

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 6

Bijections and Infinities

1 Key Ideas

This week we are going to focus on the notion of **bijection**. This is a way we have to compare two different sets, and say that they have “the same number of elements”. Of course, when sets are finite everything goes smoothly, but when they are not... things get spiced up! But let us start, as usual, with some definitions.

Definition 1. A **bijection** between a set X and a set Y consists of matching elements of X with elements of Y in such a way that every element of X has a unique match, and every element of Y has a unique match.

Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. An example of a bijection between X and Y consists of matching 1 with a , 2 with b and 3 with c .

A NOT example would match 1 and 2 with a and 3 with c . In this case, two things fail: a has two partners, and b has no partner.

Note: if you are familiar with these concepts, you may have recognized a bijection as a function between X and Y which is both injective (1:1) and surjective (onto). We will revisit the notion of bijection in this language in a few weeks. For now, we content ourselves of a slightly lower level approach. However, you may be a little unhappy with the use of the word “matching” in a mathematical definition. Here is a more formal definition for the same concept.

Definition 2. A **bijection** between a set X and a set Y is a subset $B \subseteq X \times Y$ such that every element of X is the first term of exactly one element of B and every element of Y is the second term of exactly one element of B .

Bijections are intimately related with the concept of cardinality of finite sets.

Theorem 1. *Assume X and Y are two finite sets. There exists a bijection between X and Y if and only if X and Y have the same cardinality (i.e. the same number of elements).*

We therefore make the following definition, regardless of whether the sets are finite or not.

Definition 3. Two sets X and Y have the same cardinality if there exists a bijection between them.

The “first” infinite set that most people encounter is the set of natural numbers. In a sense that I don’t want to make too precise at this point, the natural numbers are the “smallest” type of infinity there is. We make the following definition.

Definition 4. A set X is **countable** if it has the same cardinality as $\mathbb{N}$. An infinite set which is not countable is called **uncountable**.

The reason that historically the word countable was used is that giving a bijection between $\mathbb{N}$ and X is the same as choosing an element of X to be number 1, another to be number 2 and so on... in other words it is like “counting” the elements of X .

Theorem 2. Let X and Y be two countable sets. Then $X \cup Y$ is also countable.

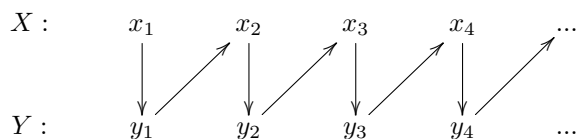
Proof. Assume that we fix a bijection between X and $\mathbb{N}$, and let us call x_n the element of X which is matched with the natural number $n \in \mathbb{N}$. Similarly we fix a bijection between Y and $\mathbb{N}$, and call y_n the element of Y which is matched with the natural number $n \in \mathbb{N}$.

We now define a bijection between $X \cup Y$ and $\mathbb{N}$ as follows: match each x_n with the natural number $2n - 1$, and each y_n with the natural number $2n$.

Via this bijection we can arrange the elements of $X \cup Y$ in the following ordered list:

$$X \cup Y = \{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}.$$

Another way to visualize this is by noting that the the bijection we described between $X \cup Y$ and $\mathbb{N}$ is obtained by following the arrows in the diagram below.



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2 Groupwork

Problem 1. Make sure that you are comfortable with how Definitions 1 and 2 are equivalent.

Problem 2. True or false: given any non-empty set X , there exists a bijection between X and itself. Prove your answer.

Problem 3. True or false: If X is a finite set and $A \subset X$ is a proper subset of X , then there can be a bijection between A and X .

Problem 4 (Hilbert’s hotel). Hilbert’s hotel has a countable number of rooms, indexed by the natural numbers. The hotel is currently full.

1. A traveler shows up at the reception. Can the receptionist accomodate the traveler, without asking anyone to shack up with somebody else, and without leaving any of the current clients without a room?

2. An infinite bus with a countable number of passengers shows up. Can the receptionist accomodate the passengers of the bus, without asking anyone to shack up with somebody else, and without leaving any of the current clients without a room?

Problem 5. True or false: If X is a set and $A \subset X$ is a proper subset of X , then there can be a bijection between A and X .

Problem 6. Show that a non-empty subset of a countable set is either finite or countable.

Problem 7. Show that if X and Y are countable, then the cartesian product $X \times Y$ is countable.

Hint: Assume that you have given bijections for X and Y , and organize the elements of the cartesian product $X \times Y$ in a square, infinite, array. Then try and find a pattern for “counting” the elements of $X \times Y$, similar to what was done in the proof of Theorem 2.

Now we prove a fun fact: even though there seem to be way more rational numbers than natural numbers, in fact $\mathbb{Q}$ is a countable set.

Problem 8. The set of rational numbers $\mathbb{Q}$ is countable. Use Problems 6 and 7, and Theorem 2 to prove this fact.

At this point we should have at least one example of a non-countable set. Here we go.

Problem 9. The set of infinite sequences of 0’s and 1’s is not countable.

Hint: We prove this fact by contradiction. Assume that the set of such sequences is countable, and imagine you can write all infinite sequences in a list. Now you need to devise a procedure to construct a sequence of 0’s and 1’s that cannot possibly be in that list.

Now try to adapt the same argument to prove the following statement.

Problem 10. The set of real numbers $\mathbb{R}$ is not countable.

3 Sunday Homework

Exercise 1. Prove Theorem 1.

Exercise 2. Consider a finite set X with n elements. What is the number of bijections between X and itself? Why?

Exercise 3. Explain how Problems 4 and 5 are related.

Exercise 4. Generalize Theorem 2 as follows: for $n \in \mathbb{N}$, let X_n be a countable set. Show that

$$\bigcup_{n \in \mathbb{N}} X_n = X_1 \cup X_2 \cup X_3 \cup \dots$$

is a countable set.

Exercise 5. Given a set X , the **power set** of X , denoted $\mathcal{P}(X)$, is the set of all subsets of X . Prove that if X has n elements, then $\mathcal{P}(X)$ has 2^n elements, in two different ways:

1. Do a proof by induction.
2. Establish a bijection between $\mathcal{P}(X)$ and the set of ordered lists of 0's and 1's of length n .

Exercise 6. Prove that if X is countable, then $\mathcal{P}(X)$ is uncountable.

Exercise 7. Let us get comfortable with bijections.

- Establish a bijection between the set $(-1, 1)$ and all of $\mathbb{R}$.
- Establish a bijection between $\mathbb{R} \cup \{\star\}$ and $\mathbb{R}$.
- Establish a bijection between the set $[-1, 1]$ and $\mathbb{R}$.

Exercise 8. Let X be an uncountable set, and $Y \subset X$ be a countable subset of X . Prove that $X \setminus Y$ is uncountable.