

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 2

Definitions and Quantifiers

1 Key Ideas

In real life, we seldom use definitions. For example, you never worried about what is the definition of a *table*. Over the course of your life you were shown a bunch of examples of tables and you abstracted the notion of *table*, so that now you recognize a *table* when you see one. In a sense, in elementary mathematics, we also do that. We don't ever really define what numbers **are**, we just become familiar with them by using them.

In more advanced mathematics, definitions are the starting point of everything. Given any mathematical statement, in order to stand a chance to prove it, you have to be familiar with the definitions of all concepts involved in the statement. Let us start with some simple examples.

Definition 1 *A natural number n is:*

even *if $n = 2 \cdot q$, where q is a natural number;*

prime *if it is different from 1, and its only divisors are 1 and itself;*

foofy *if it is even, prime, and greater than 7.*

Theorem 1 *If a natural number is divisible by 4, then it is even.*

PROOF: Let n be a natural number divisible by 4. We can write $n = 4 \cdot k$, where k is also a natural number.

Since $4 = (2 \cdot 2)$, we have $n = (2 \cdot 2) \cdot k$.

Multiplication is associative, therefore

$$n = 2 \cdot (2 \cdot k).$$

Since $2 \cdot k$ is a natural number, this shows that n satisfies the definition of being even.

□

Theorem 2 *There are no foofy numbers.*

PROOF: The only even prime number is 2, since, by definition, a prime number is divisible only 1 and itself. So 2 is the only candidate to be a foofy number. Since 2 is less than 7, 2 is not foofy. Therefore there are no foofy numbers.

□

Pay close attention to quantifiers in mathematical sentences. Notice the big difference between the following two statements:

1. **There exists** a prime divisor equal to 2 in the prime factorization of n .
2. **For every** prime divisor p in the prime factorization of n , $p = 2$.

The first statement is telling us that n is an **even** number, the second statement is telling us that $n = 2^k$.

Suppose that a number n walks up to you and you want to assess whether it satisfies statements 1. or 2.

For statement 1.

- If you want to establish 1. to be true, you must look at the prime factorization of n until you find the factor 2. If you find it, you can stop.
- If you want to establish 1. to be false, you must look at ALL the prime factorization of n , and check that all prime divisors are different from 2.

For statement 2.

- If you want to establish 2. to be true, you must look at ALL the prime factorization of n , and check that all prime divisors are equal to 2.
- If you want to establish 2. to be false, you must look at the prime factorization of n until you find a factor different from 2. If you find it, you can stop.

In other words, the negative of a statement with a “there exists” contains a “for every” and viceversa. In this specific case, let us write the negatives of these statements. Oh, and since we are talking about definitions, let us define the negative.

Definition 2 *The **negative** of a mathematical statement S is another mathematical statement not S with the property that S is true when not S is false and S is false when not S is true.*

With this definition in place, we have:

Not 1. **For every** prime divisor p in the prime factorization of n , $p \neq 2$

Not 2. **There exists** a prime divisor different from 2 in the prime factorization of n .

2 Groupwork

We are going to train our skills by making up some random definitions, and running with them.

Definition 3 Given a set U of words, a **bas** is a subset B of U such that every element of B starts with the same letter.

Definition 4 Given a set U of words, a **bes** is a subset B of U such that there exists an element of B whose first and last letter are the same.

The first thing to do is getting comfortable with definitions by giving some examples and counterexamples.

Problem 1 Give an example of a bas, a bes, of something which is a bas and a bes, a bas but not a bes, a bes but not a bas, and neither a bas nor a bes.

Then we start making some statements about our newly introduced concepts.

Problem 2 For each of the following statements, decide whether they are true or false. If they are true, give a proof. If they are false, provide a counterexample showing it.

1. If U is a bas, then every subset of U is a bas.
2. If U is a bes, then every subset of U is a bes.
3. If there exists $B \subset U$ (read: a subset B of U) such that B is a bas, then U is a bas.
4. If there exists $B \subset U$ (read: a subset B of U) such that B is a bes, then U is a bes.
5. If B is a bas, then every subset of B is a bas.
6. If B is a bes, then every subset of B is a bes.
7. If B is a bas, then every set containing B is a bas.
8. If B is a bes, then every set containing B is a bes.
9. The empty set is a bas.
10. The empty set is a bes.

We now introduce some attributes of bes and bas.

Definition 5 A bas B is **closed** if B is the empty set or if there is no subset B' strictly containing B which is also a bas.

Definition 6 A bes B is **full** if every subset of B is also a bes.

Problem 3 Give some examples of closed bas's, not closed ones, full bes's and not full ones. Notice that being a closed bas doesn't only depend on B , but it depends essentially on U . Give an example of a set B and two choices of U such that in one case B is a closed bas, in the other case it isn't.

Problem 4 Can you figure out equivalent statements for Definitions 5 and 6? Formulate these statements as Theorems of the form: "A bas is closed if and only if...", "A bes is full if and only if..."; then prove such statements.

Definition 7 Given a bas $B \subseteq U$, the **closure** of B , denoted $\overline{B}$, is the unique closed bas containing B .

Problem 5 Show that Definition 7 makes sense: for any bas B , there exists a unique closed bas containing B .

Problem 6 Prove that for any bas B , $\overline{\overline{B}} = \overline{B}$.

Definition 8 Given a bes B , the **core** of B , denoted B° , is the largest subset of B which is a full bes.

Problem 7 Show that Definition 8 makes sense: for any bes B , there exist a unique largest full bes contained in B .

Problem 8 Show that taking the core commutes with unions. I.e., for every pair of bes's B_1, B_2 ,

$$B_1^\circ \cup B_2^\circ = (B_1 \cup B_2)^\circ$$

3 Sunday Homework

Exercise 1 Refer back to Statements 1. and 2. at the end of the Key Ideas section. Decide, for every $n \leq 20$ the truth or falsehood of the statements in the leftmost column of the table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1.																				
Not 1.																				
2.																				
Not 2.																				

Observe that the negative of each statement has indeed always opposite answer than the statement.

Exercise 2 Here is another fun definition to play with.

Definition 9 A set B is **babadabubi** if all of its elements are italian words.

In mathemathese, this is written:

B is **babadabubi** if for every b in B , b is an italian word.

Answer the following questions about babadabubi sets:

1. Write down the definition of not being babadabubi.
2. Give an example of a babadabubi set.
3. Give an example of a set which is not babadabubi.
4. Is the empty set babadabubi?
5. TRUE OR FALSE: the union of two babadabubi sets is babadabubi. Prove your answer.
6. TRUE OR FALSE: the intersection of two not babadabubi sets is not babadabubi. Prove your answer.
7. TRUE OR FALSE: if a set contains an english word then it is not babadabubi. Prove your answer. (Pay attention: is belonging to one language an exclusive property?)
8. TRUE OR FALSE: if a set is not babadabubi then it contains an english word. Prove your answer.

Exercise 3 (contrapositive) We introduce the notion of the contrapositive of a mathematical statement.

Definition 10 Let S_1, S_2 be two sentences and consider a statement (S) of the form: “If S_1 , then S_2 ”. We define the **contrapositive** statement to (S) to be:

If not S_2 , then not S_1 .

Write down the contrapositive statement to each of the following:

1. If it rains, then I carry an umbrella.
2. If it doesn't rain, then I don't carry an umbrella
3. If a natural number n is a multiple of m , then $n \geq m$.
4. If $x = y$, then $x^2 = y^2$.

Exercise 4 The interesting thing is that the contrapositive of a statement is equivalent to the statement, in the sense that they are true or false together (in the sense that there cannot be any situation in which a statement is true and its contrapositive is false, or viceversa). Observe this by filling in the table below. Put T if a scenario supports a statement, F if a scenario shows that the statement is false, and X if the scenario doesn't allow you to decide. Let 1. and 2. refer to the first two statements in the previous exercise.

	1.	contrapositive to 1.	2.	contrapositive to 2.
It rains and I have my umbrella				
It rains and I don't have my umbrella				
It doesn't rain and I have my umbrella				
It doesn't rain and I don't have my umbrella				
It doesn't rain and I tip-tap dance				
I tip-tap dance with an umbrella				

Exercise 5 Sometimes in order to prove a mathematical statement to be true, it is easier to prove the contrapositive statement. This proof technique is called **proof by contradiction**. Fill in the blanks to complete the proof.

Theorem 3 There are infinitely many prime numbers.

Comment: This is another case where all the hypothesis are hidden: they consist of knowing the definition of prime numbers, and the usual axioms of arithmetics. So you should think of this theorem as the statement: “If the hidden hypothesis hold, then there are infinitely many prime numbers”. Therefore the contrapositive statement is: “If there are finitely many prime numbers, then something goes wrong in arithmetics”. What goes wrong is the following: if we assume there are finitely many prime numbers, and we take them all, then we can always produce one more prime number. Now let us write down this idea formally.

PROOF: Assume there are _____ prime numbers, in particular we can assume there are n of them and we can write them $p_1, \dots, p_n$. Consider the number $q = p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$, obtained by multiplying all the prime numbers together, and then adding 1. We claim that q is not divisible by _____ of the p_i 's. We prove this claim by contradiction.

Assume that q _____ by one of the p_i 's, without loss of generality let us assume q _____ by p_1 .

This means $q = p_1 \cdot k$, with k a natural number.

Then

$$q - p_1 \cdot p_2 \cdot \dots \cdot p_n = \text{_____},$$

which shows that $q - p_1 \cdot p_2 \cdot \dots \cdot p_n$ is divisible by _____. But $q - p_1 \cdot p_2 \cdot \dots \cdot p_n = \text{_____}$, which is not divisible by any prime. Assuming that q is divisible by one of the p_i 's led to a contradiction, which allows us to conclude the claim: q is not divisible by _____ of the p_i 's. But then there are two possibilities: either q is a new prime number, or q has a prime factorization with primes that are different from the p_i 's. In either case, we are contradicting the statement that the list $p_1, \dots, p_n$ contains _____ prime numbers. Assuming that there are finitely many primes led us to a contradiction, which shows that it must be the case that there are infinitely many prime numbers.

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