

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 7

Functions

1 Key Ideas

We now investigate the concept of function, which is central to mathematics. Functions is how you make different mathematical objects “communicate”. In order to have a function, you need the following ingredients:

- a set X called the **domain** (or **input set**, or **starting set**), and
- a set Y called **codomain** (or **output set**, or **arrival set**).

We then talk about a function **from** X **to** Y .

Definition 1. A function $f : X \rightarrow Y$ is a *prescription* that assigns, to each element x of X , one element of Y that we call the **image** of x and denote $f(x)$.

A few important observations:

1. You can think of f as some machine with an input hole and an output hole. You can put elements of x into the input hole, and the machine will spit out elements of y . You can put in any elements of X , and the machine will always work to give you elements of Y . If you put the same element of X in the machine a bunch of times, the machine will always produce the same output.
2. It is possible that the machine will produce the same output for different inputs.
3. It is possible that not all elements of Y actually appear as outputs for some inputs.

Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. A couple examples of functions from X to Y :

- $f_1 : X \rightarrow Y$ defined by $f_1(1) = a, f_1(2) = b, f_1(3) = c$;
- $f_2 : X \rightarrow Y$ defined by $f_2(1) = c, f_2(2) = c, f_2(3) = b$.

And a couple examples of things that are NOT functions from X to Y :

- $f : X \rightarrow Y$ defined by $f(1) = a, f(2) = b$;
- $f : X \rightarrow Y$ defined by $f(1) = c, f(1) = b, f(2) = c, f(3) = b$.

As we did for bijections, we can give a slightly more formal definition of function, as a subset of the cartesian product $X \times Y$.

Definition 2. A **function** $f : X \rightarrow Y$ is equivalent information to a subset $F \subseteq X \times Y$ with the property that for any element $x \in X$, there is a unique element of F whose first term is x . We also call F the **graph** of f .

For every element of F , the first term is the input, and the second term the corresponding output.

Example 2. For the two functions f_1 and f_2 in Example 3, we have:

- $F_1 = \{(1, a), (2, b), (3, c)\};$
- $F_2 = \{(1, c), (2, c), (3, b)\}.$

Definition 3. Given any set X , there is always a special function from X to X , called the **identity function**, which we will denote Id_X . This function is defined by sending each element $x \in X$ to itself: $Id_X(x) = x$.

One important features of functions is that they can be composed. In the analogy of the machines, suppose that you have a machine f whose input hole accepts elements of X and produces outputs in a set Y , and then another machine g that accepts inputs belonging to Y and produces outputs in some set Z . Then you can connect the output hole of the first machine with the input hole of the second, to create a new machine that “eats” elements of X and produces elements of Z .

Definition 4. Given $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we define their **composition** $g \circ f : X \rightarrow Z$ to be the function

$$g \circ f(x) = g(f(x)).$$

Example 3. Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$, $Z = \{\star, \bullet\}$. Let

- $f : X \rightarrow Y$ defined by $f(1) = a, f(2) = b, f(3) = c$;
- $g : Y \rightarrow Z$ defined by $g(a) = \star, g(b) = \star, g(c) = \bullet, g(d) = \star$.

Then the function $g \circ f : X \rightarrow Z$ is defined to be:

$$g \circ f(1) = \star, g \circ f(2) = \star, g \circ f(3) = \bullet.$$

Some “machines” are special in the sense that you can run them also in the opposite direction: feed in an element $y \in Y$, turn the crank backwards, and produce the element of X that y was the output of. Such special machines are called **invertible functions**.

Definition 5. A function $f : X \rightarrow Y$ is called **invertible** if there exists a function $g : Y \rightarrow X$ such that:

$$g \circ f = Id_X, \quad f \circ g = Id_Y.$$

The function g is called the **inverse** of f and denoted f^{-1} .

Example 4. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. If $f : X \rightarrow Y$ is given by

$$f(1) = b, \quad f(2) = a, \quad f(3) = c,$$

then $g : Y \rightarrow X$ defined by

$$g(a) = 2, \quad g(b) = 1, \quad g(c) = 3,$$

is the inverse of f .

2 Groupwork

Problem 1. Check that in Example 4 the function g is indeed the inverse of f . Show that also f is the inverse of g . In fact, convince yourself that according to Definition 5, you can always talk about two functions being inverses of each other.

Problem 2. What happens when you compose a function with the identity function? More precisely, let $f : X \rightarrow Y$ be a function. What are the functions $f \circ Id_X$ and $Id_Y \circ f$?

Problem 3. Prove that if a function $f : X \rightarrow Y$ is invertible, then its inverse function is unique.

We now define the concept of inverse image. It is often the case that inverse image and inverse functions are confused. Partly, because the standard notation for the two concepts is the same. So please pay close attention!

Definition 6. Given a function $f : X \rightarrow Y$ and a subset $U \subseteq Y$, the **inverse image** of U , denoted $f^{-1}(U)$, is a subset of X that consists of **all** elements of X whose output via f belongs to U . In symbols:

$$f^{-1}(U) = \{x \in X \text{ such that } f(x) \in U\}.$$

Here are the things to pay attention to: first of all, in order to talk about the inverse image, you must also specify a subset of Y . Secondly, the inverse image is a subset of X : it may be empty, it may be a single element, or it may contain multiple elements. The reason that it is acceptable to use the same notation for inverse image and inverse function, is that, when f is an invertible function AND $U = \{y\}$ is a singleton, then the two concepts coincide, in the sense that the inverse image of the one element subset U is the same as the output of the inverse function for the element y :

$$f^{-1}(\{y\}) = \{f^{-1}(y)\}.$$

Problem 4. Let $X = 1, 2, 3, 4, 5$, $Y = \{a, b, c\}$ and $f : X \rightarrow Y$ be defined by:

$$f(1) = b, \quad f(2) = b, \quad f(3) = c, \quad f(4) = c, \quad f(5) = b.$$

For every subset $U \in \mathcal{P}(Y)$, write down $f^{-1}(U)$.

Problem 5. Given any function $f : X \rightarrow Y$, what are $f^{-1}(\phi)$, $f^{-1}(Y)$?

Problem 6. Give an example of a function $f : X \rightarrow Y$ and a subset $\phi \neq U \subseteq Y$ such that $f^{-1}(U) = \phi$.

Give an example of a function $f : X \rightarrow Y$ and a proper subset $U \subset Y$ such that $f^{-1}(U) = X$.

Problem 7. Prove that if $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and $U \subseteq Z$:

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

3 Sunday Homework

Exercise 1. Decide which of the following are functions. For those that are not, write why they fail to be functions:

$f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$.

$g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $g(n) = n/2$.

$h : \mathbb{Z} \rightarrow \mathbb{R}$ defined by $h(n) = n/2$.

$i : \mathbb{R} \rightarrow \mathbb{R}$ defined by $i(x) = e^x$.

$j : \mathbb{R} \rightarrow \mathbb{R}$ defined by $j(x) = \log_2(x)$.

$k : \mathbb{R} \rightarrow \mathbb{R}$ defined by $k(x) = \pm\sqrt{x}$.

Exercise 2. Consider the following prescriptions that assign to a human, another human. Decide which of them is a function from the set of humans to the set of humans.

birthmom: assigns to each human their birth mother;

son: assigns to each human their son;

president: assigns to each human the president of the country they are a citizen of;

birthpresident: assigns to each human the president of the country they were born in, at the time they were born.

Exercise 3. Consider the function “birthmom” defined in the previous exercise. What is the function $\text{birthmom} \circ \text{birthmom}$?

Now define a function “biodad: Humans $\rightarrow$ Humans”, which assigns to each human their biological father. Are the compositions $\text{birthmom} \circ \text{biodad}$ and $\text{biodad} \circ \text{birthmom}$ the same or different functions? Conclude whether the operation of composition of functions is commutative or not.

Exercise 4. Use Definition 2 to show that a function is invertible if and only if it is a bijection.

Exercise 5. Consider the following functions from $\mathbb{Z}$ to $\mathbb{Z}$. For each of them decide if they are invertible. If they are, write down the inverse function.

- $f(n) = n + 3$;
- $g(n) = 2n$;
- $h(n) = n^2$;
- $l(n) = \begin{cases} n & \text{if } n \text{ is even;} \\ n + 2 & \text{if } n \text{ is odd.} \end{cases}$

Exercise 6. In each of the cases below, compute the inverse image $f^{-1}(U)$. All functions are from $\mathbb{R}$ to $\mathbb{R}$

- $f(x) = 3x, U = (-1, 2]$;
- $f(x) = x^2, U = (-1, 2]$;
- $f(x) = e^x, U = (-1, 2]$;
- $f(x) = \sin(x), U = (0, 1/2]$;
- $f(x) = \tan(x), U = (-1, 1)$.