

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 8

Functions: take 2

1 Key Ideas

We now introduce some attributes of functions that are extremely useful in mathematics.

Definition 1. A function $f : X \rightarrow Y$ is called **injective** (or **1 : 1**) if f maps distinct inputs to distinct outputs.

In other words, a function is injective if no two different inputs are sent to the same output.

Example 1. Let $X = \{1, 2, 3\}$, and $Y = \{a, b, c, d\}$. Then $f : X \rightarrow Y$ defined by

$$f(1) = b, \quad f(2) = c, \quad f(3) = a$$

is injective. But $g : X \rightarrow Y$ defined by

$$g(1) = b, \quad g(2) = b, \quad g(3) = c$$

is NOT injective.

The intuition here is that injective functions use elements of Y as identifications for elements of X . For example, think of the function that assigns to each CSU student, its CSU-id number. The use of such a function is that numbers are more easily entered into a database, for example. Not every number corresponds to a student (I believe CSU-id's are 9 digits long or so, and we definitely don't have a billion students at CSU), but it is essential that no two students have the same CSU-id. This is the prototypical example of an injective function!

Definition 2. A function $f : X \rightarrow Y$ is called **surjective** (or **onto**) if every element of Y is the output via f for some input element of X .

In other words, a function is surjective if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Example 2. Let $X = \{1, 2, 3\}$, and $Y = \{a, b\}$. Then $f : X \rightarrow Y$ defined by

$$f(1) = b, \quad f(2) = a, \quad f(3) = a$$

is surjective. But $g : X \rightarrow Y$ defined by

$$g(1) = b, \quad g(2) = b, \quad g(3) = b$$

is NOT surjective.

Surjective functions come into play when you only want to remember certain information about elements of X . For example, you might need to perform a task that depends only on the nationality of a person (say decide the color of their passport). Then, instead of working with the set of all living humans (which has about 6 billion elements), you may want to work with the set of nations (which has about 200 elements). Then the function that assigns to each person their nationality (assume that there is no double citizenship, just for the sake of this actually being a function) is a surjective function. It is surjective because there is no nation which has no inhabitant.

2 Groupwork

Problem 1. Give an example of a function which is both injective and surjective, of one which is injective but not surjective, of one which is surjective but not injective and of one which is neither injective nor surjective.

Problem 2. 1. Suppose X and Y are finite sets, and $f : X \rightarrow Y$ is an injective function. What can you say about the relationship between $|X|$ and $|Y|$?

2. Suppose X and Y are finite sets, and $f : X \rightarrow Y$ is a surjective function. What can you say about the relationship between $|X|$ and $|Y|$?

3. Let $f : X \rightarrow Y$ be an injective function. Prove that if Y is countable, then X is either finite or countable.

4. Let $f : X \rightarrow Y$ be a surjective function. Prove that if X is countable, then Y is either finite or countable.

Problem 3. 1. Let $f : X \rightarrow Y$ be an injective function. What can you say about the inverse images of subsets of Y ? In particular, if $U = \{y\}$ is a singleton, what can $f^{-1}(U)$ be?

2. Let $f : X \rightarrow Y$ be a surjective function. What can you say about the inverse images of subsets of Y ? In particular, if $U = \{y\}$ is a singleton, what can $f^{-1}(U)$ be?

Problem 4. Prove that the composition of two injective functions is injective.

Problem 5. Given a subset $U \subseteq Y$, there is a natural function $i_U : U \rightarrow Y$ defined by $i_U(u) = u$. Convince yourself that this function is injective. Viceversa, any injective function $f : X \rightarrow Y$ realizes a bijection between X and the image of f . Make sure this makes sense to you. This means that the notion of injective function is in some sense equivalent to the notion of subsets of a set Y . Spend a little time to understand this statement, and discuss it with the other members of the group.

Definition 3. Given a function $f : X \rightarrow Y$, the **image** (or **range**) of f is the set $f(X) \subseteq Y$ of all elements of Y that are outputs for some element of X , i.e.

$$\text{Im}(f) = f(X) = \{y \in Y \text{ such that there exists some } x \in X \text{ with } f(x) = y\}.$$

Problem 6. Give a characterization of f being surjective in terms of the image of f . Use this to prove that the composition of surjective functions is surjective.

Problem 7. Let $f : X \rightarrow Y$ be a surjective function, $|Y| = n$ and for every $y \in Y$ suppose you have $|f^{-1}(\{y\})| = m$. Then what is $|X|$?

3 Sunday Homework

Exercise 1. For each of the following situations, provide an example if it is possible, or explain why it is not possible.

1. $f : X \rightarrow Y$ not injective, $g : Y \rightarrow Z$ a function and $g \circ f$ injective;
2. $f : X \rightarrow Y$ a function, $g : Y \rightarrow Z$ not injective, and $g \circ f$ injective;
3. $f : X \rightarrow Y$ not surjective, $g : Y \rightarrow Z$ a function and $g \circ f$ surjective;
4. $f : X \rightarrow Y$ a function, $g : Y \rightarrow Z$ not surjective, and $g \circ f$ surjective

Exercise 2. Use Problem 7 to count (yet again) the number of handshakes among n persons. Consider the following sets:

- X = the set of ordered pairs of distinct persons in the group.
- Y = the set of handshakes among those persons.

Define a natural, surjective function $f : X \rightarrow Y$. What is the number of elements of X ? For each element $y \in Y$, what is $|f^{-1}(y)|$? Use the conclusion of Problem 7 to deduce $|Y|$.

Exercise 3. Given $f : X \rightarrow Y$, prove the following:

1. If $y_1 \neq y_2$, then $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$;
- 2.

$$\bigcup_{y \in Y} f^{-1}(\{y\}) = X$$

Exercise 4. Use the previous exercise to interpret the equality

$$2^n = \sum_{k=0}^n \binom{n}{k} \tag{1}$$

as follows. Let X be a set with n elements, and consider its power set $\mathcal{P}(X)$. Define a surjective function $f : \mathcal{P}(X) \rightarrow \{0, 1, 2, \dots, n\}$ by $f(U) = |U|$. What is $|\mathcal{P}(X)|$? For any element $i \in \{0, 1, 2, \dots, n\}$, what is $|f^{-1}(i)|$? Now use Exercise 3 to prove (1).

Exercise 5. Given the cartesian product $X \times Y$, we have two natural functions, called projections, defined as follows:

- $p_1 : X \times Y \rightarrow X$ is defined by $p_1((x, y)) = x$.
- $p_2 : X \times Y \rightarrow Y$ is defined by $p_2((x, y)) = y$.

Show that if X and Y are non empty, then the two projections are surjective.

Exercise 6. Use the concept of **graph of a function** introduced last week to show that any function $f : X \rightarrow Y$ may be written as the composition $s \circ i$, where i is an injective function and s is a surjective function.