

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 10

Continuity

1 Key Ideas

This week we focus on the kind of functions that we encounter in an analysis class (M317). In the simplest case, we deal with functions where both the input and output sets are the real numbers $\mathbb{R}$. We call x a coordinate for the input set, and y a coordinate for the output set. A function $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is also denoted with the familiar notation $y = f(x)$.

The notion of continuity is most intuitive in terms of the graph of a function: f is continuous if and only if its graph can be traced on a piece of paper representing $\mathbb{R} \times \mathbb{R}$ without ever needing to lift the pen from the paper. We also would like to say that a function f is continuous if when we vary the input “just a little bit”, then the output also varies “just a little bit”. The problem with this is, besides the fact that “just a little bit” does not sound like a very mathematical notion, that if you start by thinking of the x variation, there is really no way to bound the variation of the output. Think of a linear function, of the form $y = mx$. Its graph is a line, so it is as continuous as it gets. However, by growing the slope m , we can make the y -increment corresponding to a fixed x -increment be as large as we want.

So to get a precise definition of our intuitive notion, we have to start by thinking about the y -variation first. Suppose that we choose an input x_0 , with output $f(x_0)$. The function f is continuous at the point x_0 , if we can always play the following game: you can choose however you want a small interval around $f(x_0)$, let us say it has radius ε . After you do that, I must be able to choose an interval around x_0 in such a way that if any point is chosen in my interval as an input, then the output is guaranteed to land in your interval. This idea gives rise to the following initially confusing definition, which is mostly just an exercise in being comfortable with quantifiers.

Definition 1. A function $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is continuous at the point x_0 if, **for every** $\varepsilon > 0$, **there exists** δ (depending on ε), such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

We say f is a **continuous function** if it is continuous at every point x_0 .

The notion of continuity is extremely important if you imagine that f is describing or modeling some experiment that you are running: it is saying that you can control the margin of error of the output of your experiment so long as you can control sufficiently the precision of your input.

2 Groupwork

Problem 1. Prove, using the definition, that the function $y = x^3$ is a continuous function.

Solution:

Let $x_0 \in \mathbb{R}$ be a point in the input set for the function $f(x) = x^3$. x_0 is fixed and from now on we treat it just as a fixed number.

Fix an $\varepsilon > 0$. ε is fixed and from now on we treat it just as a fixed number.

We want to see for what values of x the following inequality is satisfied, hoping to show that there is an interval around x_0 where it is always verified.

Consider the inequality:

$$|f(x) - f(x_0)| < \varepsilon$$

By the definition of f this is:

$$|x^3 - x_0^3| < \varepsilon$$

Because of the absolute value, this breaks into two inequalities:

(A) For $x > x_0$,

$$x^3 - x_0^3 < \varepsilon$$

which is equivalent to

$$x^3 < \varepsilon + x_0^3.$$

and,

$$x < \sqrt[3]{\varepsilon + x_0^3}.$$

(B) For $x < x_0$,

$$x_0^3 - x^3 < \varepsilon$$

which is equivalent to

$$x^3 > x_0^3 - \varepsilon.$$

and,

$$x > \sqrt[3]{x_0^3 - \varepsilon}.$$

Now comes the interesting part. $\sqrt[3]{\varepsilon + x_0^3}$ is some fixed number just a little bigger than x_0 . $\sqrt[3]{x_0^3 - \varepsilon}$ is some fixed number, just a little smaller than x_0 . The result of the discussion in (A) and (B) is that for any x in the interval determined by these two numbers the inequality $|x^3 - x_0^3| < \varepsilon$ is satisfied. This is exactly what we want. Let us write this formally.

The discussion in (A) and (B) shows that if we choose δ to be the minimum between $\sqrt[3]{\varepsilon + x_0^3} - x_0$ and $x_0 - \sqrt[3]{x_0^3 - \varepsilon}$, then for every x such that $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. This shows that $f(x) = x^3$ is continuous at the point x_0 . Since x_0 was an arbitrary point in the input set (domain), then f is a continuous function.

Problem 2. Show, using the definition, that the function $y = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ is not continuous.

Solution:

Here we need to show that f is NOT continuous. So we must carefully choose an x_0 and an ε for which no corresponding δ exists.

Let $x_0 = 0$ and $\varepsilon = 1/7$. For any $\delta > 0$, consider the point $x = \delta/2$. We have $|x - x_0| = \delta/2 - 0 < \delta$. But

$$|f(x) - f(x_0)| = |f(\delta/2) - f(0)| = |0 - 1| = 1 > 1/7.$$

Therefore f is not continuous at $x_0 = 0$.

It is evident that checking whether a function is or is not continuous using the definition is a rather painful and cumbersome exercise. That is why we prove theorems that allow us to conclude that functions are continuous without having to go through the definition every time. Here are some examples.

Problem 3. Prove that if f_1 and f_2 are two continuous functions, then $f_1 + f_2$ is a continuous function.

Prove that a finite sum of continuous functions is a continuous function.

Solution:

Assume f_1 and f_2 are continuous functions. We want to show that $f_1 + f_2$ is continuous. Fix a point x_0 in the input set for the function $(f_1 + f_2)$ and an $\varepsilon > 0$. The idea is to split the margin of error that we are allowed (ε) between the two functions f_1 and f_2 .

Choose two positive numbers $\varepsilon_1, \varepsilon_2$ with the property that $\varepsilon_1 + \varepsilon_2 = \varepsilon$.

Now it is time to recall our hypotheses.

By hypothesis, since f_1 is continuous at x_0 , there exists a δ_1 such that, if $|x - x_0| < \delta_1$, then $|f_1(x) - f_1(x_0)| < \varepsilon_1$.

By hypothesis, since f_2 is continuous at x_0 , there exists a δ_2 such that, if $|x - x_0| < \delta_2$, then $|f_2(x) - f_2(x_0)| < \varepsilon_2$.

And now let us check that we get what we want.

Set δ equal to the minimum between δ_1 and δ_2 . If $|x - x_0| < \delta$,

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| = |f_1(x) - f_1(x_0) + f_2(x) - f_2(x_0)| \leq$$

$$|f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon_1 + \varepsilon_2 = \varepsilon.$$

But this is precisely the definition of $f_1 + f_2$ being continuous at the point x_0 .

In the Sunday homework you will be asked to prove that a product of continuous function is a continuous function. For now, assume this as a true fact.

Problem 4. Prove that all polynomials are continuous functions.

Realize that polynomials are sums and products of constant functions and functions of the form $f(x) = x$.