

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 3

Sets and Logics

1 Key Ideas

This week we are going to explore an interesting dictionary between sets and the logics we introduced to study mathematical statements. I find this dictionary useful because it allows to visualize very concretely logical statements that may feel confusing (for example, that the contrapositive of an implication is equivalent to the implication).

The main idea is the following: let the universe set $\mathcal{U}$ denote all things or events that you want to consider. To any statement, we associate the set of “things” for which the statement is true. According to our main idea, since the set $\mathcal{U}$ contains all elements that we want to consider, then $\mathcal{U}$ corresponds to the notion of a statement being true. We will call $\mathcal{U}$ also the *TRUE* set.

Question 1. What set do you think corresponds to the notion of a statement being *FALSE*?

Example 1. Let $\mathcal{U}$ be the set whose elements are all animals. Consider the mathematical statements:

S_1 : x is a cat;

S_2 : x is a black animal.

Then to S_1 corresponds the set $A_1 \subseteq \mathcal{U}$ whose elements are cats, and to S_2 corresponds the set $A_2 \subseteq \mathcal{U}$ whose elements are black animals.

Now consider the statement:

S_1 **AND** S_2 : x is a cat **AND** a black animal.

It corresponds to the set $A_1 \cap A_2$. We have discovered that the conjunction “AND” in logics corresponds to the set operation of intersection: $\cap$.

Things get more interesting once we start studying implications. Let us look at another example.

Example 2. Let $\mathcal{U}$ be the set whose elements are all animals. Consider the mathematical statements:

S_1 : x is a cat;

S_2 : x has a tail.

Now we make the statement:

$S_1 \implies S_2$ (read S_1 implies S_2 , or “If S_1 , then S_2 ”): IF x is a cat, THEN x has a tail.

For the statement $S_1 \implies S_2$ to be *TRUE*, it has to be the case that every element of $\mathcal{U}$ verifies the statement. But this is equivalent to the fact that any element of the set A_1 (i.e. every cat) must belong to the set A_2 (i.e. it must have a tail). Which means that A_1 is a subset of A_2 ($A_1 \subseteq A_2$). We have therefore learned that the notion of implication ($\implies$, or IF...THEN) corresponds to the notion of inclusion $\subseteq$.

2 Groupwork

The main goal of this groupwork is to continue building this dictionary, and get comfortable with how to use it.

Problem 1. If a statement S corresponds to a set A , what does the statement *not* S correspond to? (Hint: look at an example).

Problem 2. Use the information from Example 2 and Problem 1 to show that the contrapositive of an implication is equivalent to an implication. This means, consider a statement of the form “if S_1 , then S_2 ” and translate it into a statements about sets. Then consider the contrapositive “if not S_2 , then not S_1 ” and also translate this into a statement about sets. Then observe that the two statements about sets are equivalent, in the sense that one is true if and only if the other is.

Problem 3. For each of the statements below indicate what the sets involved are and what is the translation of the statement to the realm of sets. For example, if the statement is:

IF it rains, THEN I take my umbrella.

Then the two relevant sets are A_1 = the set of times when it rains, A_2 = the set of times I have my umbrella. And the statement translates to $A_1 \subseteq A_2$.

1. x is a dog OR a black animal.
2. I EITHER eat a slice of pie, OR a scoop of ice cream.
3. I take my umbrella ONLY IF it rains.
4. I take my umbrella IF AND ONLY IF it rains.
5. IF you are 18 years of age AND you are a citizen, THEN you can vote.
6. IF you are Italian OR French, THEN you are European.

Problem 4. Now let us formalize what we have observed up until now into a dictionary between sets and logic.

Logic	Sets
TRUE	
FALSE	
NOT	
AND	
OR	
EITHER...OR (exclusive or)	
IF...THEN	
...ONLY IF...	
IF AND ONLY IF	

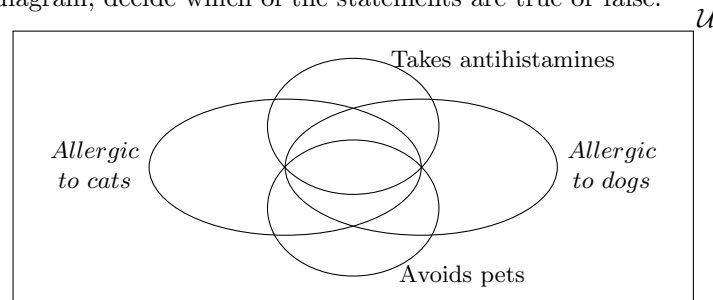
Problem 5. Translate the following statements to set language:

1. There exists a dog that likes broccoli;
2. Every dog likes bones.

Describe the translation of the quantifiers “there exists” and ”for every” in terms of sets.

3 Sunday Homework

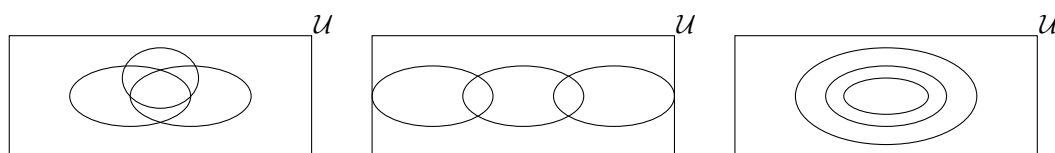
Exercise 1. In the following diagram, we have drawn four sets of people. Based on the diagram, decide which of the statements are true or false.



1. If someone is allergic to cats and dogs, then they take antihistamines or they avoid pets;
2. If someone is allergic to cats or dogs, then they take antihistamines or they avoid pets;
3. If someone takes antihistamines and is not allergic to dogs, then they are allergic to cats;
4. If someone takes antihistamines and they avoid pets, then they are allergic to cats and dogs;
5. If someone is not allergic to cats or dogs, then they don't avoid pets.

Exercise 2. Match the statements below with the set diagrams that make them true. Note: in each of the diagrams, the three bubbles represent the set of people that like Adele, Lady Gaga, Beyonce.

1. If one likes Lady Gaga, then they like Adele; if one likes Adele, then they like Beyonce.
2. Nobody likes Lady Gaga and Adele and Beyonce.
3. There are people that like Lady Gaga and Adele and Beyonce.



Exercise 3. Explain, by translating to the language of sets, the following fact: for a statement S : “If A , then B ”, when A is false then the statement S is true.

Exercise 4. Use Problem 5 to show that a statement of the form “For every element of A , then blah blah blah happens” is verified when A is equal to the empty set.

Exercise 5. In the groupwork we explored the following fact: *the contrapositive of an implication $S_1 \implies S_2$ is equivalent to the original implication* is translated to the statement about sets (that we saw last week in Exercise 1): $A \subseteq B \iff B^c \subseteq A^c$.

Use the dictionary between sets and logic to help you write the contrapositive to the following statements:

1. If it walks like a duck and it quacks like a duck, then it is a duck.
2. If you don’t behave, then you won’t get any ice cream or you won’t watch TV.
3. If you like yoga and you like goats, then you like goat-yoga and you are a funny person.

Exercise 6 (Proof by contradiction). Let us talk again about proof by contradiction. Proving that a theorem is TRUE, means to show that every event in the universe (or true) set $\mathcal{U}$ verifies the statement of the theorem. In set language, this means:

$$\mathcal{U} \subseteq \{\text{Events that verify the statement of the theorem}\}.$$

Applying complement to this inclusion (as seen in the previous exercise), we see that it is equivalent to:

$$\{\text{Events that falsify the statement of the theorem}\} \subseteq \phi.$$

So if we re-translate this back from sets to logics, we have the following: if we assume that something falsifies the statement of the theorem, then some false statement follows.

Let us now try our hands at a specific example.

Theorem 1. *The number $\sqrt{2}$ is irrational.*

Before you start proving the theorem, do you know what irrational means? If not, look at the following definitions.

Definition 1. .

- A number $x \in \mathbb{R}$ is **rational** if it can be written as quotient of two integers.
- A number $x \in \mathbb{R}$ is **irrational** if it is not rational.

We will be using the following fact.

Fact. Every integer has a unique prime factorization.

OK, here comes the exercise. Prove the theorem through the following steps:

1. Assume the statement of the theorem false (write an equation that shows that);
2. Square both terms of the equation and clear denominators to obtain an equality of integers;
3. Use the “Fact” and look at the number of prime factors equal to 2 on both sides of the equation to deduce that something is wrong.