

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 5

Combinatorics and Newton's theorem

1 Key Ideas

This week we are going to explore Newton's binomial expansion theorem. This is a very useful tool in analysis, but it also offers us an opportunity to explore an interesting connection with the combinatorics of finite sets. Newton's theorem is a formula for the expansion of an expression of the form $(x+y)^n$, for any natural number n . Once one has the formula, then a proof by induction can show that the formula is true. However, the connection to combinatorics actually explains why the formula holds. So, as usual, we will "overproof" our statement.

Let us start with some definitions.

Definition 1. For any natural number n , we define $n!$ (read " n factorial") to be the product of all natural numbers from 1 to n (including n).

So for example $1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, \dots$

There are good reasons to define $0! = 1$, even though we will not get into them right now.

Definition 2. For any two natural numbers n, k , with $k < n$, we define $\binom{n}{k}$ (read " n choose k ") to be:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

So for example $\binom{4}{2} = 6, \binom{4}{1} = \binom{4}{3} = 4$.

Since we defined $0! = 1$, we can also consistently define $\binom{n}{0} = \binom{n}{n} = 1$.

With these two definitions in place, we are ready to state Newton's binomial expansion theorem.

Theorem 1. For any natural number n , we have:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Let us see this theorem in action. If you want to expand $(x+y)^6$, you can immediately write:

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

because, in this case, we have: $\binom{6}{0} = \binom{6}{6} = 1, \binom{6}{1} = \binom{6}{5} = 6, \binom{6}{2} = \binom{6}{4} = 15, \binom{6}{3} = 20$.

Let us take a side-step for a second, and observe that the number $n!$ "counts" the number of ways you can order the elements of a set with n elements.

Theorem 2. Let $X = \{x_1, \dots, x_n\}$ be a set with n elements. There are $n!$ distinct ways of ordering the elements of X .

Proof. We have n choices for what element to put in the first position. Once we have chosen an element for first position, we have $n - 1$ choices for what element to put in second position. Once we have chosen the second element, we have $n - 2$ choices for what element to put in third position. This pattern continues until we have only one element left, which must be put in the last position. All together we have $n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$ choices, which is precisely the definition of $n!$. $\square$

2 Groupwork

Let us start by giving the number $\binom{n}{k}$ a combinatorial meaning.

Problem 1. If X is a set with n elements, show that $\binom{n}{k}$ is the number of subsets of X which contain k elements.

Question 1. After solving Problem 1, answer the following:

1. Why is $\binom{n}{k} = \binom{n}{n-k}$?
2. Are the definitions $\binom{n}{0} = \binom{n}{n} = 1$ consistent with Problem 1?

Problem 2. Show that the following identity holds:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Show it in two different ways:

1. Use the algebraic definition 2 and do some algebra.
2. Try to subdivide the set of *subsets of X with k elements* into two subsets, one of cardinality $\binom{n-1}{k}$, and the other of cardinality $\binom{n-1}{k-1}$.

Now let us tackle the proof of Newton's binomial theorem. To warm up, let us do the following exercise.

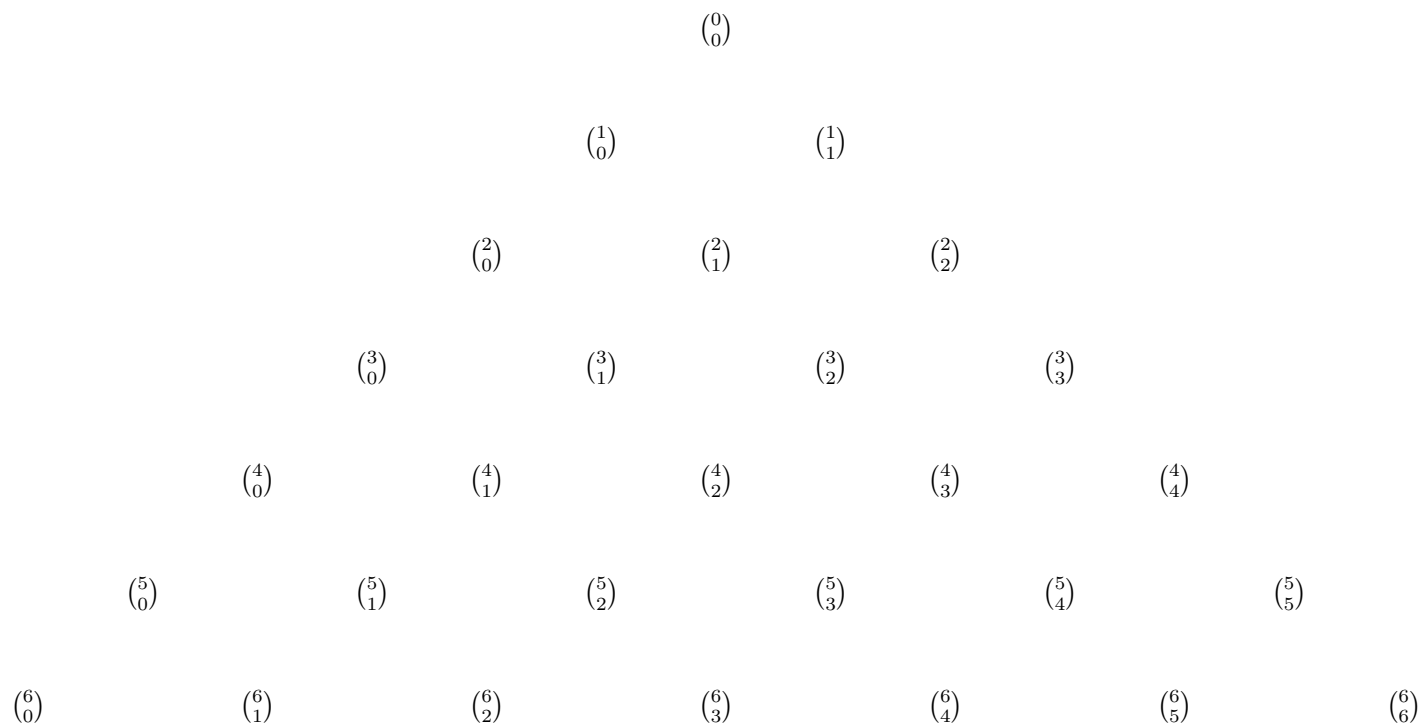
Problem 3. Write down $(x + y)(x + y)(x + y)$ in three different colors, one for each binomial. Now expand the above expression keeping track of the colors. For example, instead of writing x^3 , do write $x\text{blue}x$. Subdivide the terms into four groups, corresponding to the four monomials x^3, x^2y, xy^2, y^3 that you obtain when you forget the colors. Can you relate the number of elements in each group with a number that has to do with the set of colors $X = \{\text{black}, \text{blue}, \text{red}\}$?

Here comes the main problem in this worksheet, so make sure you spend a fair amount of time on this and understand it very well.

Problem 4. Generalize the idea from the previous problem to give a proof of Theorem 1.

Let us conclude this groupwork with some fun exploration.

Problem 5 (Pascal's triangle). Pascal's triangle is an infinite triangle of number that contains all numbers of the form $\binom{n}{k}$, as shown in the following figure:



Plugging in the numbers, you get

				1					
			1		1				
		1		2		1			
	1		3		3		1		
1		4		6		4		1	
	1		5		10		10		5
		1		6		15		20	
			1		6		15		20
				1		6		15	
					1		6		15
						1		6	
							1		6
								1	
									1

Spot as many interesting numerical patterns as you can in Pascal's triangle. Then, try to explain them!

3 Sunday Homework

Exercise 1. Give a proof by induction of Newton's Theorem (Theorem 1).

Exercise 2. Use Newton's theorem to show the following facts:

- The sum of the numbers in each row of Pascal's triangle is a power of 2.
- The alternating sum of the numbers in each row of Pascal's triangle is equal to 0.

Exercise 3. Given a natural number n , and three numbers k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = n$, what is the number of ways you can subdivide X into three disjoint subsets named U_1, U_2, U_3 , where the cardinality of U_1 is k_1 , the cardinality of U_2 is k_2 and the cardinality of U_3 is k_3 ? Prove your formula.

Exercise 4. Develop a formula for the expansion of a trinomial $(x + y + z)^n$. Prove that the formula holds in a similar way to what you did in Problem 4.

Exercise 5. Observe that the first five rows of Pascal's triangle correspond to powers of 11: $11^0 = 1, 11^1 = 11, 11^2 = 121, 11^3 = 1331, 11^4 = 14641$. Can you explain why? Why doesn't the pattern continue after the fifth row?

Exercise 6. A “hockeystick” in Pascal’s triangle is obtained by starting on the side of the triangle (at one of the 1’s), moving diagonally down for as long as you want, and then making one ninety degree downward turn and stopping. We call all the numbers that you traverse in the first direction the “rod” of the hockeystick, and the last number the “tip”. For example, if we start at $\binom{3}{0} = 1$, we could do $\{1, 4, 10, 15\}$. In this case the rod is $\{1, 4, 10\}$ and the tip is 15. Show that for any hockey stick the sum of the numbers in the rod equals the tip.