

Quotients and Operations

1 Key Ideas

A couple weeks ago we discussed quotient sets and equivalence relations. Let us recap now the important points:

1. An **equivalence relation** $\sim$ on a set X produces a partition of X into disjoint subsets, called **equivalence classes**.
2. The set whose elements are the equivalence classes is called **quotient set** and denoted $X/\sim$.
3. Given an element $x \in X$, the equivalence class that x belongs to is denoted $[x]$ (note that $[x] = [y]$ if and only if $x \sim y$).
4. The **projection function** $\pi : X \rightarrow X/\sim$ assigns to each element x the equivalence class that x belongs to. ($\pi(x) = [x]$).
5. A function $F : X \rightarrow Y$ determines a function $[F] : X/\sim \rightarrow Y$ via the prescription $[F]([x]) = f(x)$ if and only if the following condition holds:

$$x_1 \sim x_2 \implies f(x_1) = f(x_2).$$

The last point is of particular interest to us: in order for a function $F : X \rightarrow Y$ to naturally define a function $[F]$ from the quotient set, F must satisfy some special property, which we think of as a compatibility condition between F and the equivalence relation $\sim$. Now we want to boost this idea. Let us consider an equivalence relation $\sim$ on a set X and a function:

$$F : X \rightarrow X$$

We would like to define a function $[F] : X/\sim \rightarrow X/\sim$ by the prescription:

$$[F]([x]) = [F(x)]. \tag{1}$$

Let us see in an example when this works. Let

$$X = \{boy, cat, dog, bat, chloe, doe, tic, tac, toe\},$$

and $F : X \rightarrow X$ be defined by:

$$F(boy) = bat, \quad F(cat) = dog, \quad F(dog) = bat$$

$$F(bat) = boy, \quad F(chloe) = doe, \quad F(doe) = bat$$

$$F(tic) = tac, \quad F(tac) = toe, \quad F(toe) = toe,$$

Define an equivalence relation on X by saying that two elements of X are equivalent if they begin with the same letter. Then $X/\sim = \{b, c, d, t\}$ has four elements. Then the prescription given in equation (1) defines the function $[F] : X/\sim \rightarrow X/\sim$ as:

$$[F](b) = b, \quad F(c) = d, \quad F(d) = b, \quad F(t) = t.$$

Now define an equivalence relation on X by saying that two elements of X are equivalent if they end with the same letter. The quotient set $X/\sim = \{c, e, g, t, y\}$ has five elements. The prescription given in equation (1) does NOT define a function:

$$[chloe] = [doe] = e,$$

but

$$[F(chloe)] = [doe] = e \neq [F(doe)] = [bat] = t,$$

which means that $[F]$ is trying to sent the input e to two different outputs.

Definition 1. Given an equivalence relation $\sim$ on a set X , we say that a function $F : X \rightarrow X$ is **compatible with $\sim$** if

$$x_1 \sim x_2 \implies F(x_1) \sim F(x_2).$$

If this is the case, the prescription in equation (1) defines a function $[F] : X/\sim \rightarrow X/\sim$.

Our goal this week is to generalize this idea even further: if X is a set of numbers and $\sim$ an equivalence relation, when can we define operations of addition and multiplication on the quotient set $X/\sim$? We will use this idea to give a precise and formal proof of the following rule, that probably a lot of you remember from elementary school.

Theorem 1. *An integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.*

2 Groupwork

We start with a couple definitions.

Definition 2. A set X is a **number system** if there are two operations on the elements of X (addition and multiplication) that satisfy all the usual rules of arithmetics that the common addition and multiplication of ordinary numbers satisfy.

Problem 1. Let X be a set with exactly two elements denoted E, O . Let addition and multiplication be defined by the following tables:

$+$	E	O	$\cdot$	E	O
E	E	O	E	E	E
O	O	E	O	E	O

In order to prove that X is a number system we should check that such addition and multiplication are commutative, associative and they respect the distributive laws. One really should check that these properties hold in **all possible instances!** This is of course very boring, so let us instead check, for each of the properties, one particular instance.

commutativity Check that $E + O = O + E$ and $E \cdot O = O \cdot E$.

associativity of $+$ Check that $(E + O) + O = E + (O + O)$.

associativity of $\cdot$ Check that $(E \cdot O) \cdot O = E \cdot (O \cdot O)$.

distributive laws Check that $O \cdot (E + O) = (O \cdot E) + (O \cdot O)$.

Next we consider the following question. Given an equivalence relation on a number system, when is the quotient set also a number system in a natural way? We make the following definition.

Definition 3. An equivalence relation $\sim$ on a number system X is **compatible with addition** if

$$x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 + y_1 \sim x_2 + y_2.$$

An equivalence relation $\sim$ on a number system X is **compatible with multiplication** if

$$x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 \cdot y_1 \sim x_2 \cdot y_2.$$

Problem 2. Give an example of an equivalence relation on a number system which is not compatible with addition, nor multiplication.

Problem 3. Let $X = \mathbb{Z}$ be the integers, and $\sim$ the equivalence relation defined as follows:

$$m \sim n \iff m - n = 9k,$$

for some $k \in \mathbb{Z}$. Show that $\sim$ is compatible with addition and multiplication.

When an equivalence relation on a number system is compatible with addition and multiplication, the quotient set $X/\sim$ becomes a number system with the following definitions. We write here $\oplus$ and $\otimes$ to define the operations on the quotient set, so as to not confuse them with the operations $+, \cdot$ on the original number system.

$$[x] \oplus [y] := [x + y]$$

and

$$[x] \otimes [y] := [x \cdot y]$$

In other words, we are using the operations of addition and multiplication on X to define operations on $X/\sim$. The dangerous thing is that elements of the quotient set have multiple names. So what may happen is that when two different names are used, the same operation on the same elements may produce two different outcomes. Let us see one quick example.

Problem 4. Let $X = \mathbb{Z}$ and $\sim$ the equivalence relation that declares all negative integers to be equivalent, and all non-negative integers to be equivalent. Use the definition above to $[4] \oplus [-3]$ and $[2] \oplus [-3]$. What is the problem?

The condition of being compatible with addition and multiplication are precisely what is needed to make sure the problem we just witnessed does not happen. Finally, the number systems X and $X/\sim$ are very closely related, in this sense.

Theorem 2. *Given any sequence of arithmetic operations on elements of X , one obtains the same result in the following two ways:*

1. *Perform all operations in X , and at the end, put square brackets around the result.*
2. *Perform all operations in $X/\sim$, i.e., put square brackets around all numbers and circles around all operation signs.*

Formal proofs of all these statements are a bit tedious and complicated, so we will skip them for now (raincheck till your MATH 366 class). But let us see one example of the above theorem. Consider the equivalence relation from Problem 3, and denote by $\mathbb{Z}_9$ the quotient set. Then the following computations give the same result:

$$[12 + 3 \cdot (11 + 7)] = [66] = [3]$$

or

$$[12] \oplus [3] \otimes ([11] \oplus [7]) = [3] \oplus [3] \otimes ([2] \oplus [7]) = [3] \oplus [3] \otimes [0] = [3].$$

Notice that in the second case, since we get to choose different names for the same element (i.e. we can replace $[12]$ with $[3]$, since $[12] = [3]$), we can choose to work with smaller numbers. This is the idea at the base of the test for divisibility by 3. Here are all the important steps.

Problem 5. Let $\mathbb{Z}_9$ be defined as above.

1. Prove that an integer n is divisible by 3 if and only if $[n] = [0], [3]$ or $[6]$.
2. Prove that for any nonnegative integer n , $[10^n] = [1]$.
3. Prove that for any integer n , $[n]$ is equal to the equivalence class of the sum of the digits of n .

3 Sunday Homework

Exercise 1. 1. Give an example of an equivalence relation $\sim$ on a set X , and a function $F : X \rightarrow X$ which is compatible with $\sim$.

2. Give an example of an equivalence relation $\sim$ on a set X , and a function $F : X \rightarrow X$ which is NOT compatible with $\sim$.

Exercise 2. Let $\sim$ be an equivalence relation on the set X . Decide if the following statements are true or false:

1. If $F : X \rightarrow X$ is a constant function, then $[F]$ is a function.
2. If $Id_X : X \rightarrow X$ denotes the identity function of X , then $[Id_X]$ is a function.
3. If $F : X \rightarrow X$ is an injective function, then $[F]$ is a function.
4. If $F : X \rightarrow X$ is a surjective function, then $[F]$ is a function.
5. If for every $x \in X$, $[F(x)] = [x]$, then $[F] = Id_{X/\sim}$ is the identity function of the quotient set.
6. If $[F] = Id_{X/\sim}$ is the identity function of the quotient set, then $F = Id_X$.

Exercise 3. Consider the number system $X = \{E, O\}$ from Problem 1.

zero An element z of a number system is a **zero** if, for every other element x in the number system, we have $x + z = z + x = x$, and $x \cdot z = z \cdot x = z$. Show that E is a zero for X .

one An element u of a number system is a **one** if, for every other element x in the number system, we have $x \cdot u = u \cdot x = x$. Show that O is a one for X .

subtraction The operation of **subtraction** is defined as follows: $x - y$ is defined to be an element w such that $x = y + w$. What is $E - O$?

division For y NOT a zero, the operation of **division** is defined as follows: x/y is defined to be an element w such that $x = y \cdot w$. What is E/O ?

Exercise 4. Use the concepts from Problem 5 to outline a formal proof of Theorem 1. An outline means that for the details you can just refer to the work you have done in the problems. But you need to write a clear and logical sequence of steps that prove the thesis of the theorem.

Exercise 5. Devise a test to check divisibility by the number 11.