MATH 261 EXAM I PRACTICE PROBLEMS

These practice problems are pulled from actual midterms in previous semesters. Exam 1 typically has 6 problems on it, with no more than one problem of any given type (e.g., don't expect two problems about the angle between two planes). Also, **please be aware** that this is not intended as a comprehensive list of all possible problem types! In other words, you are responsible for all topics covered during the period leading up to this exam, whether they are represented in this list or not. See your notes and the suggested homework for a comprehensive list.

1. Circle **all** correct statements.

   (a) The dot product of two vectors ($\mathbf{u} \cdot \mathbf{v}$) is

   (i) a number (ii) a vector

   (b) The cross product of two vectors ($\mathbf{u} \times \mathbf{v}$) is

   (i) a number (ii) a vector

   (c) For any vector $\mathbf{v}$, $|\mathbf{v}|$ is

   (i) 1 (ii) $\sqrt{\mathbf{v} \cdot \mathbf{v}}$

   (iii) the magnitude of $\mathbf{v}$ (iv) $\mathbf{v} \cdot \mathbf{v}$

   (d) **P1** is a plane given by $2x + 3y + z = 5$,
   P2 is a plane given by $3x - 2y = 2$, and
   L is a line given by $\mathbf{r}(t) = \langle t, 1 + t, 2 - 5t \rangle$.

   (i) **P1** and **P2** are parallel (ii) **P1** and **L** intersect

   (iii) **P2** and **L** intersect (iv) **L** is the intersection of **P1** and **P2**

2. (a) Consider a plane **P1** given by the equation $Ax + By + Cz = D$. Write down a vector which is perpendicular to **P1**.

   (b) Consider a plane **P1** given by the equation $Ax+By+Cz = D$. Write the equation of a plane which is parallel to **P1** but contains the point $\langle 0, 0, 0 \rangle$.

   (c) If **P1** and **P2** have normal vectors which are not parallel, how do the two planes intersect?

   (d) Consider a line **L** described by the parameterization $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. (where $\mathbf{r}, \mathbf{v}$ are two vectors). If **L** does not intersect with a plane **P1**, is there any relationship between $\mathbf{v}$ and a vector normal to the plane?

(e) Consider a line $\mathbf{L}$ described by the parameterization $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. (where $\mathbf{r}, \mathbf{v}$ are two vectors). If $\mathbf{L}$ is contained in a plane $\mathbf{P1}$, is there any relationship between $\mathbf{v}$ and a vector normal to the plane?

3. Let

$$\mathbf{u} = \langle 2, 1, 3 \rangle, \mathbf{v} = \langle 1, -1, -1 \rangle, \mathbf{w} = \langle 1, 0, -1 \rangle, \mathbf{z} = \langle 3, 4 \rangle.$$

Compute each of the following <u>or</u> write "invalid operation" if they cannot be computed exactly as given.

(a) $\mathbf{u} + \mathbf{v}$

(b) $\mathbf{w} \cdot \mathbf{z}$

(c) $7\mathbf{v}$

(d) $\dfrac{\mathbf{z}}{|\mathbf{z}|}$

4. Given the two vectors

$$\begin{aligned} \mathbf{A} &= \langle -2, 4, -4 \rangle \\ \mathbf{B} &= \langle -1, 1, 1 \rangle \end{aligned}$$

complete the following:

(a) (8 pts) Find the projection of $\mathbf{B}$ onto $\mathbf{A}$.

(b) (8 pts) Find $\mathbf{A} \times \mathbf{B}$.

(c) (4 pts) Find the area of the triangle with vertices $(0, 0, 0)$, $(-2, 4, -4)$, $(-1, 1, 1)$.

5. Determine the distance from point $(2, 2, 3)$ to the plane $x + 2y + 2z = 4$.

6. (a) (8 pts) Find the distance from the point $(1, 2, -4)$ to the plane $x + 2y - 2z = 6$.

(b) (8 pts) Find the distance from the point $(1, 2, -4)$ to the line given by $\mathbf{r}(t) = \langle 1 + 2t, 1 - 2t, -3 + t \rangle$.

7. (a) The line given by $\mathbf{r}(t) = \langle 1+t, 2-t, 3+2t \rangle$ intersects the plane given by $x+y+z = 10$ in a point. Find the point of intersection.

(b) Find a vector parallel to the line of intersection of the planes given by $2x + z = 5$ and $x + y - z = 4$.

8. Consider the line L given by

$$L : \mathbf{r}(t) = \langle 3t + 1, t + 2, 2t - 1 \rangle$$

and the two planes P_1 and P_2 given by

$$P_1 : x + y - z = 6$$

$$P_2 : 2x - 2y + 2z = 2.$$

(a) Does the line L intersect the plane P_1? If so, find the point of intersection. If not, explain why.

(b) Find the angle between P_1 and P_2. You should leave your answer as the arccosine of some number.

(c) Find the distance between the origin and the line L.

9. For parts (a) and (b), consider the two planes P_3 and P_4 given by

$$P_3 : x + y + z = 1$$

$$P_4 : 3x + y - z = 5.$$

(a) Find a vector parallel to the line of intersection of planes P_3 and P_4.

(b) Find a point on the line of intersection of planes P_3 and P_4.

(c) Given point $(2, 4, 6)$ on the line L and vector $\langle 1, 3, 5\rangle$ parallel to L, write parametric equations for the line.

10. Given the lines below, complete the following:

(a) (8 pts) Find the points of intersection.

(b) (8 pts) Find the equation of the plane containing the lines. Your final answer must be in the form $Ax + By + Cz = D$.

$$\begin{aligned} L_1 :\ \mathbf{r}_1(t) &= \langle -2 - t, 2t, 3 + 2t\rangle \\ L_2 :\ \mathbf{r}_2(s) &= \langle 4 + s, 6 + 4s, -12 - 3s\rangle \end{aligned}$$

11. Find an equation for the plane through points $(1, 1, 0)$, $(0, 2, 1)$, and $(2, 1, 1)$. Simplify your answer to the form $Ax + By + Cz = D$.

12. The three parts of this problem are independent. Do not use anything from one part in another part.

(a) Let $\mathbf{r}(t) = \langle 5\sin(t), 12, 5\cos(t)\rangle$. Compute the unit tangent vector, $\mathbf{T}(t)$.

(b) If $\mathbf{a}(2) = \langle 4, 3, 0\rangle$ and $\mathbf{T}(2) = \langle 0, 1, 0\rangle$, compute $a_N(2)$.

(c) Suppose $\mathbf{a}(7) = \langle 1, 1, 4\rangle$, $\mathbf{T}(7) = \langle \frac{2}{9}, -\frac{1}{9}, \frac{2}{9}\rangle$, $a_T(7) = 9$, and $a_N(7) = 9$. Compute $\mathbf{N}(7)$.

13. The three parts of this problem are independent. In other words, these are three distinct problems that do not rely on each other.

(a) Suppose $\mathbf{r}(t) = (3\cos t)\mathbf{i} + 4t\mathbf{j} + (3\sin t)\mathbf{k}$. Compute $\mathbf{v}(t)$ and $\mathbf{T}(t)$.

(b) Suppose that $\mathbf{a} = \langle 2, 1, 1\rangle$ and $\mathbf{T} = \left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$ at some value of t. Compute a_T and a_N at that value of t.

(c) Find $\mathbf{N}\left(\frac{\pi}{2}\right)$, given that $\mathbf{a}\left(\frac{\pi}{2}\right) = \langle 2, 4, -2\rangle$, $\mathbf{T}\left(\frac{\pi}{2}\right) = \langle 1, 0, 0\rangle$, $a_T\left(\frac{\pi}{2}\right) = 2$, and $a_N\left(\frac{\pi}{2}\right) = -2$.

14. Let $\mathbf{a}(t) = \left\langle \frac{1}{(t-1)^2},\ e^t,\ 2\cos(t) \right\rangle$ represent acceleration.
Furthermore, let $\mathbf{v}(0) = \langle 1, 1, 1\rangle$ and $\mathbf{r}(2) = \langle 2, e^2 + 3, -2\cos(2) + 2\rangle$.

   (a) Find $\mathbf{v}(\mathbf{t})$.

   (b) Find $\mathbf{r}(\mathbf{t})$.

15. Given $\mathbf{a}(t) = \langle 12t^2 + 2, 6t - \cos(t), e^t\rangle$, $\mathbf{v}(0) = \langle 1, 0, -5\rangle$, and $\mathbf{r}(0) = \langle 3, -3, 0\rangle$, find the following:

   (a) Find $\mathbf{v}(t)$.

   (b) Find $\mathbf{r}(t)$.

16. Given $\mathbf{a}(t) = \langle e^2, 2, -1/(t+1)^2\rangle$, $\mathbf{v}(0) = \langle 1, 0, -5\rangle$, and $\mathbf{r}(0) = \langle 3, -3, 0\rangle$, find the following:

   (a) (8 pts) Find $\mathbf{v}(t)$.

   (b) (8 pts) find $\mathbf{r}(t)$.

17. Determine the length of the curve $\mathbf{r}(t) = \langle 3t, \sin(4t), \cos(4t)\rangle$ from $t = 0$ to $t = 2\pi$.

SPOILER ALERT: Solutions start on the next page.

SOLUTIONS

WARNING: Not all of these solutions are fully justified. Be sure to provide full justification with your solutions (especially where this is explicitly requested) so that we may provide partial credit, where applicable. If you are having trouble getting these answers, please come to office hours and/or exam review sessions. See the course website for details. Of course, if there seems to be an error with these solutions (which is very possible!), please let an instructor or the coordinator know.

1. (a) (i)
   (b) (ii)
   (c) (ii) and (iii)
   (d) (ii) and (iii)

2. (a) $\langle A, B, C\rangle$.
   (b) $Ax + By + Cz = 0$.
   (c) In a line.
   (d) $\mathbf{v}$ is perpendicular to the normal vector to the plane.
   (e) $\mathbf{v}$ is perpendicular to the normal vector to the plane.

3. (a) $\langle 3, 0, 2\rangle$
   (b) Invalid operation – you cannot take the dot product of vectors of different lengths (or add them, for that matter).
   (c) $\langle 7, -7, -7\rangle$
   (d) $\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

4. (a) $\left\langle -\frac{1}{9}, \frac{2}{9}, -\frac{2}{9}\right\rangle$
   (b) $\langle 8, 6, 2\rangle$
   (c) $\sqrt{26}$

5. 8/3 (using the formula $\frac{\mathbf{PS}\cdot\mathbf{n}}{|\mathbf{n}|}$).

6. (a) 7/3
   (b) 1

7. (a) $(3, 0, 7)$
   (b) $\langle -1, 3, 2\rangle$ (or any multiple of this – please don't come up with a strange multiple for the fun of it!)

8. (a) Yes, at $(4, 3, 1)$ (when $t = 1$).
   (b) $\arccos\left(-\frac{1}{3}\right)$
   (c) $\sqrt{75}/\sqrt{14}$

9. (a) $\langle -2, 4, -2 \rangle$ (or any multiple of this)
   (b) $(0, 3, -2)$ (or infinitely many others!)
   (c) $\mathbf{r}(t) = \langle 2+t, 4+3t, 6+5t \rangle$

10. (a) $\langle 1, -6, -3 \rangle$
    (b) $-14x - y - 6z = 10$

11. $x + 2y - z = 3$

12. (a) $\langle \cos(t), 0, -\sin(t) \rangle$
    (b) 4
    (c) $\langle -\frac{1}{9}, \frac{2}{9}, \frac{2}{9} \rangle$

13. (a) $\mathbf{v}(t) = \langle -3\sin(t), 4, 3\cos(t) \rangle$, $\mathbf{T}(t) = \left\langle -\frac{3}{5}\sin(t), \frac{4}{5}, \frac{3}{5}\cos(t) \right\rangle$
    (b) $a_T = 2$, $a_N = \sqrt{2}$
    (c) $\mathbf{N} = \langle 0, -2, 1 \rangle$ (Notice that this is not a unit vector as the unit normal vector $\mathbf{N}$ should be; that was an error in the writing of this exam and was handled appropriately in the grading.)

14. (a) $\left\langle \frac{-1}{t-1}, e^t, 2\sin(t) + 1 \right\rangle$
    (b) $\langle -\ln(t-1) + 2, e^t + 3, -2\cos(t) + t \rangle$

15. (a) $\langle 4t^3 + 2t + 1, 3t^2 - \sin(t), e^t - 6 \rangle$
    (b) $\langle t^4 + t^2 + t + 3, t^3 + \cos(t) - 4, e^t - 6t - 1 \rangle$

16. (a) $\left\langle e^2 t + 1, 2t, \frac{1}{t+1} - 6 \right\rangle$
    (b) $\left\langle \frac{1}{2}e^2 t^2 + t + 3, t^2 - 3, \ln(t+1) - 6t \right\rangle$

17. 10π