

## INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 10

### Continuity

## 1 Key Ideas

This week we focus on the kind of functions that we encounter in an analysis class (M317). In the simplest case, we deal with functions where both the input and output sets are the real numbers  $\mathbb{R}$ . We call  $x$  a coordinate for the input set, and  $y$  a coordinate for the output set. A function  $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$  is also denoted with the familiar notation  $y = f(x)$ .

The notion of continuity is most intuitive in terms of the graph of a function:  $f$  is continuous if and only if its graph can be traced on a piece of paper representing  $\mathbb{R} \times \mathbb{R}$  without ever needing to lift the pen from the paper. We also would like to say that a function  $f$  is continuous if when we vary the input “just a little bit”, then the output also varies “just a little bit”. The problem with this is, besides the fact that “just a little bit” does not sound like a very mathematical notion, that if you start by thinking of the  $x$  variation, there is really no way to bound the variation of the output. Think of a linear function, of the form  $y = mx$ . Its graph is a line, so it is as continuous as it gets. However, by growing the slope  $m$ , we can make the  $y$ -increment corresponding to a fixed  $x$ -increment be as large as we want.

So to get a precise definition of our intuitive notion, we have to start by thinking about the  $y$ -variation first. Suppose that we choose an input  $x_0$ , with output  $f(x_0)$ . The function  $f$  is continuous at the point  $x_0$ , if we can always play the following game: you can choose however you want a small interval around  $f(x_0)$ , let us say it has radius  $\varepsilon$ . After you do that, I must be able to choose an interval around  $x_0$  in such a way that if any point is chosen in my interval as an input, then the output is guaranteed to land in your interval. This idea gives rise to the following initially confusing definition, which is mostly just an exercise in being comfortable with quantifiers.

**Definition 1.** A function  $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$  is continuous at the point  $x_0$  if, **for every**  $\varepsilon > 0$ , **there exists**  $\delta$  (depending on  $\varepsilon$ ), such that, if  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \varepsilon$ .

We say  $f$  is a **continuous function** if it is continuous at every point  $x_0$ .

The notion of continuity is extremely important if you imagine that  $f$  is describing or modeling some experiment that you are running: it is saying that you can control the margin of error of the output of your experiment so long as you can control sufficiently the precision of your input.

## 2 Groupwork

**Problem 1.** Prove, using the definition, that the function  $y = x^3$  is a continuous function.

**Problem 2.** Show, using the definition, that the function  $y = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$  is not continuous.

It is evident that checking whether a function is or is not continuous using the definition is a rather painful and cumbersome exercise. That is why we prove theorems that allow us to conclude that functions are continuous without having to go through the definition every time. Here are some examples.

**Problem 3.** Prove that if  $f_1$  and  $f_2$  are two continuous functions, then  $f_1 + f_2$  is a continuous function.

Prove that a finite sum of continuous functions is a continuous function.

In the Sunday homework you will be asked to prove that a product of continuous function is a continuous function. For now, assume this as a true fact.

**Problem 4.** Prove that all polynomials are continuous functions.

**Definition 2.** Let  $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$  be a function, and  $x_0$  a fixed input. The **limit** of  $f$  as  $x$  approaches  $x_0$  is  $L$  if, **for every**  $\varepsilon > 0$ , **there exists**  $\delta$  (depending on  $\varepsilon$ ), such that, if  $|x - x_0| < \delta$ , then  $|f(x) - L| < \varepsilon$ . We use the following notation:

$$\lim_{x \rightarrow x_0} f(x) = L$$

**Problem 5.** Try and develop an intuitive understanding of Definition 2. Clearly it looks quite closely related to the definition of continuity. Our goal is to understand the precise relationship.

**Problem 6.** Use the definition to establish what the following limits are (if they exist):

$$\lim_{x \rightarrow 0} x^3 = ?$$

$$\lim_{x \rightarrow 0} \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} = ?$$

$$\lim_{x \rightarrow 0} \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} = ?$$

**Problem 7.** Formulate, and then prove, a statement that relates the notion of continuity of a function, to the notion of limit.

### 3 Sunday Homework

**Exercise 1.** Prove that if  $f_1$  and  $f_2$  are two continuous functions, then  $f_1 \cdot f_2$  is a continuous function.

**Exercise 2.** Prove that if  $f_1$  and  $f_2$  are two continuous functions, then  $f_1 \circ f_2$  is a continuous function.

**Exercise 3.** Suppose that  $\lim_{x \rightarrow x_0} f_1(x) = L_1$  and  $\lim_{x \rightarrow x_0} f_2(x) = L_2$ ; show that

$$\lim_{x \rightarrow x_0} f_1(x) + f_2(x) = L_1 + L_2.$$

**Exercise 4.** Suppose that  $\lim_{x \rightarrow x_0} f(x) = L_1$  and  $\lim_{x \rightarrow L_1} g(x) = L_2$ ; show that

$$\lim_{x \rightarrow x_0} g \circ f(x) = L_2.$$

**Definition 3.** An **open interval** in  $\mathbb{R}$  is the set of all numbers  $x$  which are strictly greater than a number  $a$ , and strictly less than a number  $b$ . This is denoted by the symbol  $(a, b)$ .

$$(a, b) := \{x \in \mathbb{R} \text{ such that } a < x < b\}.$$

**Exercise 5.** Prove that a subset  $S$  of the real numbers is a union of open intervals if and only if, for every  $s \in S$ , there exists an open interval which contains  $s$  and is contained in  $S$ . Such a set  $S$  is called an **open set**.

**Exercise 6.** Prove that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous if and only if, for every open set  $S$ ,  $f^{-1}(S)$  is an open set.