Worksheet 2: planes and more

Planes in 3-space

Recall $\mathbb{R}^3$ is the set of (all) ordered pairs (a_1, a_2, a_3) of real numbers. We want to think of $\mathbb{R}^3$ as an algebraic model for the geometric object **$3d$-space together with a choice of coordinate axes**. We want to understand how a plane in $3d$-space can be described algebraically in two different ways, parameterized and by equations.

Parametrized plane

A parameterization for a plane in $3d$-space is a function

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^3$$

of the form

$$\Pi(s,t) = (x(s,t), y(s,t), z(s,t)) = (a_1 + b_1 s + c_1 t, a_2 + b_2 s + c_2 t, a_3 + b_3 s + c_3 t)$$

The **plane** Π is the image of the function Π.

An alternative and completely equivalent notation that is used is:

$$\begin{cases} x(s,t) = a_1 + b_1 s + c_1 t \\ y(s,t) = a_2 + b_2 s + c_2 t \\ z(s,t) = a_3 + b_3 s + c_3 t. \end{cases}$$

Now, three vectors in $\mathbb{R}^3$ play a role in the parameterization of a line. The vector (a_1, a_2, a_3) is the **initial position vector**: it is the point where your particle is at time $(s,t) = (0,0)$. The vector (b_1, b_2, b_3) is the **s-velocity vector** and the vector (c_1, c_2, c_3) is the **t-velocity vector**.

Intuitively, you should think of this as follows: initially you are standing at the point (a_1, a_2, a_3) with two stopwatches in your hand, labeled s and t. When you start the s-stopwatch, you move (at a constant speed) in the (b_1, b_2, b_3) direction. When you start the t-stopwatch, you move (at a constant speed) in the (c_1, c_2, c_3) direction. The plane Π is the set of all places you may reach with this kind of motions.

Problem 1 *Consider the plane:*

$$\Pi(s,t) = (x(s,t), y(s,t), z(s,t)) = (1 + t, 2s, 3 + s + 3t)$$

1. If I tell you that the point $(7, b, c)$ belongs to the plane Π, do you know b, c?

2. If I tell you that the point $(1, b, 8)$ belongs to the plane Π, what is b?

3. Does the origin belong to Π?

4. If now we have a line

$$\ell(t) = (x(t), y(t), z(t)) = (t, t, t),$$

what are the points of intersections in $\Pi \cap \ell$?

Problem 2 *Consider the parameterization:*

$$F = \begin{cases} x(s,t) = 1 + s + 2t \\ y(s,t) = 1 + 2s + 4t \\ z(s,t) = 1 + 3s + 6t. \end{cases}$$

Does the image of F look like a plane? Why not? What is the condition on the s and t velocity vectors that make it so?

Discussion 1 *How do you find a parameterization for a plane if you know three points that the plane contains?*

Equations for planes in $3d$-space

A plane in $3d$-space may be described as a subset of points whose coordinates satisfy one **polynomial equation** in x, y and z of degree 1. In mathy notation:

$$p(x, y, z) = q(x, y, z)$$

where p and q are polynomials of degree one in x, y and z, i.e. expressions of the form $ax + by + cz + d$.

For example, if we give the very simple equations $z = 0$, the points that satisfy such equation give the xy-plane.

Problem 3 *Consider the plane Π:*

$$2x + y + z - 4 = 0$$

1. *Does the point $(-2, 0, 2)$ belong to Π?*

2. *Find a point on Π.*

3. *If the point $(0, 0, a)$ belongs to Π what is a?*

4. *If ℓ is the line given by*

$$\begin{cases} x + y = 0 \\ x - z = 0, \end{cases}$$

what are the points of intersections $\ell \cap \Pi$?

5. *If ℓ is the line given by*

$$\begin{cases} x(t) = 4 \\ y(t) = 2t \\ z(t) = t + 1 \end{cases}$$

what are the points of intersections $\ell \cap \Pi$?

Problem 4 *Consider the following parameterization with three parameters s, t, u:*

$$\begin{cases} x(s,t) = s + t + 2u \\ y(s,t) = s + 4t + u \\ z(s,t) = 0. \end{cases}$$

What is the image of this parameterization? Why is in not "three"-dimensional?

Discussion 2 *Time to reconvene and philosophize a bit:*

1. *Is the equation for a plane unique?*
2. *How do you find the equation of a plane if you know that the plane passes through three points?*
3. *Do any three points in $\mathbb{R}^3$ determine a plane?*

Higher dimensions

A parameterization for an **affine linear subspace of dimension** k is a function $p : \mathbb{R}^k \to \mathbb{R}^n$ of the form:

$$\begin{bmatrix} x_1(t_1, \dots, t_k) \\ x_2(t_1, \dots, t_k) \\ \dots \\ x_n(t_1, \dots, t_k) \end{bmatrix} = \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \dots \\ v_{1,n} \end{bmatrix} t_1 + \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ \dots \\ v_{2,n} \end{bmatrix} t_2 + \dots \begin{bmatrix} v_{k,1} \\ v_{k,2} \\ \dots \\ v_{k,n} \end{bmatrix} t_k$$

The vectors $\vec{v}_1 = (v_{1,1}, \dots v_{1,n}), \dots, \vec{v}_k = (v_{k,1}, \dots v_{k,n})$ give k different directions of travel, and the image of the parameterization is the set of all positions that can be reached using these directions of travel. If these k vectors are picked at random, then with probability one the resulting parameterization is indeed k-dimensional (we haven't defined dimensions formally yet, but for now let us roll with the intuition that something is k dimensional if it "needs k coordinates to be described").

Problem 5 *Give an example of a parameterization for a five dimensional affine linear space in $\mathbb{R}^7$.*

Problem 6 *Give an example of a parameterization $\mathbb{R}^5 \to \mathbb{R}^7$ that fails to produce a five dimensional affine linear space in $\mathbb{R}^7$.*

Discussion 3 *Can you say anything about when the vectors $\vec{v_1}, \dots, \vec{v_k}$ fail to produce a k-dimensional object?*

A set of equations for an **affine linear space of dimension k in $\mathbb{R}^n$** consists of $n-k$ linear equations that must be satisfied simultaneously.

Problem 7 *Give an example of a set of equations for a five dimensional affine linear space in $\mathbb{R}^7$.*

Problem 8 *Give an example of a set of two equations in seven variables that fail to produce a five dimensional affine linear space in $\mathbb{R}^7$.*

Discussion 4 *Can you say anything about when $n-k$ equations fail to produce a k-dimensional object?*

Final Questions

Question 1 *Parameterizations and equations are different ways of describing the same objects. Each has advantages and disadvantages. For each of the tasks below, decide whether it is simpler to do it via equations or via parameterizations.*

1. *Checking if a point $(a_1, \ldots, a_n)$ belongs to an affine linear subspace of dimension k.*
2. *Producing points that belong to an affine linear subspace of dimension k.*
3. *Describing (abstractly) the intersection of two affine linear subspaces.*
4. (TRICK QUESTION) *Finding the points of intersection of a line and an affine linear subspace of dimension $n-1$ in $\mathbb{R}^n$.*

Question 2 *Consider a parameterization of a plane in some $\mathbb{R}^n$. In what ways can you change the velocity vectors $\vec{v_1}, \vec{v_2}$ and still obtain the same plane?*

Question 3 *Consider the set of equations in n variables:*

$$\begin{cases} p(x_1, \ldots, x_n) = 0 \\ q(x_1, \ldots, x_n) = 0 \\ p(x_1, \ldots, x_n) + q(x_1, \ldots, x_n) = 0 \end{cases},$$

where p and q are polynomials of degree one. Do such equations define an affine linear subspace of dimension $n-3$? Why or why not?