

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 3

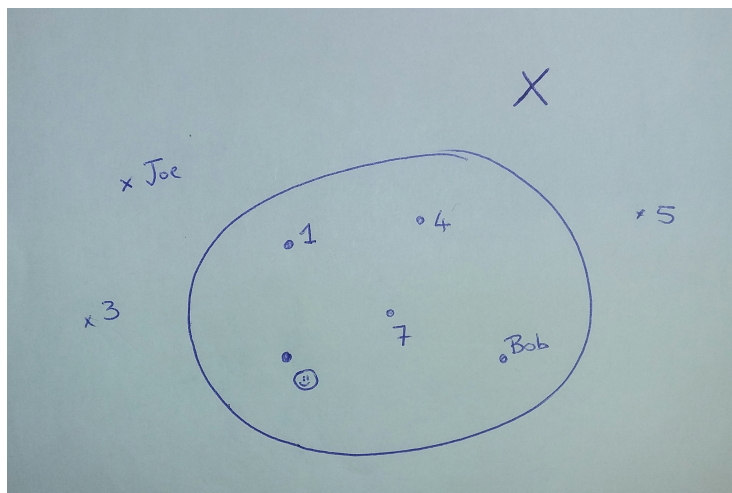
Sets

1 Key Ideas

As I mentioned on our first day of class, we are not going to define what a set is, since doing it honestly would require way too much work. We all have this intuitive notion of a set being a collection of “things”, called the elements of the set. Such intuitive notion will suffice for us to be able to work with sets, and set operations.

Here are some examples of sets:

1. The set of students of this class.
2. The set $A = \{1, 3, d, f, banana\}$.
3. The set of integers $\mathbb{Z}$.
4. The set X :



Observations:

elements: elements of a set can be anything you want. Set A containing banana as an element is just as mathematical as the set of integers.

how to describe sets: there are many ways that we can describe sets:

1. by giving properties that identify the elements of the set (Example 1).

2. by listing the elements of the set (Example 2). The order does not matter!
3. certain common mathematical sets have their own name (Example 3).
4. by drawing an oval, and putting the elements of the set inside the oval: this is called the Venn diagram representation of a set (Example 4).

cardinality of a set: the number of elements of a set is called the cardinality of the set. If a set does not contain a finite number of elements, we say its cardinality is infinite.

Definition 1. A set A is a **subset** of a set B if every element of A belongs to B . We write

$$A \subseteq B$$

to say that A is a subset of B .

Question 1. Is a set A a subset of itself?

Definition 2. A set A is a **proper subset** of a set B if $A \subseteq B$ and $A \neq B$. We write

$$A \subset B$$

to say that A is a proper subset of B .

Definition 3. Given two sets A, B we define:

Intersection. The intersection $A \cap B$ is the set of all common elements of A and B . In other words, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Union. The union $A \cup B$ is the set of all elements that belong to either A or B . In “mathematese”, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Observe that the set $A \cap B$ is a subset of A and it is a subset of B . On the other hand A is a subset of $A \cup B$, and B is a subset of $A \cup B$.

Here is an example of the kind of simple mathematical statements you can make about sets.

Theorem 1. *For any three sets A, B, C ,*

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

The basic trick here is that in order to prove an equality of sets $X = Y$, one typically proves two inclusions: that X is a subset of Y and Y is a subset of X . I also recommend, when proving statements like this, to draw Venn diagrams to see what is going on.

PROOF: *First we show that $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.* Consider an element $x \in (A \cup B) \cap C$; we want to show that $x \in (A \cap C) \cup (B \cap C)$. We know that $x \in C$ and $x \in A \cup B$. There are two possibilities:

1. $x \in C$ and $x \in A$, or
2. $x \in C$ and $x \in B$.

In the first case $x \in (A \cap C)$, and therefore $x \in (A \cap C) \cup (B \cap C)$. In the second case $x \in (B \cap C)$, and therefore $x \in (A \cap C) \cup (B \cap C)$. It follows that $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

Now we must show that $(A \cup B) \cap C \supseteq (A \cap C) \cup (B \cap C)$. Now consider an element $x \in (A \cap C) \cup (B \cap C)$. There are two possibilities:

1. $x \in A$ and $x \in C$, or
2. $x \in B$ and $x \in C$.

In both cases, $x \in C$, so we conclude that $x \in C$. Since in the first case $x \in A$ and in the second case $x \in B$, we conclude that $x \in A \cup B$. Therefore $x \in (A \cup B) \cap C$. We have shown that $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$.

Since $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$, it must be that $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$. This concludes the proof.

□

Let us conclude this part with one last definition.

Definition 4. The **universe** $\mathcal{U}$ is the set of all things that we wish to consider. Note that the universe set depends on what we are trying to do with it...in mathematics, the universe might be the set of all numbers that we want to work with. In probability, it may be the set of all events that we wish to consider. In any case, we just want to think of the universe as containing “everything”.

I like to represent the universe in Venn diagrams as a large rectangle. Then I will fit any set that I want to consider inside such rectangle.

2 Groupwork

Problem 1. Let us begin by introducing the empty set.

Definition 5. A set that contains no elements is called the **empty set** and denoted ϕ .

1. Show that the empty set is a subset of any other set.
2. For any set A , what is $A \cup \phi$? What is $A \cap \phi$?

Definition 6. Let us introduce some other operations on sets.

Complement: Given a set X , we define the **complement** of X , denoted X^c , by the following property: an element y belongs to X^c if and only if y does not belong to X .

Difference: Given two sets X, Y , the **difference** $X \setminus Y$ consists of all elements of X which do not belong to Y .

Symmetric difference: Given two sets X, Y , the **symmetric difference** $X \Delta Y$ is defined to be:

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X).$$

Problem 2. Draw Venn diagrams that illustrate the above three definitions.

Problem 3. Convince yourselves that for any set X ,

$$X^c = \mathcal{U} \setminus X.$$

Problem 4. When is $X \setminus Y = Y \setminus X$?

Problem 5. Prove, with the same level of detail as the proof of Theorem 1, that for every two sets A, B ,

$$(A \cap B)^c = A^c \cup B^c.$$

Problem 6. Decide, by drawing appropriate Venn diagrams, whether the operations of difference and symmetric difference are associative. More precisely, this means to check whether, for sets A, B, C :

1.

$$(A \setminus B) \setminus C = A \setminus (B \setminus C)?$$

2.

$$(A \Delta B) \Delta C = A \Delta (B \Delta C)?$$

3 Sunday Homework

Exercise 1. Prove, with the same level of detail as the proof of Theorem 1, that given two sets A, B ,

$$A \subseteq B \iff B^c \subseteq A^c.$$

(read: A is a subset of B if and only if B is a subset of A).

Exercise 2. Prove, with the same level of detail as the proof of Theorem 1, that for every two sets A, B ,

$$(A \cup B)^c = A^c \cap B^c.$$

Exercise 3. For any set A , prove (with the same level of detail as the proof of Theorem 1) that

$$(A^c)^c = A.$$

Exercise 4. Draw Venn diagrams representing each of the sets below:

1.

$$(A \cup B) \Delta C.$$

2.

$$A \cup (B \Delta C).$$

3.

$$(A \Delta B \Delta C)^c \setminus (A \cup B \cup C).$$

4.

$$((A \cap B) \Delta (B \cap C)) \Delta (A \cap C).$$

5.

$$((A \cap B) \cup (B \cap C) \cup (A \cap C)) \setminus (A \cap B \cap C).$$

Definition 7 (Cartesian product). Given two sets X, Y the **cartesian product** $X \times Y$ is the set of ordered pairs

$$X \times Y := \{(x, y) | x \in X, y \in Y\}.$$

Exercise 5. Let us get familiar with the cartesian product.

- If $X = \{1, 2\}$ and $Y = \{a, b, c\}$, write down the elements of $X \times Y$ and $Y \times X$. Notice that $X \times Y \neq Y \times X$.
- If X is a set with n elements, and Y a set with m elements, what is the cardinality of $X \times Y$?
- What is $X \times \phi$?

Exercise 6. More questions on the cartesian product:

- If $A \subseteq X$ and $B \subseteq Y$, prove that $A \times B \subseteq X \times Y$.
- True or false:

$$(X \times Y) \setminus (A \times B) = (X \setminus A) \times (Y \setminus B)?$$