

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet1

Examples of proofs

1 Key Ideas

Proving a mathematical statement means to give a sequence of implications that show how the thesis (what is stated) follows from the hypothesis (what is assumed to be true). There are many different ways to write a proof, and trying to constrain proof writing to an algorithm will sooner or later be limiting. Also, it is often useful to “translate” a mathematical statement into an equivalent statement where we have better tools to show its validity. So rather than blabbing abstractly, let me illustrate this with one example. Consider the mathematical statement:

The sum of the first n odd natural numbers is equal to n^2 . (S)

We can rewrite this statement symbolically as follows:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2, \quad (1)$$

or if we want to be even more slick,

$$\sum_{i=1}^n (2i - 1) = n^2. \quad (2)$$

Before we proceed any further, let us pause for a second to become aware of some hidden things to which we will pay a lot of attention during this semester:

hypotheses: In the statement above there seem to be no hypothesis, but in fact we have just hidden what we consider common knowledge: that we agree on what a natural number is, on what an odd number is, that we agree on how the operations of addition and multiplication work and that the symbol n^2 means $n \cdot n$. These are the hidden hypotheses of this mathematical statement.

quantifier: In order for statement (S) to be true, formula (2) must hold **for every possible choice of a natural number** n . So, if one wanted to be absolutely complete, one should rewrite it as

For every natural number n , the sum of the first n odd natural numbers is equal to n^2 .

clear pattern: In equation (1) we take advantage of the fact that we are communicating among humans and not to a computer. Writing $1 + 3 + 5$ establishes a pattern which is easily recognizable by just about anyone. That is why it is acceptable to put “...” to indicate that one should continue with the same pattern until the number $(2n - 1)$.

Now let us get to work and prove statement (S). OK, hold on... before we actually start the proof, there is one more important thing to do... being skeptical! We are going to test the statement for a few values of n . This is useful for two reasons:

1. It provides a check that there was not a typo, or that flatly you were given as a task to try and prove a false statement (never trust your teachers).
2. Testing a statement often allows you to understand the statement better, and often suggests how to prove it.

So let us test our statement for n up to 5:

$$n = 1$$

$$1 = 1^2,$$

$$n = 2$$

$$1 + 3 = 4 = 2^2,$$

$$n = 3$$

$$1 + 3 + 5 = 9 = 3^2,$$

$$n = 4$$

$$1 + 3 + 5 + 7 = 16 = 4^2,$$

$$n = 5$$

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2.$$

Allright, it looks like I was not trying to fool you. But here is an interesting observation that one can make by looking at the above list of checks. At each line, you have a summation which is equal to the previous line plus one more term. Which in practice means that if you have already checked that $1 + 3 + 5 + 7 = 16$ in the fourth line, in the fifth line you could save yourself some energy by computing directly $16 + 9$ as opposed to going back to doing all over $1 + 3 + 5 + 7 + 9$. This idea is at the basis of a proof technique called **induction**. Let us first show the proof of statement (S) using this technique, and then make some general comments about induction.

1.1 Proof 1: by induction

We establish the **base case**. Statement (S) is true for $n = 1$: the sum of the first odd natural number is 1, which is equal to 1^2 .

We now assume statement (S) to be true for n equal to a particular but unspecified N . With such hypothesis, we can establish the validity of statement (S) for $N + 1$. (this is called the **inductive step**).

Consider the left hand side of (S) for $n = N + 1$. By associativity of addition we have:

$$1 + 3 + 5 + \dots + (2N - 1) + (2N + 1) = [1 + 3 + 5 + \dots + (2N - 1)] + (2N + 1)$$

In the square parentheses we have the left hand side of (S) for $n = N$. Since we are assuming (S) to be a true statement for $n = N$, we can replace the square parenthesis with the right hand side of (S) for $n = N$:

$$[1 + 3 + 5 + \dots + (2N - 1)] + (2N + 1) = N^2 + (2N + 1).$$

Now by elementary arithmetics we have

$$N^2 + (2N + 1) = (N + 1)^2,$$

which is the right hand side of (S) for $n = N + 1$. This concludes the proof of statement (S) .

Induction

What just happened? How did we just prove statement (S) ? There is a perfect analogy between a proof by induction and the game of domino: you should imagine that statement (S) is in fact an infinite number of statements, one for each value of n (in the test above we saw the first 5). Imagine each of these statements is the tile of a domino game, put one after the other. A statement being true corresponds to a tile falling. When we show the **inductive step**, we are saying that if any one tile falls, then it knocks down the next one. Establishing the base case amounts to showing that the first tile falls. These two together show that every tile is gonna fall, because the first knocks down the second, the second the third, and so on forever.

Given a statement of the form:

for every natural number n , $\text{blah}(n)$ happens.

Think of blah as a formula, or as a sentence containing a formula depending on n . A proof by induction goes as follows:

base case: showing that $\text{blah}(1)$ happens.

inductive step: assume that $\text{blah}(N)$ happens, and show that $\text{blah}(N + 1)$ happens.

1.2 Proof 2: by geometric translation

We now prove statement (S) by translating it to a geometric statement which we then show to be true.

Imagine the natural numbers as counting the number of beads that we have. If we have $1 + 3 + \dots + (2n - 1)$ beads, we are going to arrange them in the plane in such a way that it is apparent that the total number of beads we have is n^2 . We put the first bead at $(0, 0)$. The next three beads will be put in position $(1, 0), (1, 1), (0, 1)$. Notice that now the beads form a square pattern of side given by 2 beads, and therefore we have a total of $4 = 2^2$ beads. We can now add five beads in position $(2, 0), (2, 1), (2, 2), (1, 2), (0, 2)$ to complete

a square of side-length 3 beads, therefore containing a total of 3^2 beads. This process continues indefinitely: when we add the n -th odd number of beads by putting them in the positions (i, j) where either i or j (or both) are equal to $n-1$, we have that the beads are arranged to fill a square of side-length n beads, and therefore we have a total of n^2 beads. Look at the Figure 1 for a reference to this construction.

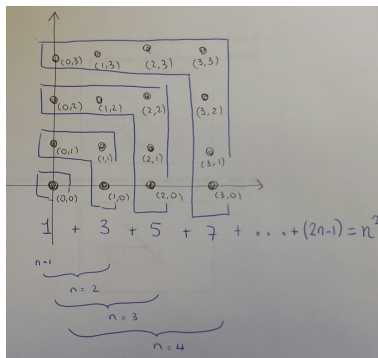


Figure 1: A figure that illustrates the argument of the geometric proof of statement (S). Remember: a figure is not a proof, but a figure helps understanding a proof by clarifying a geometric pattern which we are using in the proof of our argument.

2 Groupwork

Let us start by defining some symbols.

$H(n)$ is a number depending on an integer n . $H(n)$ is the number of handshakes that happen if, in a group of n persons, everyone shakes hands with everyone else exactly once.

$G(m)$ is a number depending on an integer m . $G(m)$ is the sum of the first m natural numbers. In symbols,

$$G(m) = \sum_{i=1}^m i.$$

Now we state some theorems.

Theorem 1 For every natural number n ,

$$H(n) = G(n-1).$$

Theorem 2 *For every natural number n ,*

$$H(n) = \frac{1}{2}n \cdot (n - 1).$$

Theorem 3 *For every natural number m ,*

$$G(m) = \frac{1}{2}m \cdot (m + 1).$$

First off, let us observe that Theorem 2 gives a formula for $H(n)$, Theorem 3 gives a formula for $G(m)$ and Theorem 1 expresses a relationship between H and G .

Question 1 *Do the three theorems have a chance of being all true? In other words, is the relationship given in Theorem 1 compatible with the formulas given in Theorems 2 and 3?*

Now let us observe that we can think of Theorem 1 as a translation between two problems: a combinatorial problem of counting handshakes is related to an algebraic problem of adding a sequence of natural numbers.

Problem 1 *Meditate on the following fact. If we establish that Theorem 1 is true, then proving one of the remaining Theorems implies that the last one is also true.*

Problem 1 makes precise what it means for Theorem 1 to be a translation between the two problems: a solution to one will immediately lead to the solution of the other.

As we said, before embarking on proving anything, we should test our statements to make sure we can't immediately show that they are false.

Problem 2 *Compute $H(n)$ for $n \leq 4$ and $G(m)$ for $m \leq 4$, and show that the statements of the three theorems are verified for these small values of n and m .*

Next, we are going to establish the truth of Theorem 1.

Problem 3 *Prove Theorem 1 by organizing the count of the handshakes $H(n)$ in a way that makes it clear that the total number of handshakes is $G(m - 1)$.*

Now, any proof of either of the remaining theorem will complete the work and show that all three theorems hold true. However, we are not going to content ourselves with that, but we will provide a few different proofs. For each of the proofs we present, ask yourselves what are the things you like and dislike about such proof (this is a subjective question, so there isn't a correct answer).

Problem 4 (Baby Gauss trick) *Prove Theorem 3 by using a trick attributed (probably an urban legend) to Gauss in kindergarden. Write the numbers 1 to m in a $m \times 2$ array. In the first row write them in increasing order, in the second row write them in decreasing order:*

$$\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & \dots & m-2 & m-1 & m \\ m & m-1 & m-2 & \dots & 3 & 2 & 1 \end{array}$$

What is the relationship between the sum of all the numbers in this array and $G(n)$? Find a way to compute the sum of all the numbers in this array that allows to give a formula for this sum.

We now give a rather complicated proof of Theorem 3. We will see however that this proof has the advantage of giving the possibility of generalizing the statement of Theorem 3. Start from the following identity, which is well known to hold true for any value of i :

$$(i+1)^2 = i^2 + 2i + 1$$

We manipulate it as follows:

$$-i^2 + (i+1)^2 = 2i + 1$$

$$-i^2 + (i+1)^2 - 1 = 2i$$

$$\frac{1}{2} (-i^2 + (i+1)^2 - 1) = i$$

This seems a bizarre thing to do, let us ponder for a second what we just did: we expressed an integer i as an expression that contains the difference of the squares of two consecutive numbers. This doesn't seem all that convenient at first, but let us remember that our task is to add integers. So let us add summation signs on either sides of the above identity:

$$\sum_{i=1}^m \frac{1}{2} (-i^2 + (i+1)^2 - 1) = \sum_{i=1}^m i$$

Since addition is commutative and distributive, we can rewrite this as:

$$\frac{1}{2} \left(\sum_{i=1}^m [-i^2 + (i+1)^2] - \sum_{i=1}^m 1 \right) = \sum_{i=1}^m i = G(m). \quad (3)$$

Problem 5 Consider the summation:

$$\sum_{i=1}^m [-i^2 + (i+1)^2]$$

Compute the value of this summation for small values of m . Guess a formula for a general value of m . Provide an argument that proves your formula.

This summation is what is called a **telescoping sum**: the second term of the first summand cancels with the first term of the second summand, the second term of the second summand cancels with the first term of the third summand, and so on; so, in the end, out of this whole big summation, the only surviving terms are the very first and the very last.

Problem 6 Use the formula from Problem 5 to compute (3), and show that this gives a proof of Theorem 3.

3 Sunday Homework

Exercise 1 Give a proof by induction of Theorem 3.

Exercise 2 Give a geometric proof of Theorem 3, again by thinking of the integers as counting the number of beads, and realizing a strategy of positioning the beads on the plane in such a way that the quantity $G(m)$ can be counted. **Hint:** look for arranging $1 + 2 + 3 + \dots + m$ beads inside a square $m \times m$ grid.

Exercise 3 Give a direct proof of Theorem 2 by relating the number of handshakes in a group of n people to the number of ordered pairs of distinct persons from that group.

Exercise 4 Use Theorem 3 to give another proof of the statement:

$$\sum_{i=1}^n (2i - 1) = n^2.$$

Exercise 5 Prove by induction that for $q \neq 1$

$$\sum_{i=0}^n q^i = \frac{q^{n+1} - 1}{q - 1}. \quad (4)$$

Exercise 6 Consider equation (4). Since $q \neq 1$, show that you obtain an equivalent equation by clearing denominators. Realize that now on the left hand side you have a telescoping sum and use that fact to give another proof of the validity of (4).

Exercise 7 Use an argument similar to the last proof in the groupwork to find a formula for the sum of the first n squares:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + (n-1)^2 + n^2 = ?$$

Exercise 8 (challenge - optional) Prove that the formula for the sum of the first n k -powers is a polynomial in n of degree $k + 1$.

$$\sum_{i=1}^n i^k = 1^k + 2^k + \dots + (n-1)^k + n^k = a_{k+1}n^{k+1} + \dots + a_0$$

Note: I am not asking you to find the formula for such a polynomial!!