)) \cup (-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes y)\right),\end{aligned}$$

since $O^1(\gamma, n)$ is S_n-invariant. Using the induction assumption twice, we get

$$\begin{aligned}&\sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup(k+1)} \cup (\gamma\alpha \otimes y)\right)\\ &\quad= (n-1)!\ O^1(1_X, n) \cup \mathfrak{p}_{-1}^{(k)}(\gamma\alpha)(y)\\ &\qquad - \sum_g \mathrm{ad}_g\left((-\xi_{n;n})^{\cup k} \cup (\gamma\alpha \otimes (O^1(1_X, n-1) \cup y))\right)\\ &\quad= (n-1)!\big(O^1(1_X, n) \cup \mathfrak{p}_{-1}^{(k)}(\gamma\alpha)(y) - \mathfrak{p}_{-1}^{(k)}(\gamma\alpha)(O^1(1_X, n-1) \cup y)\big)\\ &\quad= (n-1)!\ \mathfrak{p}_{-1}^{(k+1)}(\gamma\alpha)(y).\end{aligned}$$

By induction, we have established (5.14), and thus the theorem. □

Definition 5.23 The Heisenberg commutation relations (5.6), Theorem 5.21, and Theorem 5.22 together constitute the *LLQW axioms of Chen–Ruan cohomology*.

The central algebraic theorem is:

Theorem 5.24 *The LLQW axioms uniquely determine the Chen–Ruan cohomology ring of the symmetric product on X. That is, suppose we have an irreducible representation of the super Heisenberg algebra $\mathcal{A}(H^*(X))$ on a graded ring $\mathcal{V}$. If $\mathcal{V}$ satisfies Theorems 5.21 and 5.22, then $\mathcal{V}$ must be isomorphic as a graded ring to the Chen–Ruan cohomology $\mathcal{H} = \bigoplus_n H^*_{\mathrm{CR}}(X^n/S_n)$.*

We refer readers to the original paper for the proof.

References

[1] D. Abramovich, T. Graber, and A. Vistoli, Algebraic orbifold quantum products, *Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, *Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 1–24.

[2] D. Abramovich and A. Vistoli, Compactifying the space of stable maps, *J. Amer. Math. Soc.* **15**:1 (2002), 27–75 (electronic).

[3] A. Adem, Characters and K-theory of discrete groups, *Invent. Math.* **114**:3 (1993), 489–514.

[4] A. Adem and J. Pan, Toroidal orbifolds, gerbes and group cohomology, *Trans. Amer. Math. Soc.* **358** (2006), 3969–3983.

[5] A. Adem and Y. Ruan, Twisted orbifold K-theory, *Comm. Math. Phys.* **237**:3 (2003), 533–556.

[6] C. Allday and V. Puppe, *Cohomological Methods in Transformation Groups*, Cambridge Studies in Advanced Mathematics, vol. 32, Cambridge: Cambridge University Press, 1993.

[7] M. Artin, Versal deformations and algebraic stacks, *Invent. Math.* **27** (1974), 165–189.

[8] M. Atiyah and G. Segal, Twisted k-theory, math.KT/0407054.

[9] M. F. Atiyah, *K-theory*, second edn, Advanced Book Classics, Redwood City, CA: Addison-Wesley Publishing Company, 1989, notes by D. W. Anderson.

[10] M. F. Atiyah and G. B. Segal, Equivariant K-theory and completion, *J. Differential Geometry* **3** (1969), 1–18.

[11] — On equivariant Euler characteristics, *J. Geom. Phys.* **6**:4 (1989), 671–677.

[12] W. L. Baily, Jr., The decomposition theorem for V-manifolds, *Amer. J. Math.* **78** (1956), 862–888.

[13] W. Barth, C. Peters, and A. Van de Ven, Compact complex surfaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], vol. 4, Berlin: Springer-Verlag, 1984.

[14] V. V. Batyrev, Stringy Hodge numbers of varieties with Gorenstein canonical singularities, *Integrable Systems and Algebraic Geometry* (Kobe/Kyoto, 1997), River Edge, NJ: World Science Publishing, 1998, pp. 1–32.

[15] — Birational Calabi-Yau n-folds have equal Betti numbers, *New Trends in Algebraic Geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser.*, vol. 264, Cambridge: Cambridge University Press, 1999, pp. 1–11.

[16] — Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, *J. Eur. Math. Soc. (JEMS)* **1**:1 (1999), 5–33.

[17] V. V. Batyrev and D. I. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, *Topology* **35**:4 (1996), 901–929.

[18] P. Baum, J.-L. Brylinski, and R. MacPherson, Cohomologie équivariante délocalisée, *C. R. Acad. Sci. Paris Sér. I Math.* **300**:17 (1985), 605–608.

[19] K. Behrend, Gromov–Witten invariants in algebraic geometry, *Invent. Math.* **127**:3 (1997), 601–617.

[20] K. Behrend and B. Fantechi, The intrinsic normal cone, *Invent. Math.* **128**:1 (1997), 45–88.

[21] K. Behrend and Yu. Manin, Stacks of stable maps and Gromov–Witten invariants, *Duke Math. J.* **85**:1 (1996), 1–60.

[22] J. Block and E. Getzler, Equivariant cyclic homology and equivariant differential forms, *Ann. Sci. École Norm. Sup.* (4) **27**:4 (1994), 493–527.

[23] C. Borcea, $K3$ surfaces with involution and mirror pairs of Calabi–Yau manifolds, *Mirror Symmetry, II, AMS/IP Stud. Adv. Math.*, vol. 1, Providence, RI: American Mathematical Society, 1997, pp. 717–743.

[24] A. Borel and G. Prasad, private communication.

[25] A. Borel and J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* **48** (1973), 436–491, With an appendix: Arrondissement des variétés à coins, by A. Douady and L. Hérault.

[26] L. A. Borisov, L. Chen, and G. G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, *J. Amer. Math. Soc.* **18**:1 (2005), 193–215 (electronic).

[27] L. A. Borisov and A. Libgober, Elliptic genera of singular varieties, *Duke Math. J.* **116**:2 (2003), 319–351.

[28] L. A. Borisov and A. R. Mavlyutov, String cohomology of Calabi–Yau hypersurfaces via mirror symmetry, *Adv. Math.* **180**:1 (2003), 355–390.

[29] R. Bott and L. W. Tu, Differential Forms in Algebraic Topology, Graduate Texts in *Mathematics*, vol. 82, New York: Springer-Verlag, 1982.

[30] G. E. Bredon, Equivariant Cohomology Theories, *Lecture Notes in Mathematics*, No. 34, Berlin: Springer-Verlag, 1967.

[31] — Introduction to Compact Transformation Groups, *Pure and Applied Mathematics*, vol. 46, New York: Academic Press, 1972.

[32] T. Bridgeland, A. King, and M. Reid, The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.* **14**:3 (2001), 535–554 (electronic).

[33] K. S. Brown, Euler characteristics of discrete groups and G-spaces, *Invent. Math.* **27** (1974), 229–264.

[34] J.-L. Brylinski and V. Nistor, Cyclic cohomology of étale groupoids, *K-Theory* **8**:4 (1994), 341–365.

[35] B. Chen and S. Hu, A de Rham model for Chen-Ruan cohomology ring of abelian orbifolds, arXiv:math.SG/0408265.

[36] W. Chen, A homotopy theory of orbispaces, arXiv:math.AT/0102020.

[37] W. Chen and Y. Ruan, Orbifold Gromov–Witten theory, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 25–85.

[38] — A new cohomology theory of orbifold, *Comm. Math. Phys.* **248**:1 (2004), 1–31.

[39] M. Crainic and I. Moerdijk, A homology theory for étale groupoids, *J. Reine Angew. Math.* **521** (2000), 25–46.
[40] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *Inst. Hautes Études Sci. Publ. Math.* (1969), no. 36, 75–109.
[41] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.* **135**:1 (1999), 201–232.
[42] — Motivic integration, quotient singularities and the McKay correspondence, *Compositio Math.* **131**:3 (2002), 267–290.
[43] R. Dijkgraaf, Discrete torsion and symmetric products, arXiv:hep-th/9912101.
[44] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Strings on orbifolds. I, *Nuclear Phys. B* **261**:4 (1985), 678–686.
[45] — Strings on orbifolds. II, *Nuclear Phys. B* **274**:2 (1986), 285–314.
[46] C. Dong, K. Liu, and X. Ma, On orbifold elliptic genus, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 87–105.
[47] C. Dwyer, Twisted equivariant K-theory for proper actions of discrete groups, Ph.D. thesis, University of Wisconsin, Madison, 2005.
[48] G. Ellingsrud, L. Göttsche, and M. Lehn, On the cobordism class of the Hilbert scheme of a surface, *J. Algebraic Geom.* **10**:1 (2001), 81–100.
[49] G. Ellingsrud and S. A. Strømme, On the homology of the Hilbert scheme of points in the plane, *Invent. Math.* **87**:2 (1987), 343–352.
[50] B. Fantechi, Stacks for everybody, *European Congress of Mathematics (Barcelona, 2000), Progr. Math.*, vol. 201, Basel: Birkhäuser, 2001, pp. 349–359.
[51] B. Fantechi and L. Göttsche, The cohomology ring of the Hilbert scheme of 3 points on a smooth projective variety, *J. Reine Angew. Math.* **439** (1993), 147–158.
[52] — Orbifold cohomology for global quotients, *Duke Math. J.* **117**:2 (2003), 197–227.
[53] W. Feit, The Representation Theory of Finite Groups, *North-Holland Mathematical Library*, vol. 25, Amsterdam: North-Holland Publishing Co., 1982.
[54] D. S. Freed, The Verlinde algebra is twisted equivariant K-theory, *Turkish J. Math.* **25**:1 (2001), 159–167.
[55] — Twisted K-theory and loop groups, *Proceedings of the International Congress (Beijing, 2002) of Mathematicians*, vol. III Beijing, China: Higher Education Press, 2002, pp. 419–430.
[56] D. S. Freed, M. J. Hopkins, and C. Teleman, Twisted K-theory and loop group representations, arXiv:math.AT/0312155.
[57] I. B. Frenkel and W. Wang, Virasoro algebra and wreath product convolution, *J. Algebra* **242**:2 (2001), 656–671.
[58] P. G. Goerss, (Pre-)sheaves of ring spectra over the moduli stack of formal group laws, *Axiomatic, Enriched and Motivic Homotopy Theory, NATO Sci. Ser. II Math. Phys. Chem.*, vol. 131, Dordrecht: Kluwer Acad. Publ., 2004, pp. 101–131.
[59] M. Goresky and R. MacPherson, Intersection homology theory, *Topology* **19**:2 (1980), 135–162.
[60] L. Göttsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, *Math. Ann.* **286**:1–3 (1990), 193–207.

[61] I. Grojnowski, Instantons and affine algebras. I. The Hilbert scheme and vertex operators, *Math. Res. Lett.* **3**:2 (1996), 275–291.

[62] V. Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian T^n-Spaces, *Progress in Mathematics*, vol. 122, Boston, MA: Birkhäuser Boston Inc., 1994.

[63] J.-P. Haeberly, For $G = S^1$ there is no G-Chern character, *Group Actions on Manifolds (Boulder, Co., 1983), Contemp. Math.*, vol. 36, Providence, RI: American Mathematical Society, 1985, pp. 113–118.

[64] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal Structure of Foliations (Toulouse, 1982).

[65] — Orbi-espaces, *Sur les Groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), Progr. Math.*, vol. 83, Boston, MA: Birkhäuser Boston, 1990, pp. 203–213.

[66] — Complexes of groups and orbihedra, *Group Theory from a Geometrical Viewpoint (Trieste, 1990)*, River Edge, NJ: World Science Publishing, 1991, pp. 504–540.

[67] — Extension of complexes of groups, *Ann. Inst. Fourier (Grenoble)* **42**:1–2 (1992), 275–311.

[68] — Groupoids and foliations, *Groupoids in Analysis, Geometry, and Physics (Boulder, CO, 1999), Contemp. Math.*, vol. 282, Providence, RI: American Mathematical Society, 2001, pp. 83–100.

[69] A. Henriques and D. S. Metzler, Presentations of noneffective orbifolds, *Trans. Amer. Math. Soc.* **356**:6 (2004), 2481–2499 (electronic).

[70] M. Hilsum and G. Skandalis, Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes), *Ann. Sci. École Norm. Sup.* (4) **20**:3 (1987), 325–390.

[71] S. Illman, The equivariant triangulation theorem for actions of compact Lie groups, *Math. Ann.* **262**:4 (1983), 487–501.

[72] S. Jackowski, J. McClure, and B. Oliver, Homotopy classification of self-maps of BG via G-actions. I, *Ann. of Math.* (2) **135**:1 (1992), 183–226.

[73] — Homotopy classification of self-maps of BG via G-actions. II, *Ann. of Math.* (2) **135**:2 (1992), 227–270.

[74] Y. Jiang, The Chen–Ruan cohomology of weighted projective spaces, arXiv:math.AG/0304140.

[75] D. D. Joyce, Compact Manifolds with Special Holonomy, *Oxford Mathematical Monographs*, Oxford: Oxford University Press, 2000.

[76] D. Juan-Pineda, On the ring structure of K^* for discrete groups, *Topology Appl.* **87**:2 (1998), 79–88.

[77] A.-A. A. Jucys, Symmetric polynomials and the center of the symmetric group ring, *Rep. Mathematical Phys.* **5**:1 (1974), 107–112.

[78] V. Kac, Vertex Algebras for Beginners, second edn., *University Lecture Series*, vol. 10, Providence, RI: American Mathematical Society, 1998.

[79] G. Karpilovsky, Projective Representations of Finite Groups, *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 94, New York: Marcel Dekker Inc., 1985.

[80] R. M. Kaufmann, Orbifolding Frobenius algebras, *Internat. J. Math.* **14**:6 (2003), 573–617.

[81] Y. Kawamata, D-equivalence and K-equivalence, *J. Differential Geom.* **61**:1 (2002), 147–171.
[82] — Equivalences of derived categories of sheaves on smooth stacks, *Amer. J. Math.* **126**:5 (2004), 1057–1083.
[83] — Log crepant birational maps and derived categories, *J. Math. Sci. Univ. Tokyo* **12**:2 (2005), 211–231.
[84] T. Kawasaki, The signature theorem for V-manifolds, *Topology* **17**:1 (1978), 75–83.
[85] — The Riemann–Roch theorem for complex V-manifolds, *Osaka J. Math.* **16**:1 (1979), 151–159.
[86] M. Kontsevich, Lecture at Orsay, December 7, 1995, unpublished.
[87] M. Kontsevich and Yu. Manin, Gromov–Witten classes, quantum cohomology, and enumerative geometry, *Comm. Math. Phys.* **164**:3 (1994), 525–562.
[88] — Quantum cohomology of a product, *Invent. Math.* **124**:1–3 (1996), 313–339, With an appendix by R. Kaufmann.
[89] N. J. Kuhn, Character rings in algebraic topology, *Advances in Homotopy Theory (Cortona, 1988), London Math. Soc. Lecture Note Ser.*, vol. 139, Cambridge: Cambridge University Press, 1989, pp. 111–126.
[90] G. Laumon and L. Moret-Bailly, Champs algébriques, *Ergebnisse der Mathematik und ihrer Grenzgebiete*. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Berlin: Springer-Verlag, 2000.
[91] M. Lehn, Chern classes of tautological sheaves on Hilbert schemes of points on surfaces, *Invent. Math.* **136**:1 (1999), 157–207.
[92] M. Lehn and C. Sorger, Symmetric groups and the cup product on the cohomology of Hilbert schemes, *Duke Math. J.* **110**:2 (2001), 345–357.
[93] — The cup product of Hilbert schemes for $K3$ surfaces, *Invent. Math.* **152**:2 (2003), 305–329.
[94] A.-M. Li and Y. Ruan, Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds, *Invent. Math.* **145**:1 (2001), 151–218.
[95] W.-P. Li and Z. Qin, On 1-point Gromov–Witten invariants of the Hilbert schemes of points on surfaces, *Turkish J. Math.* **26**:1 (2002), 53–68.
[96] W.-P. Li, Z. Qin, and W. Wang, Generators for the cohomology ring of Hilbert schemes of points on surfaces, *Internat. Math. Res. Notices* (2001), no. 20, 1057–1074.
[97] — Hilbert schemes and $\mathscr{W}$ algebras, *Int. Math. Res. Not.* (2002), no. 27, 1427–1456.
[98] — Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces, *Math. Ann.* **324**:1 (2002), 105–133.
[99] — Stability of the cohomology rings of Hilbert schemes of points on surfaces, *J. Reine Angew. Math.* **554** (2003), 217–234.
[100] — Ideals of the cohomology rings of Hilbert schemes and their applications, *Trans. Amer. Math. Soc.* **356**:1 (2004), 245–265 (electronic).
[101] W. Lück and B. Oliver, Chern characters for the equivariant K-theory of proper G-CW-complexes, *Cohomological Methods in Homotopy Theory (Bellaterra, 1998), Progr. Math.*, vol. 196, Basel: Birkhäuser, 2001, pp. 217–247.

[102] E. Lupercio and M. Poddar, The global McKay-Ruan correspondence via motivic integration, *Bull. London Math. Soc.* **36**:4 (2004), 509–515.

[103] E. Lupercio and B. Uribe, Loop groupoids, gerbes, and twisted sectors on orbifolds, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 163–184.

[104] — Gerbes over orbifolds and twisted K-theory, *Comm. Math. Phys.* **245**:3 (2004), 449–489.

[105] M. Marcolli and V. Mathai, Twisted index theory on good orbifolds. I. Noncommutative Bloch theory, *Commun. Contemp. Math.* **1**:4 (1999), 553–587.

[106] Eyal Markman, Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, *J. Reine Angew. Math.* **544** (2002), 61–82.

[107] D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, University Lecture Series, vol. 6, Providence, RI: American Mathematical Society, 1994.

[108] J. McKay, Graphs, singularities, and finite groups, *The Santa Cruz Conference on Finite Groups (University of California, Santa Cruz, CA, 1979), Proc. Sympos. Pure Math.*, vol. 37, Providence, RI: American Mathematical Society, 1980, pp. 183–186.

[109] D. Metzler, Topological and smooth stacks, math.DG/0306176.

[110] J. W. Milnor and James D. Stasheff, Characteristic Classes, Press, *Annals of Mathematics Studies*, No. 76. Princeton, NJ: Princeton University Press, 1974.

[111] I. Moerdijk, Classifying toposes and foliations, *Ann. Inst. Fourier (Grenoble)* **41**:1 (1991), 189–209.

[112] — Orbifolds as groupoids: an introduction, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 205–222.

[113] I. Moerdijk and D. A. Pronk, Orbifolds, sheaves and groupoids, *K-Theory* **12**:1 (1997), 3–21.

[114] — Simplicial cohomology of orbifolds, *Indag. Math. (N.S.)* **10**:2 (1999), 269–293.

[115] J. Morava, private communication.

[116] J. Mrcun, Stability and invariants of Hilsum–Skandalis maps, `math.DG/0506484` (2005).

[117] D. Mumford, Picard groups of moduli problems, *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, New York: Harper & Row, 1965, pp. 33–81.

[118] G. E. Murphy, A new construction of Young's seminormal representation of the symmetric groups, *J. Algebra* **69**:2 (1981), 287–297.

[119] W. Nahm and K. Wendland, Mirror symmetry on Kummer type $K3$ surfaces, *Comm. Math. Phys.* **243**:3 (2003), 557–582.

[120] H. Nakajima, Lectures on Hilbert Schemes of Points on Surfaces, *University Lecture Series*, vol. 18, Providence, RI: American Mathematical Society, 1999.

[121] V. Nistor, Cyclic cohomology of crossed products by algebraic groups, *Invent. Math.* **112**:3 (1993), 615–638.

[122] B. D. Park and M. Poddar, The Chen–Ruan cohomology ring of mirror quintic, *J. Reine Angew. Math.* **578** (2005), 49–77.

[123] M. Poddar, Orbifold cohomology group of toric varieties, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 223–231.

[124] — Orbifold hodge numbers of Calabi–Yau hypersurfaces, *Pacific J. Math.* **208**:1 (2003), 151–167.

[125] D. A. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Math.* **102**:3 (1996), 243–303.

[126] Z. Qin and W. Wang, Hilbert schemes of points on the minimal resolution and soliton equations, arXiv:math.QA/0404540.

[127] — Hilbert schemes and symmetric products: a dictionary, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 233–257.

[128] D. Quillen, The spectrum of an equivariant cohomology ring. I, II, *Ann. of Math.* (2) **94** (1971), 549–572; 573–602.

[129] M. Reid, The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible, *Math. Ann.* **278**:1–4 (1987), 329–334.

[130] — *La correspondance de McKay*, Astérisque (2002), no. 276, 53–72, Séminaire Bourbaki, vol. 1999/2000.

[131] S.-S. Roan, Minimal resolutions of Gorenstein orbifolds in dimension three, *Topology* **35**:2 (1996), 489–508.

[132] — Orbifold Euler characteristic, *Mirror symmetry, II, AMS/IP Stud. Adv. Math.*, vol. 1, Providence, RI: American Mathematical Society, 1997, pp. 129–140.

[133] Y. Ruan, Cohomology ring of crepant resolutions of orbifolds, arXiv:math.AG/0108195.

[134] — Surgery, quantum cohomology and birational geometry, *Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2*, vol. 196, Providence, RI: American Mathematical Society, 1999, pp. 183–198.

[135] — Stringy geometry and topology of orbifolds, *Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math.*, vol. 312, Providence, RI: American Mathematical Society, 2002, pp. 187–233.

[136] — Stringy orbifolds, *Orbifolds in Mathematics and Physics (Madison, WI, 2001), Contemp. Math.*, vol. 310, Providence, RI: American Mathematical Society, 2002, pp. 259–299.

[137] — Discrete torsion and twisted orbifold cohomology, *J. Symplectic Geom.* **2**:1 (2003), 1–24.

[138] I. Satake, On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. USA* **42** (1956), 359–363.

[139] — The Gauss–Bonnet theorem for V-manifolds, *J. Math. Soc. Japan* **9** (1957), 464–492.

[140] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* **15**:5 (1983), 401–487.

[141] G. Segal, Classifying spaces and spectral sequences, *Inst. Hautes Études Sci. Publ. Math.* (1968), no. 34, 105–112.

[142] — Equivariant K-theory, *Inst. Hautes Études Sci. Publ. Math.* (1968), no. 34, 129–151.

[143] — Categories and cohomology theories, *Topology* **13** (1974), 293–312.

[144] J. Słomińska, On the equivariant Chern homomorphism, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **24**:10 (1976), 909–913.

[145] E. H. Spanier, *Algebraic Topology*, New York: Springer-Verlag, 1981, corrected reprint.

[146] N. Steenrod, The Topology of Fibre Bundles, *Princeton Mathematical Series*, vol. 14, Princeton, NJ: Princeton University Press, 1951.

[147] B. Steer and A. Wren, Grothendieck topology and the Picard group of a complex orbifold, *Homotopy Invariant Algebraic Structures (Baltimore, MD, 1998), Contemp. Math.*, vol. 239, Providence, RI: American Mathematical Society, 1999, pp. 251–262.

[148] W. Thurston, The geometry and topology of three-manifolds, *Princeton Lecture Notes*, 1980, unpublished.

[149] B. Toen, Théorèmes de Riemann-Roch pour les champs de Deligne–Mumford, *K-Theory* **18**:1 (1999), 33–76.

[150] B. Toen, Notes on G-theory of Deligne–Mumford stacks, arXiv:math.AG/9912172.

[151] T. tom Dieck, Transformation Groups and Representation Theory, *Lecture Notes in Mathematics*, vol. 766, Berlin: Springer, 1979.

[152] — Transformation Groups, *de Gruyter Studies in Mathematics*, vol. 8, Berlin: Walter de Gruyter & Co., 1987.

[153] B. Uribe, Orbifold cohomology of the symmetric product, arXiv:math.AT/0109125, to appear in *Comm. Ana. Geom.*

[154] — Twisted K-theory and orbifold cohomology of the symmetric product, Ph.D. thesis, University of Wisconsin, Madison, May 2002.

[155] C. Vafa and E. Witten, A strong coupling test of S-duality, *Nuclear Phys. B* **431**:1–2 (1994), 3–77.

[156] Eric Vasserot, Sur l'anneau de cohomologie du schéma de Hilbert de $\mathbf{C}^2$, *C. R. Acad. Sci. Paris Sér. I Math.* **332**:1 (2001), 7–12.

[157] G. Vezzosi and A. Vistoli, Higher algebraic K-theory of group actions with finite stabilizers, *Duke Math. J.* **113**:1 (2002), 1–55.

[158] C. Voisin, Miroirs et involutions sur les surfaces $K3$, Astérisque (1993), no. 218, 273–323, *Journées de Géométrie Algébrique d'Orsay* (Orsay, 1992).

[159] C.-L. Wang, On the topology of birational minimal models, *J. Differential Geom.* **50**:1 (1998), 129–146.

[160] W. Wang, Equivariant K-theory, generalized symmetric products, and twisted Heisenberg algebra, *Comm. Math. Phys.* **234**:1 (2003), 101–127.

[161] K. Wendland, Consistency of orbifold conformal field theories on $K3$, *Adv. Theor. Math. Phys.* **5**:3 (2001), 429–456.

[162] E. Witten, D-branes and K-theory, *J. High Energy Phys.* (1998), no. 12, Paper 19, 41 pp. (electronic).

[163] T. Yasuda, Twisted jets, motivic measures and orbifold cohomology, *Compos. Math.* **140**:2 (2004), 396–422.

[164] E. Zaslow, Topological orbifold models and quantum cohomology rings, *Comm. Math. Phys.* **156**:2 (1993), 301–331.

Index