

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet1

Examples of proofs

1 Key Ideas

Proving a mathematical statement means to give a sequence of implications that show how the thesis (what is stated) follows from the hypothesis (what is assumed to be true). There are many different ways to write a proof, and trying to constrain proof writing to an algorithm will sooner or later be limiting. Also, it is often useful to “translate” a mathematical statement into an equivalent statement where we have better tools to show its validity. So rather than blabbing abstractly, let me illustrate this with one example. Consider the mathematical statement:

The sum of the first n odd natural numbers is equal to n^2 . (S)

We can rewrite this statement symbolically as follows:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2, \quad (1)$$

or if we want to be even more slick,

$$\sum_{i=1}^n (2i - 1) = n^2. \quad (2)$$

Before we proceed any further, let us pause for a second to become aware of some hidden things to which we will pay a lot of attention during this semester:

hypotheses: In the statement above there seem to be no hypothesis, but in fact we have just hidden what we consider common knowledge: that we agree on what a natural number is, on what an odd number is, that we agree on how the operations of addition and multiplication work and that the symbol n^2 means $n \cdot n$. These are the hidden hypotheses of this mathematical statement.

quantifier: In order for statement (S) to be true, formula (2) must hold **for every possible choice of a natural number** n . So, if one wanted to be absolutely complete, one should rewrite it as

For every natural number n , the sum of the first n odd natural numbers is equal to n^2 .

clear pattern: In equation (1) we take advantage of the fact that we are communicating among humans and not to a computer. Writing $1 + 3 + 5$ establishes a pattern which is easily recognizable by just about anyone. That is why it is acceptable to put “...” to indicate that one should continue with the same pattern until the number $(2n - 1)$.

Now let us get to work and prove statement (S). OK, hold on... before we actually start the proof, there is one more important thing to do... being skeptical! We are going to test the statement for a few values of n . This is useful for two reasons:

1. It provides a check that there was not a typo, or that flatly you were given as a task to try and prove a false statement (never trust your teachers).
2. Testing a statement often allows you to understand the statement better, and often suggests how to prove it.

So let us test our statement for n up to 5:

$$n = 1$$

$$1 = 1^2,$$

$$n = 2$$

$$1 + 3 = 4 = 2^2,$$

$$n = 3$$

$$1 + 3 + 5 = 9 = 3^2,$$

$$n = 4$$

$$1 + 3 + 5 + 7 = 16 = 4^2,$$

$$n = 5$$

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2.$$

Allright, it looks like I was not trying to fool you. But here is an interesting observation that one can make by looking at the above list of checks. At each line, you have a summation which is equal to the previous line plus one more term. Which in practice means that if you have already checked that $1 + 3 + 5 + 7 = 16$ in the fourth line, in the fifth line you could save yourself some energy by computing directly $16 + 9$ as opposed to going back to doing all over $1 + 3 + 5 + 7 + 9$. This idea is at the basis of a proof technique called **induction**. Let us first show the proof of statement (S) using this technique, and then make some general comments about induction.

1.1 Proof 1: by induction

We establish the **base case**. Statement (S) is true for $n = 1$: the sum of the first odd natural number is 1, which is equal to 1^2 .

We now assume statement (S) to be true for n equal to a particular but unspecified N . With such hypothesis, we can establish the validity of statement (S) for $N + 1$. (this is called the **inductive step**).

Consider the left hand side of (S) for $n = N + 1$. By associativity of addition we have:

$$1 + 3 + 5 + \dots + (2N - 1) + (2N + 1) = [1 + 3 + 5 + \dots + (2N - 1)] + (2N + 1)$$

In the square parentheses we have the left hand side of (S) for $n = N$. Since we are assuming (S) to be a true statement for $n = N$, we can replace the square parenthesis with the right hand side of (S) for $n = N$:

$$[1 + 3 + 5 + \dots + (2N - 1)] + (2N + 1) = N^2 + (2N + 1).$$

Now by elementary arithmetics we have

$$N^2 + (2N + 1) = (N + 1)^2,$$

which is the right hand side of (S) for $n = N + 1$. This concludes the proof of statement (S) .

Induction

What just happened? How did we just prove statement (S) ? There is a perfect analogy between a proof by induction and the game of domino: you should imagine that statement (S) is in fact an infinite number of statements, one for each value of n (in the test above we saw the first 5). Imagine each of these statements is the tile of a domino game, put one after the other. A statement being true corresponds to a tile falling. When we show the **inductive step**, we are saying that if any one tile falls, then it knocks down the next one. Establishing the base case amounts to showing that the first tile falls. These two together show that every tile is gonna fall, because the first knocks down the second, the second the third, and so on forever.

Given a statement of the form:

for every natural number n , $\text{blah}(n)$ happens.

Think of blah as a formula, or as a sentence containing a formula depending on n . A proof by induction goes as follows:

base case: showing that $\text{blah}(1)$ happens.

inductive step: assume that $\text{blah}(N)$ happens, and show that $\text{blah}(N + 1)$ happens.

1.2 Proof 2: by geometric translation

We now prove statement (S) by translating it to a geometric statement which we then show to be true.

Imagine the natural numbers as counting the number of beads that we have. If we have $1 + 3 + \dots + (2n - 1)$ beads, we are going to arrange them in the plane in such a way that it is apparent that the total number of beads we have is n^2 . We put the first bead at $(0, 0)$. The next three beads will be put in position $(1, 0), (1, 1), (0, 1)$. Notice that now the beads form a square pattern of side given by 2 beads, and therefore we have a total of $4 = 2^2$ beads. We can now add five beads in position $(2, 0), (2, 1), (2, 2), (1, 2), (0, 2)$ to complete

a square of side-length 3 beads, therefore containing a total of 3^2 beads. This process continues indefinitely: when we add the n -th odd number of beads by putting them in the positions (i, j) where either i or j (or both) are equal to $n-1$, we have that the beads are arranged to fill a square of side-length n beads, and therefore we have a total of n^2 beads. Look at the Figure 1 for a reference to this construction.

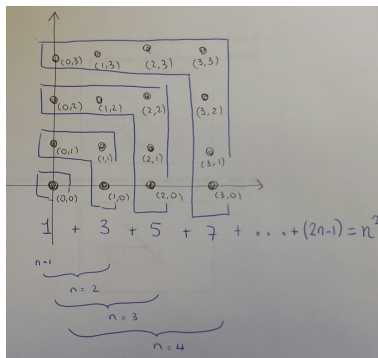


Figure 1: A figure that illustrates the argument of the geometric proof of statement (S). Remember: a figure is not a proof, but a figure helps understanding a proof by clarifying a geometric pattern which we are using in the proof of our argument.

2 Groupwork

Let us start by defining some symbols.

$H(n)$ is a number depending on an integer n . $H(n)$ is the number of handshakes that happen if, in a group of n persons, everyone shakes hands with everyone else exactly once.

$G(m)$ is a number depending on an integer m . $G(m)$ is the sum of the first m natural numbers. In symbols,

$$G(m) = \sum_{i=1}^m i.$$

Now we state some theorems.

Theorem 1 For every natural number n ,

$$H(n) = G(n-1).$$

Theorem 2 *For every natural number n ,*

$$H(n) = \frac{1}{2}n \cdot (n - 1).$$

Theorem 3 *For every natural number m ,*

$$G(m) = \frac{1}{2}m \cdot (m + 1).$$

First off, let us observe that Theorem 2 gives a formula for $H(n)$, Theorem 3 gives a formula for $G(m)$ and Theorem 1 expresses a relationship between H and G .

Question 1 *Do the three theorems have a chance of being all true? In other words, is the relationship given in Theorem 1 compatible with the formulas given in Theorems 2 and 3?*

Now let us observe that we can think of Theorem 1 as a translation between two problems: a combinatorial problem of counting handshakes is related to an algebraic problem of adding a sequence of natural numbers.

Problem 1 *Meditate on the following fact. If we establish that Theorem 1 is true, then proving one of the remaining Theorems implies that the last one is also true.*

Problem 1 makes precise what it means for Theorem 1 to be a translation between the two problems: a solution to one will immediately lead to the solution of the other.

As we said, before embarking on proving anything, we should test our statements to make sure we can't immediately show that they are false.

Problem 2 *Compute $H(n)$ for $n \leq 4$ and $G(m)$ for $m \leq 4$, and show that the statements of the three theorems are verified for these small values of n and m .*

Next, we are going to establish the truth of Theorem 1.

Problem 3 *Prove Theorem 1 by organizing the count of the handshakes $H(n)$ in a way that makes it clear that the total number of handshakes is $G(m - 1)$.*

Now, any proof of either of the remaining theorem will complete the work and show that all three theorems hold true. However, we are not going to content ourselves with that, but we will provide a few different proofs. For each of the proofs we present, ask yourselves what are the things you like and dislike about such proof (this is a subjective question, so there isn't a correct answer).

Problem 4 (Baby Gauss trick) *Prove Theorem 3 by using a trick attributed (probably an urban legend) to Gauss in kindergarden. Write the numbers 1 to m in a $m \times 2$ array. In the first row write them in increasing order, in the second row write them in decreasing order:*

$$\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & \dots & m-2 & m-1 & m \\ m & m-1 & m-2 & \dots & 3 & 2 & 1 \end{array}$$

What is the relationship between the sum of all the numbers in this array and $G(n)$? Find a way to compute the sum of all the numbers in this array that allows to give a formula for this sum.

We now give a rather complicated proof of Theorem 3. We will see however that this proof has the advantage of giving the possibility of generalizing the statement of Theorem 3. Start from the following identity, which is well known to hold true for any value of i :

$$(i+1)^2 = i^2 + 2i + 1$$

We manipulate it as follows:

$$-i^2 + (i+1)^2 = 2i + 1$$

$$-i^2 + (i+1)^2 - 1 = 2i$$

$$\frac{1}{2} (-i^2 + (i+1)^2 - 1) = i$$

This seems a bizarre thing to do, let us ponder for a second what we just did: we expressed an integer i as an expression that contains the difference of the squares of two consecutive numbers. This doesn't seem all that convenient at first, but let us remember that our task is to add integers. So let us add summation signs on either sides of the above identity:

$$\sum_{i=1}^m \frac{1}{2} (-i^2 + (i+1)^2 - 1) = \sum_{i=1}^m i$$

Since addition is commutative and distributive, we can rewrite this as:

$$\frac{1}{2} \left(\sum_{i=1}^m [-i^2 + (i+1)^2] - \sum_{i=1}^m 1 \right) = \sum_{i=1}^m i = G(m). \quad (3)$$

Problem 5 Consider the summation:

$$\sum_{i=1}^m [-i^2 + (i+1)^2]$$

Compute the value of this summation for small values of m . Guess a formula for a general value of m . Provide an argument that proves your formula.

This summation is what is called a **telescoping sum**: the second term of the first summand cancels with the first term of the second summand, the second term of the second summand cancels with the first term of the third summand, and so on; so, in the end, out of this whole big summation, the only surviving terms are the very first and the very last.

Problem 6 Use the formula from Problem 5 to compute (3), and show that this gives a proof of Theorem 3.

3 Sunday Homework

Exercise 1 Give a proof by induction of Theorem 3.

Exercise 2 Give a geometric proof of Theorem 3, again by thinking of the integers as counting the number of beads, and realizing a strategy of positioning the beads on the plane in such a way that the quantity $G(m)$ can be counted. **Hint:** look for arranging $1 + 2 + 3 + \dots + m$ beads inside a square $m \times m$ grid.

Exercise 3 Give a direct proof of Theorem 2 by relating the number of handshakes in a group of n people to the number of ordered pairs of distinct persons from that group.

Exercise 4 Use Theorem 3 to give another proof of the statement:

$$\sum_{i=1}^n (2i - 1) = n^2.$$

Exercise 5 Prove by induction that for $q \neq 1$

$$\sum_{i=0}^n q^i = \frac{q^{n+1} - 1}{q - 1}. \quad (4)$$

Exercise 6 Consider equation (4). Since $q \neq 1$, show that you obtain an equivalent equation by clearing denominators. Realize that now on the left hand side you have a telescoping sum and use that fact to give another proof of the validity of (4).

Exercise 7 Use an argument similar to the last proof in the groupwork to find a formula for the sum of the first n squares:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + (n-1)^2 + n^2 = ?$$

Exercise 8 (challenge - optional) Prove that the formula for the sum of the first n k -powers is a polynomial in n of degree $k + 1$.

$$\sum_{i=1}^n i^k = 1^k + 2^k + \dots + (n-1)^k + n^k = a_{k+1}n^{k+1} + \dots + a_0$$

Note: I am not asking you to find the formula for such a polynomial!!

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 2

Definitions and Quantifiers

1 Key Ideas

In real life, we seldom use definitions. For example, you never worried about what is the definition of a *table*. Over the course of your life you were shown a bunch of examples of tables and you abstracted the notion of *table*, so that now you recognize a *table* when you see one. In a sense, in elementary mathematics, we also do that. We don't ever really define what numbers **are**, we just become familiar with them by using them.

In more advanced mathematics, definitions are the starting point of everything. Given any mathematical statement, in order to stand a chance to prove it, you have to be familiar with the definitions of all concepts involved in the statement. Let us start with some simple examples.

Definition 1 *A natural number n is:*

even *if $n = 2 \cdot q$, where q is a natural number;*

prime *if it is different from 1, and its only divisors are 1 and itself;*

foofy *if it is even, prime, and greater than 7.*

Theorem 1 *If a natural number is divisible by 4, then it is even.*

PROOF: Let n be a natural number divisible by 4. We can write $n = 4 \cdot k$, where k is also a natural number.

Since $4 = (2 \cdot 2)$, we have $n = (2 \cdot 2) \cdot k$.

Multiplication is associative, therefore

$$n = 2 \cdot (2 \cdot k).$$

Since $2 \cdot k$ is a natural number, this shows that n satisfies the definition of being even.

□

Theorem 2 *There are no foofy numbers.*

PROOF: The only even prime number is 2, since, by definition, a prime number is divisible only 1 and itself. So 2 is the only candidate to be a foofy number. Since 2 is less than 7, 2 is not foofy. Therefore there are no foofy numbers.

□

Pay close attention to quantifiers in mathematical sentences. Notice the big difference between the following two statements:

1. **There exists** a prime divisor equal to 2 in the prime factorization of n .
2. **For every** prime divisor p in the prime factorization of n , $p = 2$.

The first statement is telling us that n is an **even** number, the second statement is telling us that $n = 2^k$.

Suppose that a number n walks up to you and you want to assess whether it satisfies statements 1. or 2.

For statement 1.

- If you want to establish 1. to be true, you must look at the prime factorization of n until you find the factor 2. If you find it, you can stop.
- If you want to establish 1. to be false, you must look at ALL the prime factorization of n , and check that all prime divisors are different from 2.

For statement 2.

- If you want to establish 2. to be true, you must look at ALL the prime factorization of n , and check that all prime divisors are equal to 2.
- If you want to establish 2. to be false, you must look at the prime factorization of n until you find a factor different from 2. If you find it, you can stop.

In other words, the negative of a statement with a “there exists” contains a “for every” and viceversa. In this specific case, let us write the negatives of these statements. Oh, and since we are talking about definitions, let us define the negative.

Definition 2 *The **negative** of a mathematical statement S is another mathematical statement not S with the property that S is true when not S is false and S is false when not S is true.*

With this definition in place, we have:

Not 1. **For every** prime divisor p in the prime factorization of n , $p \neq 2$

Not 2. **There exists** a prime divisor different from 2 in the prime factorization of n .

2 Groupwork

We are going to train our skills by making up some random definitions, and running with them.

Definition 3 *Given a set U of words, a **bas** is a subset B of U such that every element of B starts with the same letter.*

Definition 4 *Given a set U of words, a **bes** is a subset B of U such that there exists an element of B whose first and last letter are the same.*

The first thing to do is getting comfortable with definitions by giving some examples and counterexamples.

Problem 1 *Give an example of a bas, a bes, of something which is a bas and a bes, a bas but not a bes, a bes but not a bas, and neither a bas nor a bes.*

Then we start making some statements about our newly introduced concepts.

Problem 2 *For each of the following statements, decide whether they are true or false. If they are true, give a proof. If they are false, provide a counterexample showing it.*

1. *If U is a bas, then every subset of U is a bas.*
2. *If U is a bes, then every subset of U is a bes.*
3. *If there exists $B \subset U$ (read: a subset B of U) such that B is a bas, then U is a bas.*
4. *If there exists $B \subset U$ (read: a subset B of U) such that B is a bes, then U is a bes.*
5. *If B is a bas, then every subset of B is a bas.*
6. *If B is a bes, then every subset of B is a bes.*
7. *If B is a bas, then every set containing B is a bas.*
8. *If B is a bes, then every set containing B is a bes.*
9. *The empty set is a bas.*
10. *The empty set is a bes.*

We now introduce some attributes of bes and bas.

Definition 5 *A bas B is **closed** if B is the empty set or if there is no subset B' strictly containing B which is also a bas.*

Definition 6 *A bes B is **full** if every subset of B is also a bes.*

Problem 3 Give some examples of closed bas's, not closed ones, full bes's and not full ones. Notice that being a closed bas doesn't only depend on B , but it depends essentially on U . Give an example of a set B and two choices of U such that in one case B is a closed bas, in the other case it isn't.

Problem 4 Can you figure out equivalent statements for Definitions 5 and 6? Formulate these statements as Theorems of the form: "A bas is closed if and only if...", "A bes is full if and only if..."; then prove such statements.

Definition 7 Given a bas $B \subseteq U$, the **closure** of B , denoted $\overline{B}$, is the unique closed bas containing B .

Problem 5 Show that Definition 7 makes sense: for any bas B , there exists a unique closed bas containing B .

Problem 6 Prove that for any bas B , $\overline{\overline{B}} = \overline{B}$.

Definition 8 Given a bes B , the **core** of B , denoted B° , is the largest subset of B which is a full bes.

Problem 7 Show that Definition 8 makes sense: for any bes B , there exist a unique largest full bes contained in B .

Problem 8 Show that taking the core commutes with unions. I.e., for every pair of bes's B_1, B_2 ,

$$B_1^\circ \cup B_2^\circ = (B_1 \cup B_2)^\circ$$

3 Sunday Homework

Exercise 1 Refer back to Statements 1. and 2. at the end of the Key Ideas section. Decide, for every $n \leq 20$ the truth or falsehood of the statements in the leftmost column of the table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1.																				
Not 1.																				
2.																				
Not 2.																				

Observe that the negative of each statement has indeed always opposite answer than the statement.

Exercise 2 Here is another fun definition to play with.

Definition 9 A set B is **babadabubi** if all of its elements are italian words.

In mathemathese, this is written:

B is **babadabubi** if for every b in B , b is an italian word.

Answer the following questions about babadabubi sets:

1. Write down the definition of not being babadabubi.
2. Give an example of a babadabubi set.
3. Give an example of a set which is not babadabubi.
4. Is the empty set babadabubi?
5. TRUE OR FALSE: the union of two babadabubi sets is babadabubi. Prove your answer.
6. TRUE OR FALSE: the intersection of two not babadabubi sets is not babadabubi. Prove your answer.
7. TRUE OR FALSE: if a set contains an english word then it is not babadabubi. Prove your answer. (Pay attention: is belonging to one language an exclusive property?)
8. TRUE OR FALSE: if a set is not babadabubi then it contains an english word. Prove your answer.

Exercise 3 (contrapositive) We introduce the notion of the contrapositive of a mathematical statement.

Definition 10 Let S_1, S_2 be two sentences and consider a statement (S) of the form: “If S_1 , then S_2 ”. We define the **contrapositive** statement to (S) to be:

If not S_2 , then not S_1 .

Write down the contrapositive statement to each of the following:

1. If it rains, then I carry an umbrella.
2. If it doesn't rain, then I don't carry an umbrella
3. If a natural number n is a multiple of m , then $n \geq m$.
4. If $x = y$, then $x^2 = y^2$.

Exercise 4 The interesting thing is that the contrapositive of a statement is equivalent to the statement, in the sense that they are true or false together (in the sense that there cannot be any situation in which a statement is true and its contrapositive is false, or viceversa). Observe this by filling in the table below. Put T if a scenario supports a statement, F if a scenario shows that the statement is false, and X if the scenario doesn't allow you to decide. Let 1. and 2. refer to the first two statements in the previous exercise.

	1.	contrapositive to 1.	2.	contrapositive to 2.
It rains and I have my umbrella				
It rains and I don't have my umbrella				
It doesn't rain and I have my umbrella				
It doesn't rain and I don't have my umbrella				
It doesn't rain and I tip-tap dance				
I tip-tap dance with an umbrella				

Exercise 5 Sometimes in order to prove a mathematical statement to be true, it is easier to prove the contrapositive statement. This proof technique is called **proof by contradiction**. Fill in the blanks to complete the proof.

Theorem 3 There are infinitely many prime numbers.

Comment: This is another case where all the hypothesis are hidden: they consist of knowing the definition of prime numbers, and the usual axioms of arithmetics. So you should think of this theorem as the statement: “If the hidden hypothesis hold, then there are infinitely many prime numbers”. Therefore the contrapositive statement is: “If there are finitely many prime numbers, then something goes wrong in arithmetics”. What goes wrong is the following: if we assume there are finitely many prime numbers, and we take them all, then we can always produce one more prime number. Now let us write down this idea formally.

PROOF: Assume there are _____ prime numbers, in particular we can assume there are n of them and we can write them $p_1, \dots, p_n$. Consider the number $q = p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$, obtained by multiplying all the prime numbers together, and then adding 1. We claim that q is not divisible by _____ of the p_i 's. We prove this claim by contradiction.

Assume that q _____ by one of the p_i 's, without loss of generality let us assume q _____ by p_1 .

This means $q = p_1 \cdot k$, with k a natural number.

Then

$$q - p_1 \cdot p_2 \cdot \dots \cdot p_n = \text{_____},$$

which shows that $q - p_1 \cdot p_2 \cdot \dots \cdot p_n$ is divisible by _____. But $q - p_1 \cdot p_2 \cdot \dots \cdot p_n = \text{_____}$, which is not divisible by any prime. Assuming that q is divisible by one of the p_i 's led to a contradiction, which allows us to conclude the claim: q is not divisible by _____ of the p_i 's. But then there are two possibilities: either q is a new prime number, or q has a prime factorization with primes that are different from the p_i 's. In either case, we are contradicting the statement that the list $p_1, \dots, p_n$ contains _____ prime numbers. Assuming that there are finitely many primes led us to a contradiction, which shows that it must be the case that there are infinitely many prime numbers.

□

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 3

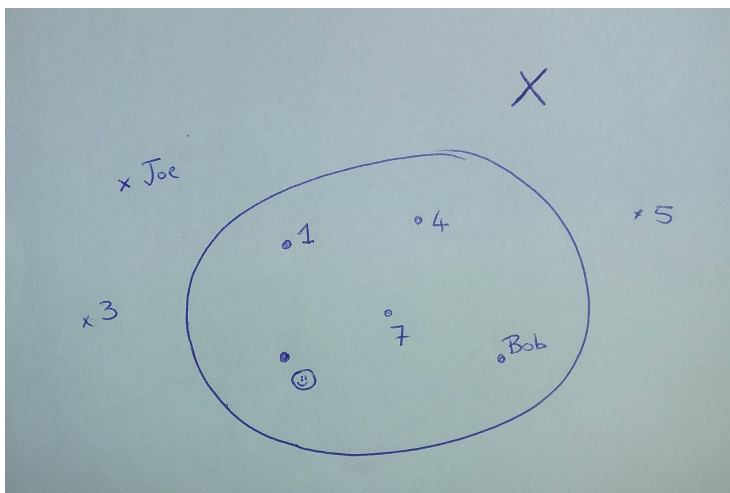
Sets

1 Key Ideas

As I mentioned on our first day of class, we are not going to define what a set is, since doing it honestly would require way too much work. We all have this intuitive notion of a set being a collection of “things”, called the elements of the set. Such intuitive notion will suffice for us to be able to work with sets, and set operations.

Here are some examples of sets:

1. The set of students of this class.
2. The set $A = \{1, 3, d, f, banana\}$.
3. The set of integers $\mathbb{Z}$.
4. The set X :



Observations:

elements: elements of a set can be anything you want. Set A containing banana as an element is just as mathematical as the set of integers.

how to describe sets: there are many ways that we can describe sets:

1. by giving properties that identify the elements of the set (Example 1).

2. by listing the elements of the set (Example 2). The order does not matter!
3. certain common mathematical sets have their own name (Example 3).
4. by drawing an oval, and putting the elements of the set inside the oval: this is called the Venn diagram representation of a set (Example 4).

cardinality of a set: the number of elements of a set is called the cardinality of the set. If a set does not contain a finite number of elements, we say its cardinality is infinite.

Definition 1. A set A is a **subset** of a set B if every element of A belongs to B . We write

$$A \subseteq B$$

to say that A is a subset of B .

Question 1. Is a set A a subset of itself?

Definition 2. A set A is a **proper subset** of a set B if $A \subseteq B$ and $A \neq B$. We write

$$A \subset B$$

to say that A is a proper subset of B .

Definition 3. Given two sets A, B we define:

Intersection. The intersection $A \cap B$ is the set of all common elements of A and B . In other words, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Union. The union $A \cup B$ is the set of all elements that belong to either A or B . In “mathematese”, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Observe that the set $A \cap B$ is a subset of A and it is a subset of B . On the other hand A is a subset of $A \cup B$, and B is a subset of $A \cup B$.

Here is an example of the kind of simple mathematical statements you can make about sets.

Theorem 1. *For any three sets A, B, C ,*

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

The basic trick here is that in order to prove an equality of sets $X = Y$, one typically proves two inclusions: that X is a subset of Y and Y is a subset of X . I also recommend, when proving statements like this, to draw Venn diagrams to see what is going on.

PROOF: *First we show that $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.* Consider an element $x \in (A \cup B) \cap C$; we want to show that $x \in (A \cap C) \cup (B \cap C)$. We know that $x \in C$ and $x \in A \cup B$. There are two possibilities:

1. $x \in C$ and $x \in A$, or
2. $x \in C$ and $x \in B$.

In the first case $x \in (A \cap C)$, and therefore $x \in (A \cap C) \cup (B \cap C)$. In the second case $x \in (B \cap C)$, and therefore $x \in (A \cap C) \cup (B \cap C)$. It follows that $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

Now we must show that $(A \cup B) \cap C \supseteq (A \cap C) \cup (B \cap C)$. Now consider an element $x \in (A \cap C) \cup (B \cap C)$. There are two possibilities:

1. $x \in A$ and $x \in C$, or
2. $x \in B$ and $x \in C$.

In both cases, $x \in C$, so we conclude that $x \in C$. Since in the first case $x \in A$ and in the second case $x \in B$, we conclude that $x \in A \cup B$. Therefore $x \in (A \cup B) \cap C$. We have shown that $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$.

Since $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$, it must be that $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$. This concludes the proof.

□

Let us conclude this part with one last definition.

Definition 4. The **universe** $\mathcal{U}$ is the set of all things that we wish to consider. Note that the universe set depends on what we are trying to do with it...in mathematics, the universe might be the set of all numbers that we want to work with. In probability, it may be the set of all events that we wish to consider. In any case, we just want to think of the universe as containing “everything”.

I like to represent the universe in Venn diagrams as a large rectangle. Then I will fit any set that I want to consider inside such rectangle.

2 Groupwork

Problem 1. Let us begin by introducing the empty set.

Definition 5. A set that contains no elements is called the **empty set** and denoted ϕ .

1. Show that the empty set is a subset of any other set.
2. For any set A , what is $A \cup \phi$? What is $A \cap \phi$?

Definition 6. Let us introduce some other operations on sets.

Complement: Given a set X , we define the **complement** of X , denoted X^c , by the following property: an element y belongs to X^c if and only if y does not belong to X .

Difference: Given two sets X, Y , the **difference** $X \setminus Y$ consists of all elements of X which do not belong to Y .

Symmetric difference: Given two sets X, Y , the **symmetric difference** $X \Delta Y$ is defined to be:

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X).$$

Problem 2. Draw Venn diagrams that illustrate the above three definitions.

Problem 3. Convince yourselves that for any set X ,

$$X^c = \mathcal{U} \setminus X.$$

Problem 4. When is $X \setminus Y = Y \setminus X$?

Problem 5. Prove, with the same level of detail as the proof of Theorem 1, that for every two sets A, B ,

$$(A \cap B)^c = A^c \cup B^c.$$

Problem 6. Decide, by drawing appropriate Venn diagrams, whether the operations of difference and symmetric difference are associative. More precisely, this means to check whether, for sets A, B, C :

1.

$$(A \setminus B) \setminus C = A \setminus (B \setminus C)?$$

2.

$$(A \Delta B) \Delta C = A \Delta (B \Delta C)?$$

3 Sunday Homework

Exercise 1. Prove, with the same level of detail as the proof of Theorem 1, that given two sets A, B ,

$$A \subseteq B \iff B^c \subseteq A^c.$$

(read: A is a subset of B if and only if B is a subset of A).

Exercise 2. Prove, with the same level of detail as the proof of Theorem 1, that for every two sets A, B ,

$$(A \cup B)^c = A^c \cap B^c.$$

Exercise 3. For any set A , prove (with the same level of detail as the proof of Theorem 1) that

$$(A^c)^c = A.$$

Exercise 4. Draw Venn diagrams representing each of the sets below:

1.

$$(A \cup B) \Delta C.$$

2.

$$A \cup (B \Delta C).$$

3.

$$(A \Delta B \Delta C)^c \setminus (A \cup B \cup C).$$

4.

$$((A \cap B) \Delta (B \cap C)) \Delta (A \cap C).$$

5.

$$((A \cap B) \cup (B \cap C) \cup (A \cap C)) \setminus (A \cap B \cap C).$$

Definition 7 (Cartesian product). Given two sets X, Y the **cartesian product** $X \times Y$ is the set of ordered pairs

$$X \times Y := \{(x, y) | x \in X, y \in Y\}.$$

Exercise 5. Let us get familiar with the cartesian product.

- If $X = \{1, 2\}$ and $Y = \{a, b, c\}$, write down the elements of $X \times Y$ and $Y \times X$. Notice that $X \times Y \neq Y \times X$.
- If X is a set with n elements, and Y a set with m elements, what is the cardinality of $X \times Y$?
- What is $X \times \phi$?

Exercise 6. More questions on the cartesian product:

- If $A \subseteq X$ and $B \subseteq Y$, prove that $A \times B \subseteq X \times Y$.
- True or false:

$$(X \times Y) \setminus (A \times B) = (X \setminus A) \times (Y \setminus B)?$$

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 3

Sets and Logics

1 Key Ideas

This week we are going to explore an interesting dictionary between sets and the logics we introduced to study mathematical statements. I find this dictionary useful because it allows to visualize very concretely logical statements that may feel confusing (for example, that the contrapositive of an implication is equivalent to the implication).

The main idea is the following: let the universe set $\mathcal{U}$ denote all things or events that you want to consider. To any statement, we associate the set of “things” for which the statement is true. According to our main idea, since the set $\mathcal{U}$ contains all elements that we want to consider, then $\mathcal{U}$ corresponds to the notion of a statement being true. We will call $\mathcal{U}$ also the *TRUE* set.

Question 1. What set do you think corresponds to the notion of a statement being *FALSE*?

Example 1. Let $\mathcal{U}$ be the set whose elements are all animals. Consider the mathematical statements:

S_1 : x is a cat;

S_2 : x is a black animal.

Then to S_1 corresponds the set $A_1 \subseteq \mathcal{U}$ whose elements are cats, and to S_2 correspondes the set $A_2 \subseteq \mathcal{U}$ whose elements are black animals.

Now consider the statement:

S_1 **AND** S_2 : x is a cat **AND** a black animal.

It corresponds to the set $A_1 \cap A_2$. We have discovered that the conjunction “AND” in logics corresponds to the set operation of intersection: $\cap$.

Things get more interesting once we start studying implications. Let us look at another example.

Example 2. Let $\mathcal{U}$ be the set whose elements are all animals. Consider the mathematical statements:

S_1 : x is a cat;

S_2 : x has a tail.

Now we make the statement:

$S_1 \implies S_2$ (read S_1 implies S_2 , or “If S_1 , then S_2 ”): IF x is a cat, THEN x has a tail.

For the statement $S_1 \implies S_2$ to be *TRUE*, it has to be the case that every element of $\mathcal{U}$ verifies the statement. But this is equivalent to the fact that any element of the set A_1 (i.e. every cat) must belong to the set A_2 (i.e. it must have a tail). Which means that A_1 is a subset of A_2 ($A_1 \subseteq A_2$). We have therefore learned that the notion of implication ($\implies$, or IF...THEN) corresponds to the notion of inclusion $\subseteq$.

2 Groupwork

The main goal of this groupwork is to continue building this dictionary, and get comfortable with how to use it.

Problem 1. If a statement S corresponds to a set A , what does the statement *not* S correspond to? (Hint: look at an example).

Problem 2. Use the information from Example 2 and Problem 1 to show that the contrapositive of an implication is equivalent to an implication. This means, consider a statement of the form “if S_1 , then S_2 ” and translate it into a statements about sets. Then consider the contrapositive “if not S_2 , then not S_1 ” and also translate this into a statement about sets. Then observe that the two statements about sets are equivalent, in the sense that one is true if and only if the other is.

Problem 3. For each of the statements below indicate what the sets involved are and what is the translation of the statement to the realm of sets. For example, if the statement is:

IF it rains, THEN I take my umbrella.

Then the two relevant sets are A_1 = the set of times when it rains, A_2 = the set of times I have my umbrella. And the statement translates to $A_1 \subseteq A_2$.

1. x is a dog OR a black animal.
2. I EITHER eat a slice of pie, OR a scoop of ice cream.
3. I take my umbrella ONLY IF it rains.
4. I take my umbrella IF AND ONLY IF it rains.
5. IF you are 18 years of age AND you are a citizen, THEN you can vote.
6. IF you are Italian OR French, THEN you are European.

Problem 4. Now let us formalize what we have observed up until now into a dictionary between sets and logic.

Logic	Sets
TRUE	
FALSE	
NOT	
AND	
OR	
EITHER...OR (exclusive or)	
IF...THEN	
...ONLY IF...	
IF AND ONLY IF	

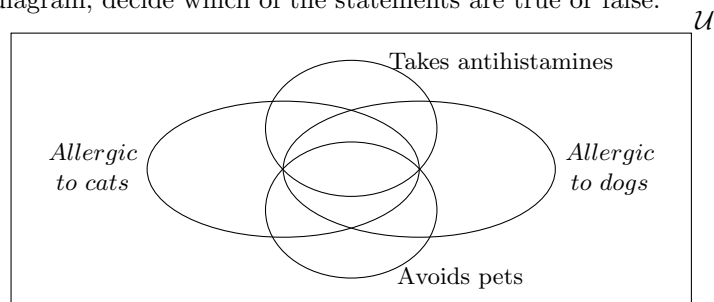
Problem 5. Translate the following statements to set language:

1. There exists a dog that likes broccoli;
2. Every dog likes bones.

Describe the translation of the quantifiers “there exists” and ”for every” in terms of sets.

3 Sunday Homework

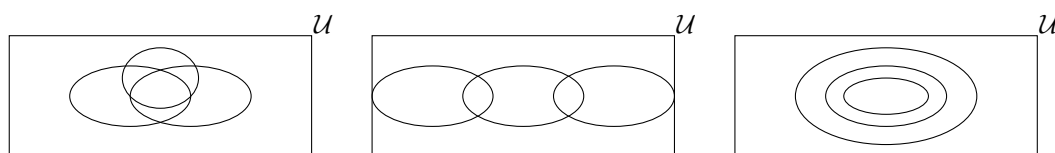
Exercise 1. In the following diagram, we have drawn four sets of people. Based on the diagram, decide which of the statements are true or false.



1. If someone is allergic to cats and dogs, then they take antihistamines or they avoid pets;
2. If someone is allergic to cats or dogs, then they take antihistamines or they avoid pets;
3. If someone takes antihistamines and is not allergic to dogs, then they are allergic to cats;
4. If someone takes antihistamines and they avoid pets, then they are allergic to cats and dogs;
5. If someone is not allergic to cats or dogs, then they don't avoid pets.

Exercise 2. Match the statements below with the set diagrams that make them true. Note: in each of the diagrams, the three bubbles represent the set of people that like Adele, Lady Gaga, Beyonce.

1. If one likes Lady Gaga, then they like Adele; if one likes Adele, then they like Beyonce.
2. Nobody likes Lady Gaga and Adele and Beyonce.
3. There are people that like Lady Gaga and Adele and Beyonce.



Exercise 3. Explain, by translating to the language of sets, the following fact: for a statement S : “If A , then B ”, when A is false then the statement S is true.

Exercise 4. Use Problem 5 to show that a statement of the form “For every element of A , then blah blah blah happens” is verified when A is equal to the empty set.

Exercise 5. In the groupwork we explored the following fact: *the contrapositive of an implication $S_1 \implies S_2$ is equivalent to the original implication* is translated to the statement about sets (that we saw last week in Exercise 1): $A \subseteq B \iff B^c \subseteq A^c$.

Use the dictionary between sets and logic to help you write the contrapositive to the following statements:

1. If it walks like a duck and it quacks like a duck, then it is a duck.
2. If you don’t behave, then you won’t get any ice cream or you won’t watch TV.
3. If you like yoga and you like goats, then you like goat-yoga and you are a funny person.

Exercise 6 (Proof by contradiction). Let us talk again about proof by contradiction. Proving that a theorem is TRUE, means to show that every event in the universe (or true) set $\mathcal{U}$ verifies the statement of the theorem. In set language, this means:

$$\mathcal{U} \subseteq \{\text{Events that verify the statement of the theorem}\}.$$

Applying complement to this inclusion (as seen in the previous exercise), we see that it is equivalent to:

$$\{\text{Events that falsify the statement of the theorem}\} \subseteq \phi.$$

So if we re-translate this back from sets to logics, we have the following: if we assume that something falsifies the statement of the theorem, then some false statement follows.

Let us now try our hands at a specific example.

Theorem 1. *The number $\sqrt{2}$ is irrational.*

Before you start proving the theorem, do you know what irrational means? If not, look at the following definitions.

Definition 1. .

- A number $x \in \mathbb{R}$ is **rational** if it can be written as quotient of two integers.
- A number $x \in \mathbb{R}$ is **irrational** if it is not rational.

We will be using the following fact.

Fact. Every integer has a unique prime factorization.

OK, here comes the exercise. Prove the theorem through the following steps:

1. Assume the statement of the theorem false (write an equation that shows that);
2. Square both terms of the equation and clear denominators to obtain an equality of integers;
3. Use the “Fact” and look at the number of prime factors equal to 2 on both sides of the equation to deduce that something is wrong.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 5

Combinatorics and Newton's theorem

1 Key Ideas

This week we are going to explore Newton's binomial expansion theorem. This is a very useful tool in analysis, but it also offers us an opportunity to explore an interesting connection with the combinatorics of finite sets. Newton's theorem is a formula for the expansion of an expression of the form $(x+y)^n$, for any natural number n . Once one has the formula, then a proof by induction can show that the formula is true. However, the connection to combinatorics actually explains why the formula holds. So, as usual, we will "overproof" our statement.

Let us start with some definitions.

Definition 1. For any natural number n , we define $n!$ (read " n factorial") to be the product of all natural numbers from 1 to n (including n).

So for example $1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, \dots$

There are good reasons to define $0! = 1$, even though we will not get into them right now.

Definition 2. For any two natural numbers n, k , with $k < n$, we define $\binom{n}{k}$ (read " n choose k ") to be:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

So for example $\binom{4}{2} = 6, \binom{4}{1} = \binom{4}{3} = 4$.

Since we defined $0! = 1$, we can also consistently define $\binom{n}{0} = \binom{n}{n} = 1$.

With these two definitions in place, we are ready to state Newton's binomial expansion theorem.

Theorem 1. For any natural number n , we have:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Let us see this theorem in action. If you want to expand $(x+y)^6$, you can immediately write:

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

because, in this case, we have: $\binom{6}{0} = \binom{6}{6} = 1, \binom{6}{1} = \binom{6}{5} = 6, \binom{6}{2} = \binom{6}{4} = 15, \binom{6}{3} = 20$.

Let us take a side-step for a second, and observe that the number $n!$ "counts" the number of ways you can order the elements of a set with n elements.

Theorem 2. Let $X = \{x_1, \dots, x_n\}$ be a set with n elements. There are $n!$ distinct ways of ordering the elements of X .

Proof. We have n choices for what element to put in the first position. Once we have chosen an element for first position, we have $n - 1$ choices for what element to put in second position. Once we have chosen the second element, we have $n - 2$ choices for what element to put in third position. This pattern continues until we have only one element left, which must be put in the last position. All together we have $n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$ choices, which is precisely the definition of $n!$. $\square$

2 Groupwork

Let us start by giving the number $\binom{n}{k}$ a combinatorial meaning.

Problem 1. If X is a set with n elements, show that $\binom{n}{k}$ is the number of subsets of X which contain k elements.

Question 1. After solving Problem 1, answer the following:

1. Why is $\binom{n}{k} = \binom{n}{n-k}$?
2. Are the definitions $\binom{n}{0} = \binom{n}{n} = 1$ consistent with Problem 1?

Problem 2. Show that the following identity holds:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Show it in two different ways:

1. Use the algebraic definition 2 and do some algebra.
2. Try to subdivide the set of *subsets of X with k elements* into two subsets, one of cardinality $\binom{n-1}{k}$, and the other of cardinality $\binom{n-1}{k-1}$.

Now let us tackle the proof of Newton's binomial theorem. To warm up, let us do the following exercise.

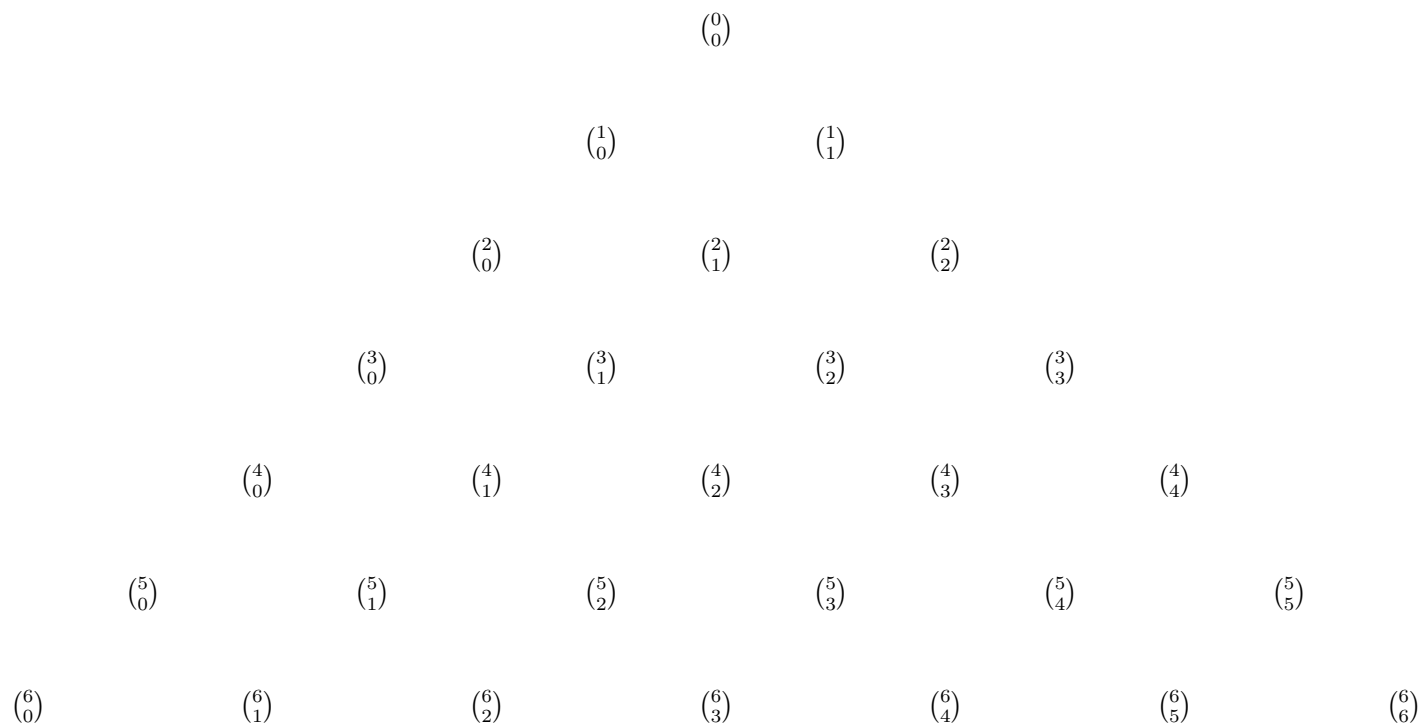
Problem 3. Write down $(x + y)(x + y)(x + y)$ in three different colors, one for each binomial. Now expand the above expression keeping track of the colors. For example, instead of writing x^3 , do write $x\textcolor{blue}{x}\textcolor{red}{x}$. Subdivide the terms into four groups, corresponding to the four monomials x^3, x^2y, xy^2, y^3 that you obtain when you forget the colors. Can you relate the number of elements in each group with a number that has to do with the set of colors $X = \{\textit{black}, \textit{blue}, \textit{red}\}$?

Here comes the main problem in this worksheet, so make sure you spend a fair amount of time on this and understand it very well.

Problem 4. Generalize the idea from the previous problem to give a proof of Theorem 1.

Let us conclude this groupwork with some fun exploration.

Problem 5 (Pascal's triangle). Pascal's triangle is an infinite triangle of number that contains all numbers of the form $\binom{n}{k}$, as shown in the following figure:



Plugging in the numbers, you get

				1					
			1		1				
		1		2		1			
	1		3		3		1		
1		4		6		4		1	
	1		5		10		10		5
		1		6		15		20	
			1		10		20		15
				1		6		15	
					1		6		1

Spot as many interesting numerical patterns as you can in Pascal's triangle. Then, try to explain them!

3 Sunday Homework

Exercise 1. Give a proof by induction of Newton's Theorem (Theorem 1).

Exercise 2. Use Newton's theorem to show the following facts:

- The sum of the numbers in each row of Pascal's triangle is a power of 2.
- The alternating sum of the numbers in each row of Pascal's triangle is equal to 0.

Exercise 3. Given a natural number n , and three numbers k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = n$, what is the number of ways you can subdivide X into three disjoint subsets named U_1, U_2, U_3 , where the cardinality of U_1 is k_1 , the cardinality of U_2 is k_2 and the cardinality of U_3 is k_3 ? Prove your formula.

Exercise 4. Develop a formula for the expansion of a trinomial $(x + y + z)^n$. Prove that the formula holds in a similar way to what you did in Problem 4.

Exercise 5. Observe that the first five rows of Pascal's triangle correspond to powers of 11: $11^0 = 1, 11^1 = 11, 11^2 = 121, 11^3 = 1331, 11^4 = 14641$. Can you explain why? Why doesn't the pattern continue after the fifth row?

Exercise 6. A “hockeystick” in Pascal’s triangle is obtained by starting on the side of the triangle (at one of the 1’s), moving diagonally down for as long as you want, and then making one ninety degree downward turn and stopping. We call all the numbers that you traverse in the first direction the “rod” of the hockeystick, and the last number the “tip”. For example, if we start at $\binom{3}{0} = 1$, we could do $\{1, 4, 10, 15\}$. In this case the rod is $\{1, 4, 10\}$ and the tip is 15. Show that for any hockey stick the sum of the numbers in the rod equals the tip.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 6

Bijections and Infinities

1 Key Ideas

This week we are going to focus on the notion of **bijection**. This is a way we have to compare two different sets, and say that they have “the same number of elements”. Of course, when sets are finite everything goes smoothly, but when they are not... things get spiced up! But let us start, as usual, with some definitions.

Definition 1. A **bijection** between a set X and a set Y consists of matching elements of X with elements of Y in such a way that every element of X has a unique match, and every element of Y has a unique match.

Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. An example of a bijection between X and Y consists of matching 1 with a , 2 with b and 3 with c .

A NOT example would match 1 and 2 with a and 3 with c . In this case, two things fail: a has two partners, and b has no partner.

Note: if you are familiar with these concepts, you may have recognized a bijection as a function between X and Y which is both injective (1:1) and surjective (onto). We will revisit the notion of bijection in this language in a few weeks. For now, we content ourselves of a slightly lower level approach. However, you may be a little unhappy with the use of the word “matching” in a mathematical definition. Here is a more formal definition for the same concept.

Definition 2. A **bijection** between a set X and a set Y is a subset $B \subseteq X \times Y$ such that every element of X is the first term of exactly one element of B and every element of Y is the second term of exactly one element of B .

Bijections are intimately related with the concept of cardinality of finite sets.

Theorem 1. *Assume X and Y are two finite sets. There exists a bijection between X and Y if and only if X and Y have the same cardinality (i.e. the same number of elements).*

We therefore make the following definition, regardless of whether the sets are finite or not.

Definition 3. Two sets X and Y have the same cardinality if there exists a bijection between them.

The “first” infinite set that most people encounter is the set of natural numbers. In a sense that I don’t want to make too precise at this point, the natural numbers are the “smallest” type of infinity there is. We make the following definition.

Definition 4. A set X is **countable** if it has the same cardinality as $\mathbb{N}$. An infinite set which is not countable is called **uncountable**.

The reason that historically the word countable was used is that giving a bijection between $\mathbb{N}$ and X is the same as choosing an element of X to be number 1, another to be number 2 and so on... in other words it is like “counting” the elements of X .

Theorem 2. Let X and Y be two countable sets. Then $X \cup Y$ is also countable.

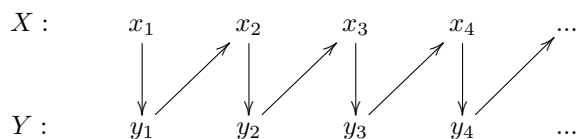
Proof. Assume that we fix a bijection between X and $\mathbb{N}$, and let us call x_n the element of X which is matched with the natural number $n \in \mathbb{N}$. Similarly we fix a bijection between Y and $\mathbb{N}$, and call y_n the element of Y which is matched with the natural number $n \in \mathbb{N}$.

We now define a bijection between $X \cup Y$ and $\mathbb{N}$ as follows: match each x_n with the natural number $2n - 1$, and each y_n with the natural number $2n$.

Via this bijection we can arrange the elements of $X \cup Y$ in the following ordered list:

$$X \cup Y = \{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}.$$

Another way to visualize this is by noting that the the bijection we described between $X \cup Y$ and $\mathbb{N}$ is obtained by following the arrows in the diagram below.



□

2 Groupwork

Problem 1. Make sure that you are comfortable with how Definitions 1 and 2 are equivalent.

Problem 2. True or false: given any non-empty set X , there exists a bijection between X and itself. Prove your answer.

Problem 3. True or false: If X is a finite set and $A \subset X$ is a proper subset of X , then there can be a bijection between A and X .

Problem 4 (Hilbert’s hotel). Hilbert’s hotel has a countable number of rooms, indexed by the natural numbers. The hotel is currently full.

1. A traveler shows up at the reception. Can the receptionist accomodate the traveler, without asking anyone to shack up with somebody else, and without leaving any of the current clients without a room?

2. An infinite bus with a countable number of passengers shows up. Can the receptionist accomodate the passengers of the bus, without asking anyone to shack up with somebody else, and without leaving any of the current clients without a room?

Problem 5. True or false: If X is a set and $A \subset X$ is a proper subset of X , then there can be a bijection between A and X .

Problem 6. Show that a non-empty subset of a countable set is either finite or countable.

Problem 7. Show that if X and Y are countable, then the cartesian product $X \times Y$ is countable.

Hint: Assume that you have given bijections for X and Y , and organize the elements of the cartesian product $X \times Y$ in a square, infinite, array. Then try and find a pattern for “counting” the elements of $X \times Y$, similar to what was done in the proof of Theorem 2.

Now we prove a fun fact: even though there seem to be way more rational numbers than natural numbers, in fact $\mathbb{Q}$ is a countable set.

Problem 8. The set of rational numbers $\mathbb{Q}$ is countable. Use Problems 6 and 7, and Theorem 2 to prove this fact.

At this point we should have at least one example of a non-countable set. Here we go.

Problem 9. The set of infinite sequences of 0’s and 1’s is not countable.

Hint: We prove this fact by contradiction. Assume that the set of such sequences is countable, and imagine you can write all infinite sequences in a list. Now you need to devise a procedure to construct a sequence of 0’s and 1’s that cannot possibly be in that list.

Now try to adapt the same argument to prove the following statement.

Problem 10. The set of real numbers $\mathbb{R}$ is not countable.

3 Sunday Homework

Exercise 1. Prove Theorem 1.

Exercise 2. Consider a finite set X with n elements. What is the number of bijections between X and itself? Why?

Exercise 3. Explain how Problems 4 and 5 are related.

Exercise 4. Generalize Theorem 2 as follows: for $n \in \mathbb{N}$, let X_n be a countable set. Show that

$$\bigcup_{n \in \mathbb{N}} X_n = X_1 \cup X_2 \cup X_3 \cup \dots$$

is a countable set.

Exercise 5. Given a set X , the **power set** of X , denoted $\mathcal{P}(X)$, is the set of all subsets of X . Prove that if X has n elements, then $\mathcal{P}(X)$ has 2^n elements, in two different ways:

1. Do a proof by induction.
2. Establish a bijection between $\mathcal{P}(X)$ and the set of ordered lists of 0's and 1's of length n .

Exercise 6. Prove that if X is countable, then $\mathcal{P}(X)$ is uncountable.

Exercise 7. Let us get comfortable with bijections.

- Establish a bijection between the set $(-1, 1)$ and all of $\mathbb{R}$.
- Establish a bijection between $\mathbb{R} \cup \{\star\}$ and $\mathbb{R}$.
- Establish a bijection between the set $[-1, 1]$ and $\mathbb{R}$.

Exercise 8. Let X be an uncountable set, and $Y \subset X$ be a countable subset of X . Prove that $X \setminus Y$ is uncountable.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 7

Functions

1 Key Ideas

We now investigate the concept of function, which is central to mathematics. Functions is how you make different mathematical objects “communicate”. In order to have a function, you need the following ingredients:

- a set X called the **domain** (or **input set**, or **starting set**), and
- a set Y called **codomain** (or **output set**, or **arrival set**).

We then talk about a function **from** X **to** Y .

Definition 1. A function $f : X \rightarrow Y$ is a *prescription* that assigns, to each element x of X , one element of Y that we call the **image** of x and denote $f(x)$.

A few important observations:

1. You can think of f as some machine with an input hole and an output hole. You can put elements of x into the input hole, and the machine will spit out elements of y . You can put in any elements of X , and the machine will always work to give you elements of Y . If you put the same element of X in the machine a bunch of times, the machine will always produce the same output.
2. It is possible that the machine will produce the same output for different inputs.
3. It is possible that not all elements of Y actually appear as outputs for some inputs.

Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. A couple examples of functions from X to Y :

- $f_1 : X \rightarrow Y$ defined by $f_1(1) = a, f_1(2) = b, f_1(3) = c$;
- $f_2 : X \rightarrow Y$ defined by $f_2(1) = c, f_2(2) = c, f_2(3) = b$.

And a couple examples of things that are NOT functions from X to Y :

- $f : X \rightarrow Y$ defined by $f(1) = a, f(2) = b$;
- $f : X \rightarrow Y$ defined by $f(1) = c, f(1) = b, f(2) = c, f(3) = b$.

As we did for bijections, we can give a slightly more formal definition of function, as a subset of the cartesian product $X \times Y$.

Definition 2. A **function** $f : X \rightarrow Y$ is equivalent information to a subset $F \subseteq X \times Y$ with the property that for any element $x \in X$, there is a unique element of F whose first term is x . We also call F the **graph** of f .

For every element of F , the first term is the input, and the second term the corresponding output.

Example 2. For the two functions f_1 and f_2 in Example 3, we have:

- $F_1 = \{(1, a), (2, b), (3, c)\};$
- $F_2 = \{(1, c), (2, c), (3, b)\}.$

Definition 3. Given any set X , there is always a special function from X to X , called the **identity function**, which we will denote Id_X . This function is defined by sending each element $x \in X$ to itself: $Id_X(x) = x$.

One important features of functions is that they can be composed. In the analogy of the machines, suppose that you have a machine f whose input hole accepts elements of X and produces outputs in a set Y , and then another machine g that accepts inputs belonging to Y and produces outputs in some set Z . Then you can connect the output hole of the first machine with the input hole of the second, to create a new machine that “eats” elements of X and produces elements of Z .

Definition 4. Given $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we define their **composition** $g \circ f : X \rightarrow Z$ to be the function

$$g \circ f(x) = g(f(x)).$$

Example 3. Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$, $Z = \{\star, \bullet\}$. Let

- $f : X \rightarrow Y$ defined by $f(1) = a, f(2) = b, f(3) = c$;
- $g : Y \rightarrow Z$ defined by $g(a) = \star, g(b) = \star, g(c) = \bullet, g(d) = \star$.

Then the function $g \circ f : X \rightarrow Z$ is defined to be:

$$g \circ f(1) = \star, g \circ f(2) = \star, g \circ f(3) = \bullet.$$

Some “machines” are special in the sense that you can run them also in the opposite direction: feed in an element $y \in Y$, turn the crank backwards, and produce the element of X that y was the output of. Such special machines are called **invertible functions**.

Definition 5. A function $f : X \rightarrow Y$ is called **invertible** if there exists a function $g : Y \rightarrow X$ such that:

$$g \circ f = Id_X, \quad f \circ g = Id_Y.$$

The function g is called the **inverse** of f and denoted f^{-1} .

Example 4. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. If $f : X \rightarrow Y$ is given by

$$f(1) = b, \quad f(2) = a, \quad f(3) = c,$$

then $g : Y \rightarrow X$ defined by

$$g(a) = 2, \quad g(b) = 1, \quad g(c) = 3,$$

is the inverse of f .

2 Groupwork

Problem 1. Check that in Example 4 the function g is indeed the inverse of f . Show that also f is the inverse of g . In fact, convince yourself that according to Definition 5, you can always talk about two functions being inverses of each other.

Problem 2. What happens when you compose a function with the identity function? More precisely, let $f : X \rightarrow Y$ be a function. What are the functions $f \circ Id_X$ and $Id_Y \circ f$?

Problem 3. Prove that if a function $f : X \rightarrow Y$ is invertible, then its inverse function is unique.

We now define the concept of inverse image. It is often the case that inverse image and inverse functions are confused. Partly, because the standard notation for the two concepts is the same. So please pay close attention!

Definition 6. Given a function $f : X \rightarrow Y$ and a subset $U \subseteq Y$, the **inverse image** of U , denoted $f^{-1}(U)$, is a subset of X that consists of **all** elements of X whose output via f belongs to U . In symbols:

$$f^{-1}(U) = \{x \in X \text{ such that } f(x) \in U\}.$$

Here are the things to pay attention to: first of all, in order to talk about the inverse image, you must also specify a subset of Y . Secondly, the inverse image is a subset of X : it may be empty, it may be a single element, or it may contain multiple elements. The reason that it is acceptable to use the same notation for inverse image and inverse function, is that, when f is an invertible function AND $U = \{y\}$ is a singleton, then the two concepts coincide, in the sense that the inverse image of the one element subset U is the same as the output of the inverse function for the element y :

$$f^{-1}(\{y\}) = \{f^{-1}(y)\}.$$

Problem 4. Let $X = 1, 2, 3, 4, 5$, $Y = \{a, b, c\}$ and $f : X \rightarrow Y$ be defined by:

$$f(1) = b, \quad f(2) = b, \quad f(3) = c, \quad f(4) = c, \quad f(5) = b.$$

For every subset $U \in \mathcal{P}(Y)$, write down $f^{-1}(U)$.

Problem 5. Given any function $f : X \rightarrow Y$, what are $f^{-1}(\phi)$, $f^{-1}(Y)$?

Problem 6. Give an example of a function $f : X \rightarrow Y$ and a subset $\phi \neq U \subseteq Y$ such that $f^{-1}(U) = \phi$.

Give an example of a function $f : X \rightarrow Y$ and a proper subset $U \subset Y$ such that $f^{-1}(U) = X$.

Problem 7. Prove that if $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and $U \subseteq Z$:

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

3 Sunday Homework

Exercise 1. Decide which of the following are functions. For those that are not, write why they fail to be functions:

$f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$.

$g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $g(n) = n/2$.

$h : \mathbb{Z} \rightarrow \mathbb{R}$ defined by $h(n) = n/2$.

$i : \mathbb{R} \rightarrow \mathbb{R}$ defined by $i(x) = e^x$.

$j : \mathbb{R} \rightarrow \mathbb{R}$ defined by $j(x) = \log_2(x)$.

$k : \mathbb{R} \rightarrow \mathbb{R}$ defined by $k(x) = \pm\sqrt{x}$.

Exercise 2. Consider the following prescriptions that assign to a human, another human. Decide which of them is a function from the set of humans to the set of humans.

birthmom: assigns to each human their birth mother;

son: assigns to each human their son;

president: assigns to each human the president of the country they are a citizen of;

birthpresident: assigns to each human the president of the country they were born in, at the time they were born.

Exercise 3. Consider the function “birthmom” defined in the previous exercise. What is the function $\text{birthmom} \circ \text{birthmom}$?

Now define a function “biodad: Humans $\rightarrow$ Humans”, which assigns to each human their biological father. Are the compositions $\text{birthmom} \circ \text{biodad}$ and $\text{biodad} \circ \text{birthmom}$ the same or different functions? Conclude whether the operation of composition of functions is commutative or not.

Exercise 4. Use Definition 2 to show that a function is invertible if and only if it is a bijection.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 8

Functions: take 2

1 Key Ideas

We now introduce some attributes of functions that are extremely useful in mathematics.

Definition 1. A function $f : X \rightarrow Y$ is called **injective** (or **1 : 1**) if f maps distinct inputs to distinct outputs.

In other words, a function is injective if no two different inputs are sent to the same output.

Example 1. Let $X = \{1, 2, 3\}$, and $Y = \{a, b, c, d\}$. Then $f : X \rightarrow Y$ defined by

$$f(1) = b, \quad f(2) = c, \quad f(3) = a$$

is injective. But $g : X \rightarrow Y$ defined by

$$g(1) = b, \quad g(2) = b, \quad g(3) = c$$

is NOT injective.

The intuition here is that injective functions use elements of Y as identifications for elements of X . For example, think of the function that assigns to each CSU student, its CSU-id number. The use of such a function is that numbers are more easily entered into a database, for example. Not every number corresponds to a student (I believe CSU-id's are 9 digits long or so, and we definitely don't have a billion students at CSU), but it is essential that no two students have the same CSU-id. This is the prototypical example of an injective function!

Definition 2. A function $f : X \rightarrow Y$ is called **surjective** (or **onto**) if every element of Y is the output via f for some input element of X .

In other words, a function is surjective if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Example 2. Let $X = \{1, 2, 3\}$, and $Y = \{a, b\}$. Then $f : X \rightarrow Y$ defined by

$$f(1) = b, \quad f(2) = a, \quad f(3) = a$$

is surjective. But $g : X \rightarrow Y$ defined by

$$g(1) = b, \quad g(2) = b, \quad g(3) = b$$

is NOT surjective.

Surjective functions come into play when you only want to remember certain information about elements of X . For example, you might need to perform a task that depends only on the nationality of a person (say decide the color of their passport). Then, instead of working with the set of all living humans (which has about 6 billion elements), you may want to work with the set of nations (which has about 200 elements). Then the function that assigns to each person their nationality (assume that there is no double citizenship, just for the sake of this actually being a function) is a surjective function. It is surjective because there is no nation which has no inhabitant.

2 Groupwork

Problem 1. Give an example of a function which is both injective and surjective, of one which is injective but not surjective, of one which is surjective but not injective and of one which is neither injective nor surjective.

Problem 2. 1. Suppose X and Y are finite sets, and $f : X \rightarrow Y$ is an injective function. What can you say about the relationship between $|X|$ and $|Y|$?

2. Suppose X and Y are finite sets, and $f : X \rightarrow Y$ is a surjective function. What can you say about the relationship between $|X|$ and $|Y|$?

3. Let $f : X \rightarrow Y$ be an injective function. Prove that if Y is countable, then X is either finite or countable.

4. Let $f : X \rightarrow Y$ be a surjective function. Prove that if X is countable, then Y is either finite or countable.

Problem 3. 1. Let $f : X \rightarrow Y$ be an injective function. What can you say about the inverse images of subsets of Y ? In particular, if $U = \{y\}$ is a singleton, what can $f^{-1}(U)$ be?

2. Let $f : X \rightarrow Y$ be a surjective function. What can you say about the inverse images of subsets of Y ? In particular, if $U = \{y\}$ is a singleton, what can $f^{-1}(U)$ be?

Problem 4. Prove that the composition of two injective functions is injective.

Problem 5. Given a subset $U \subseteq Y$, there is a natural function $i_U : U \rightarrow Y$ defined by $i_U(u) = u$. Convince yourself that this function is injective. Viceversa, any injective function $f : X \rightarrow Y$ realizes a bijection between X and the image of f . Make sure this makes sense to you. This means that the notion of injective function is in some sense equivalent to the notion of subsets of a set Y . Spend a little time to understand this statement, and discuss it with the other members of the group.

Definition 3. Given a function $f : X \rightarrow Y$, the **image** (or **range**) of f is the set $f(X) \subseteq Y$ of all elements of Y that are outputs for some element of X , i.e.

$$\text{Im}(f) = f(X) = \{y \in Y \text{ such that there exists some } x \in X \text{ with } f(x) = y\}.$$

Problem 6. Give a characterization of f being surjective in terms of the image of f . Use this to prove that the composition of surjective functions is surjective.

Problem 7. Let $f : X \rightarrow Y$ be a surjective function, $|Y| = n$ and for every $y \in Y$ suppose you have $|f^{-1}(\{y\})| = m$. Then what is $|X|$?

3 Sunday Homework

Exercise 1. For each of the following situations, provide an example if it is possible, or explain why it is not possible.

1. $f : X \rightarrow Y$ not injective, $g : Y \rightarrow Z$ a function and $g \circ f$ injective;
2. $f : X \rightarrow Y$ a function, $g : Y \rightarrow Z$ not injective, and $g \circ f$ injective;
3. $f : X \rightarrow Y$ not surjective, $g : Y \rightarrow Z$ a function and $g \circ f$ surjective;
4. $f : X \rightarrow Y$ a function, $g : Y \rightarrow Z$ not surjective, and $g \circ f$ surjective

Exercise 2. Use Problem 7 to count (yet again) the number of handshakes among n persons. Consider the following sets:

- X = the set of ordered pairs of distinct persons in the group.
- Y = the set of handshakes among those persons.

Define a natural, surjective function $f : X \rightarrow Y$. What is the number of elements of X ? For each element $y \in Y$, what is $|f^{-1}(y)|$? Use the conclusion of Problem 7 to deduce $|Y|$.

Exercise 3. Given $f : X \rightarrow Y$, prove the following:

1. If $y_1 \neq y_2$, then $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$;
- 2.

$$\bigcup_{y \in Y} f^{-1}(\{y\}) = X$$

Exercise 4. Use the previous exercise to interpret the equality

$$2^n = \sum_{k=0}^n \binom{n}{k} \tag{1}$$

as follows. Let X be a set with n elements, and consider its power set $\mathcal{P}(X)$. Define a surjective function $f : \mathcal{P}(X) \rightarrow \{0, 1, 2, \dots, n\}$ by $f(U) = |U|$. What is $|\mathcal{P}(X)|$? For any element $i \in \{0, 1, 2, \dots, n\}$, what is $|f^{-1}(i)|$? Now use Exercise 3 to prove (1).

Exercise 5. Given the cartesian product $X \times Y$, we have two natural functions, called projections, defined as follows:

- $p_1 : X \times Y \rightarrow X$ is defined by $p_1((x, y)) = x$.
- $p_2 : X \times Y \rightarrow Y$ is defined by $p_2((x, y)) = y$.

Show that if X and Y are non empty, then the two projections are surjective.

Exercise 6. Use the concept of **graph of a function** introduced last week to show that any function $f : X \rightarrow Y$ may be written as the composition $s \circ i$, where i is an injective function and s is a surjective function.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 9

Equivalence relations and quotient sets

1 Key Ideas

The notion of a quotient set is one of the really confusing things that students struggle with in classes like M366, M369 and M317. So try to really pay attention to what happens this week, as it will be coming back over and over in your future classes.

Let us start by saying something that should feel pretty natural:

- (i1) Given a subset $X \subset Y$, we can think of it as an injective function $i_X : X \rightarrow Y$.
- (i2) Viceversa, given any injective function $f : X \rightarrow Y$, f realizes a bijection between X and the subset $f(X) \subseteq Y$.

These two statements together mean that the concept of injective function contains the same amount of information as the concept of subset, in the sense that there is a procedure that, given an injective function, produces a subset, and an inverse procedure that, given a subset, produces an injective function.

Now we would like to do the same for the concept of surjective function. Surprisingly, things get quite a bit more complicated and confusing. Let us start by writing right away the analogous statements to (i1) and (i2), and then we will spend a fair amount of time making sense of them.

- (s1) Given an equivalence relation on a set X , we can get a surjective function $\pi : X \rightarrow X/\sim$.
- (s2) Viceversa, given any surjective function $f : X \rightarrow Y$, f induces a natural bijection between $X/\sim$ and Y .

Now we must define two new concepts: the notion of an equivalence relation, and the notion of a quotient set.

Definition 1. Given a set X , an **equivalence relation** $\sim$ on X is either one of the following concepts:

- (e1) A partition of X into disjoint subsets:

$$X = \bigcup_{\alpha \in A} X_\alpha \quad \text{and} \quad X_{\alpha_1} \cap X_{\alpha_2} = \emptyset \quad (\text{when } \alpha_1 \neq \alpha_2)$$

- (e2) A procedure that decides when two elements x_1 and x_2 are equivalent (in which case we write $x_1 \sim x_2$), which satisfies the following three requirements:

- r:** for every x , $x \sim x$;
- s:** if $x_1 \sim x_2$, then $x_2 \sim x_1$;
- t:** if $x_1 \sim x_2$ and $x_2 \sim x_3$, then $x_1 \sim x_3$.

The first way to define an equivalence relation is more slick, and ultimately, more useful. The second one however explains better why we use the name.

Definition 2. Given an equivalence relation on a set X defined in the (e1) fashion, a subset X_α is called an **equivalence class** for $\sim$. The set of equivalence classes (in the notation above this is the set A) is called the **quotient set** and denoted $X/\sim$.

If $x \in X$, then we denote the equivalence class of x by $[x]$.

So the quotient set is a set whose elements are subsets of the set X . There are two very confusing things going on here:

1. sometimes we want to think of an equivalence class as a subset of X , sometimes we want to think of it as an element of the quotient set. Unfortunately, the notation $[x]$ leaves the distinction to the context of the statement.
2. the symbol $[x]$ is just one possible name for the equivalence class of x . For any other element $y \in [x]$, we have $[x] = [y]$. Therefore one equivalence class has as many different names as it has elements that it contains.

Finally we can define the projection function.

Definition 3. The **projection function** $\pi : X \rightarrow X/\sim$ is defined by

$$\pi(x) = [x].$$

In other words, π assigns to each element of X the equivalence class it belongs to.

Now that we have all relevant definitions in place, let us get to work to understand things!

2 Groupwork

We begin by understanding why the two definitions (e1) and (e2) are equivalent, by looking at a couple examples:

1. Consider the set $X = \{a, b, c, d, e\}$, and the partition $X = \{a, c, e\} \cup \{b, d\}$. To such a partition, we can assign the following notion of equivalence of elements: $a \sim c \sim e$ and $b \sim d$.
2. Let $X = \mathbb{Z}$ and let $\sim$ be defined as follows: $m \sim n$ when their difference is even. Then $\mathbb{Z} = E \cup O$, where E is the set of even numbers, and O is the set of odd numbers.

Problem 1. Check that the prescription $\sim$ in 2. satisfies the three requirements **r,s,t**.

Problem 2. By observing the two examples 1. and 2., come up with general procedures to go from a partition of a set to a notion of $\sim$ and viceversa. What are equivalence classes in the point of view (e2)?

Problem 3. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $Y = \{a, b, c\}$ and $f : X \rightarrow Y$ be defined by:

$$f(1) = a \quad f(2) = a \quad f(3) = c \quad f(4) = b \quad f(5) = a \quad f(6) = b \quad f(7) = c \quad f(8) = a.$$

1. Write down $f^{-1}(\{a\})$, $f^{-1}(\{b\})$, $f^{-1}(\{c\})$. Observe that they realize a partition of X .
2. Define $x_1 \sim x_2$ when $f(x_1) = f(x_2)$. Check that $\sim$ satisfies **r,s,m**. What are the equivalence classes?
3. We have seen now that we have constructed an equivalence relation on X . Write down the projection function $\pi : X \rightarrow X/\sim$.
4. Consider the function $[f] : X/\sim \rightarrow Y$, defined by $[f]([x]) = f(x)$. Show that $[f]$ is a well defined function and that it is a bijection.
5. Show that $f = [f] \circ \pi$.

Problem 4. Abstract the example in the previous problem, and show that any surjective function $f : X \rightarrow Y$ gives rise to an equivalence relation and a natural bijection between $X/\sim$ and Y .

It looks like in the definition of $[f]$ nothing is really happening...we seem to only be rearranging brackets and then using f . This is a quite subtle point however. The next problem illustrates what could go wrong.

Problem 5. Consider $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with the equivalence relation $X/\sim = \{1, 2, 4\} \cup \{3, 7\} \cup \{6, 8\} \cup \{5\}$. Then consider the function $F : X \rightarrow \{a, b\}$ that sends all even numbers to a and odd numbers to b . Does the prescription

$$[F]([x]) = F(x)$$

define a function $[F] : X/\sim \rightarrow \{a, b\}$?

Problem 6. Let $\sim$ be an equivalence relation on a set X and $f : X \rightarrow Y$ a function. When is there a function $[f] : X/\sim \rightarrow Y$ such that $f = [f] \circ \pi$?

3 Sunday Homework

Exercise 1. Consider the set $X = \{a, b, c, d, e, g, h\}$ and the equivalence relation given by the partition $X = \{a, c, e\} \cup \{b, h\} \cup \{d, g\}$. Answer the following questions:

1. Is $a \sim b$?
2. How many elements does the quotient set have?
3. Is $[a] = [e]$?
4. Write down the projection function $\pi : X \rightarrow X/\sim$.
5. Is $\pi(b) = [h]$?

Exercise 2. Consider the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the equivalence relation prescribed by $m \sim n$ when $m - n$ is a multiple of 4.

1. How many elements does $X/\sim$ have?
2. Write down the equivalence classes.
3. Write down the projection function.

Exercise 3. Consider the equivalence relation on $\mathbb{Z}$ defined by the prescription that all positive numbers are equivalent, all negative numbers are equivalent, and 0 is only equivalent to itself. Let $f : \mathbb{Z} \rightarrow \{a, b\}$ be the function that maps all negative numbers to a and all non-negative numbers to b . Does there exist a function $F : X/\sim \rightarrow \{a, b\}$ such that $f = F \circ \pi$? If so, describe it.

Exercise 4. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ defined by $n \sim m$ when $n - m$ is a multiple of 5. Show that if $n_1 \sim n_2$ and $m_1 \sim m_2$, then $n_1 + m_1 \sim n_2 + m_2$.

Exercise 5. Consider the equivalence relation on $\mathbb{Z}$ defined by the prescription that all positive numbers are equivalent, all negative numbers are equivalent, and 0 is only equivalent to itself. Show that you can find $n_1 \sim n_2$ and $m_1 \sim m_2$ such that $n_1 + m_1 \not\sim n_2 + m_2$.

INTRODUCTION TO MATHEMATICAL REASONING

Worksheet 10

Continuity

1 Key Ideas

This week we focus on the kind of functions that we encounter in an analysis class (M317). In the simplest case, we deal with functions where both the input and output sets are the real numbers $\mathbb{R}$. We call x a coordinate for the input set, and y a coordinate for the output set. A function $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is also denoted with the familiar notation $y = f(x)$.

The notion of continuity is most intuitive in terms of the graph of a function: f is continuous if and only if its graph can be traced on a piece of paper representing $\mathbb{R} \times \mathbb{R}$ without ever needing to lift the pen from the paper. We also would like to say that a function f is continuous if when we vary the input “just a little bit”, then the output also varies “just a little bit”. The problem with this is, besides the fact that “just a little bit” does not sound like a very mathematical notion, that if you start by thinking of the x variation, there is really no way to bound the variation of the output. Think of a linear function, of the form $y = mx$. Its graph is a line, so it is as continuous as it gets. However, by growing the slope m , we can make the y -increment corresponding to a fixed x -increment be as large as we want.

So to get a precise definition of our intuitive notion, we have to start by thinking about the y -variation first. Suppose that we choose an input x_0 , with output $f(x_0)$. The function f is continuous at the point x_0 , if we can always play the following game: you can choose however you want a small interval around $f(x_0)$, let us say it has radius ε . After you do that, I must be able to choose an interval around x_0 in such a way that if any point is chosen in my interval as an input, then the output is guaranteed to land in your interval. This idea gives rise to the following initially confusing definition, which is mostly just an exercise in being comfortable with quantifiers.

Definition 1. A function $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is continuous at the point x_0 if, **for every** $\varepsilon > 0$, **there exists** δ (depending on ε), such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

We say f is a **continuous function** if it is continuous at every point x_0 .

The notion of continuity is extremely important if you imagine that f is describing or modeling some experiment that you are running: it is saying that you can control the margin of error of the output of your experiment so long as you can control sufficiently the precision of your input.

2 Groupwork

Problem 1. Prove, using the definition, that the function $y = x^3$ is a continuous function.

Problem 2. Show, using the definition, that the function $y = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ is not continuous.

It is evident that checking whether a function is or is not continuous using the definition is a rather painful and cumbersome exercise. That is why we prove theorems that allow us to conclude that functions are continuous without having to go through the definition every time. Here are some examples.

Problem 3. Prove that if f_1 and f_2 are two continuous functions, then $f_1 + f_2$ is a continuous function.

Prove that a finite sum of continuous functions is a continuous function.

In the Sunday homework you will be asked to prove that a product of continuous function is a continuous function. For now, assume this as a true fact.

Problem 4. Prove that all polynomials are continuous functions.

Definition 2. Let $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ be a function, and x_0 a fixed input. The **limit** of f as x approaches x_0 is L if, **for every** $\varepsilon > 0$, **there exists** δ (depending on ε), such that, if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. We use the following notation:

$$\lim_{x \rightarrow x_0} f(x) = L$$

Problem 5. Try and develop an intuitive understanding of Definition 2. Clearly it looks quite closely related to the definition of continuity. Our goal is to understand the precise relationship.

Problem 6. Use the definition to establish what the following limits are (if they exist):

$$\lim_{x \rightarrow 0} x^3 = ?$$

$$\lim_{x \rightarrow 0} \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} = ?$$

$$\lim_{x \rightarrow 0} \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} = ?$$

Problem 7. Formulate, and then prove, a statement that relates the notion of continuity of a function, to the notion of limit.

3 Sunday Homework

Exercise 1. Prove that if f_1 and f_2 are two continuous functions, then $f_1 \cdot f_2$ is a continuous function.

Exercise 2. Prove that if f_1 and f_2 are two continuous functions, then $f_1 \circ f_2$ is a continuous function.

Exercise 3. Suppose that $\lim_{x \rightarrow x_0} f_1(x) = L_1$ and $\lim_{x \rightarrow x_0} f_2(x) = L_2$; show that

$$\lim_{x \rightarrow x_0} f_1(x) + f_2(x) = L_1 + L_2.$$

Exercise 4. Suppose that $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow L_1} g(x) = L_2$; show that

$$\lim_{x \rightarrow x_0} g \circ f(x) = L_2.$$

Definition 3. An **open interval** in $\mathbb{R}$ is the set of all numbers x which are strictly greater than a number a , and strictly less than a number b . This is denoted by the symbol (a, b) .

$$(a, b) := \{x \in \mathbb{R} \text{ such that } a < x < b\}.$$

Exercise 5. Prove that a subset S of the real numbers is a union of open intervals if and only if, for every $s \in S$, there exists an open interval which contains s and is contained in S . Such a set S is called an **open set**.

Exercise 6. Prove that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if, for every open set S , $f^{-1}(S)$ is an open set.

Quotients and Operations

1 Key Ideas

A couple weeks ago we discussed quotient sets and equivalence relations. Let us recap now the important points:

1. An **equivalence relation** $\sim$ on a set X produces a partition of X into disjoint subsets, called **equivalence classes**.
2. The set whose elements are the equivalence classes is called **quotient set** and denoted $X/\sim$.
3. Given an element $x \in X$, the equivalence class that x belongs to is denoted $[x]$ (note that $[x] = [y]$ if and only if $x \sim y$).
4. The **projection function** $\pi : X \rightarrow X/\sim$ assigns to each element x the equivalence class that x belongs to. ($\pi(x) = [x]$).
5. A function $F : X \rightarrow Y$ determines a function $[F] : X/\sim \rightarrow Y$ via the prescription $[F]([x]) = f(x)$ if and only if the following condition holds:

$$x_1 \sim x_2 \implies f(x_1) = f(x_2).$$

The last point is of particular interest to us: in order for a function $F : X \rightarrow Y$ to naturally define a function $[F]$ from the quotient set, F must satisfy some special property, which we think of as a compatibility condition between F and the equivalence relation $\sim$. Now we want to boost this idea. Let us consider an equivalence relation $\sim$ on a set X and a function:

$$F : X \rightarrow X$$

We would like to define a function $[F] : X/\sim \rightarrow X/\sim$ by the prescription:

$$[F]([x]) = [F(x)]. \tag{1}$$

Let us see in an example when this works. Let

$$X = \{boy, cat, dog, bat, chloe, doe, tic, tac, toe\},$$

and $F : X \rightarrow X$ be defined by:

$$F(boy) = bat, \quad F(cat) = dog, \quad F(dog) = bat$$

$$F(bat) = boy, \quad F(chloe) = doe, \quad F(doe) = bat$$

$$F(tic) = tac, \quad F(tac) = toe, \quad F(toe) = toe,$$

Define an equivalence relation on X by saying that two elements of X are equivalent if they begin with the same letter. Then $X/\sim = \{b, c, d, t\}$ has four elements. Then the prescription given in equation (1) defines the function $[F] : X/\sim \rightarrow X/\sim$ as:

$$[F](b) = b, \quad F(c) = d, \quad F(d) = b, \quad F(t) = t.$$

Now define an equivalence relation on X by saying that two elements of X are equivalent if they end with the same letter. The quotient set $X/\sim = \{c, e, g, t, y\}$ has five elements. The prescription given in equation (1) does NOT define a function:

$$[chloe] = [doe] = e,$$

but

$$[F(chloe)] = [doe] = e \neq [F(doe)] = [bat] = t,$$

which means that $[F]$ is trying to sent the input e to two different outputs.

Definition 1. Given an equivalence relation $\sim$ on a set X , we say that a function $F : X \rightarrow X$ is **compatible with** $\sim$ if

$$x_1 \sim x_2 \implies F(x_1) \sim F(x_2).$$

If this is the case, the prescription in equation (1) defines a function $[F] : X/\sim \rightarrow X/\sim$.

Our goal this week is to generalize this idea even further: if X is a set of numbers and $\sim$ an equivalence relation, when can we define operations of addition and multiplication on the quotient set $X/\sim$? We will use this idea to give a precise and formal proof of the following rule, that probably a lot of you remember from elementary school.

Theorem 1. *An integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.*

2 Groupwork

We start with a couple definitions.

Definition 2. A set X is a **number system** if there are two operations on the elements of X (addition and multiplication) that satisfy all the usual rules of arithmetics that the common addition and multiplication of ordinary numbers satisfy.

Problem 1. Let X be a set with exactly two elements denoted E, O . Let addition and multiplication be defined by the following tables:

$+$	E	O	$\cdot$	E	O
E	E	O	E	E	E
O	O	E	O	E	O

In order to prove that X is a number system we should check that such addition and multiplication are commutative, associative and they respect the distributive laws. One really should check that these properties hold in **all possible instances!** This is of course very boring, so let us instead check, for each of the properties, one particular instance.

commutativity Check that $E + O = O + E$ and $E \cdot O = O \cdot E$.

associativity of $+$ Check that $(E + O) + O = E + (O + O)$.

associativity of $\cdot$ Check that $(E \cdot O) \cdot O = E \cdot (O \cdot O)$.

distributive laws Check that $O \cdot (E + O) = (O \cdot E) + (O \cdot O)$.

Next we consider the following question. Given an equivalence relation on a number system, when is the quotient set also a number system in a natural way? We make the following definition.

Definition 3. An equivalence relation $\sim$ on a number system X is **compatible with addition** if

$$x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 + y_1 \sim x_2 + y_2.$$

An equivalence relation $\sim$ on a number system X is **compatible with multiplication** if

$$x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 \cdot y_1 \sim x_2 \cdot y_2.$$

Problem 2. Give an example of an equivalence relation on a number system which is not compatible with addition, nor multiplication.

Problem 3. Let $X = \mathbb{Z}$ be the integers, and $\sim$ the equivalence relation defined as follows:

$$m \sim n \iff m - n = 9k,$$

for some $k \in \mathbb{Z}$. Show that $\sim$ is compatible with addition and multiplication.

When an equivalence relation on a number system is compatible with addition and multiplication, the quotient set $X/\sim$ becomes a number system with the following definitions. We write here $\oplus$ and $\otimes$ to define the operations on the quotient set, so as to not confuse them with the operations $+, \cdot$ on the original number system.

$$[x] \oplus [y] := [x + y]$$

and

$$[x] \otimes [y] := [x \cdot y]$$

In other words, we are using the operations of addition and multiplication on X to define operations on $X/\sim$. The dangerous thing is that elements of the quotient set have multiple names. So what may happen is that when two different names are used, the same operation on the same elements may produce two different outcomes. Let us see one quick example.

Problem 4. Let $X = \mathbb{Z}$ and $\sim$ the equivalence relation that declares all negative integers to be equivalent, and all non-negative integers to be equivalent. Use the definition above to $[4] \oplus [-3]$ and $[2] \oplus [-3]$. What is the problem?

The condition of being compatible with addition and multiplication are precisely what is needed to make sure the problem we just witnessed does not happen. Finally, the number systems X and $X/\sim$ are very closely related, in this sense.

Theorem 2. *Given any sequence of arithmetic operations on elements of X , one obtains the same result in the following two ways:*

1. *Perform all operations in X , and at the end, put square brackets around the result.*
2. *Perform all operations in $X/\sim$, i.e., put square brackets around all numbers and circles around all operation signs.*

Formal proofs of all these statements are a bit tedious and complicated, so we will skip them for now (raincheck till your MATH 366 class). But let us see one example of the above theorem. Consider the equivalence relation from Problem 3, and denote by $\mathbb{Z}_9$ the quotient set. Then the following computations give the same result:

$$[12 + 3 \cdot (11 + 7)] = [66] = [3]$$

or

$$[12] \oplus [3] \otimes ([11] \oplus [7]) = [3] \oplus [3] \otimes ([2] \oplus [7]) = [3] \oplus [3] \otimes [0] = [3].$$

Notice that in the second case, since we get to choose different names for the same element (i.e. we can replace $[12]$ with $[3]$, since $[12] = [3]$), we can choose to work with smaller numbers. This is the idea at the base of the test for divisibility by 3. Here are all the important steps.

Problem 5. Let $\mathbb{Z}_9$ be defined as above.

1. Prove that an integer n is divisible by 3 if and only if $[n] = [0], [3]$ or $[6]$.
2. Prove that for any nonnegative integer n , $[10^n] = [1]$.
3. Prove that for any integer n , $[n]$ is equal to the equivalence class of the sum of the digits of n .

3 Sunday Homework

Exercise 1. 1. Give an example of an equivalence relation $\sim$ on a set X , and a function $F : X \rightarrow X$ which is compatible with $\sim$.

2. Give an example of an equivalence relation $\sim$ on a set X , and a function $F : X \rightarrow X$ which is NOT compatible with $\sim$.

Exercise 2. Let $\sim$ be an equivalence relation on the set X . Decide if the following statements are true or false:

1. If $F : X \rightarrow X$ is a constant function, then $[F]$ is a function.
2. If $Id_X : X \rightarrow X$ denotes the identity function of X , then $[Id_X]$ is a function.
3. If $F : X \rightarrow X$ is an injective function, then $[F]$ is a function.
4. If $F : X \rightarrow X$ is a surjective function, then $[F]$ is a function.
5. If for every $x \in X$, $[F(x)] = [x]$, then $[F] = Id_{X/\sim}$ is the identity function of the quotient set.
6. If $[F] = Id_{X/\sim}$ is the identity function of the quotient set, then $F = Id_X$.

Exercise 3. Consider the number system $X = \{E, O\}$ from Problem 1.

zero An element z of a number system is a **zero** if, for every other element x in the number system, we have $x + z = z + x = x$, and $x \cdot z = z \cdot x = z$. Show that E is a zero for X .

one An element u of a number system is a **one** if, for every other element x in the number system, we have $x \cdot u = u \cdot x = x$. Show that O is a one for X .

subtraction The operation of **subtraction** is defined as follows: $x - y$ is defined to be an element w such that $x = y + w$. What is $E - O$?

division For y NOT a zero, the operation of **division** is defined as follows: x/y is defined to be an element w such that $x = y \cdot w$. What is E/O ?

Exercise 4. Use the concepts from Problem 5 to outline a formal proof of Theorem 1. An outline means that for the details you can just refer to the work you have done in the problems. But you need to write a clear and logical sequence of steps that prove the thesis of the theorem.

Exercise 5. Devise a test to check divisibility by the number 11.

Exam #1

M235

Fall 2018

Name: _____ Score: _____

Read the problems. Read them again. Make sure to express your answers in a clear and logical way. Don't rush!

1. (20 pts.) Prove by induction the following identity. For all natural numbers n ,

$$\sum_{i=1}^n (2i + 1) = n^2 + 2n.$$

Careful! A proof by induction is not just a string of equations. Please supply the proper amount of connecting sentences that makes the proof readable and understandable by your non-mathematician cousin.

2. (20 pts.) Let $\mathcal{U}$ denote the set of all animals. Consider the following definitions:

Definition 1: A set $X \subseteq \mathcal{U}$ is **flick** if there exists one element of X that can fly.

Definition 2: A set $X \subseteq \mathcal{U}$ is **fluck** if every element of X can fly.

Give an example of:

- (a) A set which is flick and fluck.
- (b) A set which is flick but not fluck.
- (c) A set which is fluck but not flick.
- (d) A set which is neither flick nor fluck.

3. (20 pts.) TRUE-FALSE: no explanation is needed for this problem:

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

(c) If $A \setminus B = \phi$, then $B \setminus A = \phi$.

(d) $A \subseteq B$ if and only if $B^c \subseteq A^c$.

(e) If $A \subseteq C$ and $B \subseteq C$, then $A \cap B \subseteq C$.

(f) If $A \cap B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

4. (20 pts.) Match each of the statements below with the set theoretical statement it corresponds to:

- (a) If I don't put sunscreen, I burn.
- (b) If $2 + 2 = 5$, then moooo!
- (c) If it rains, it pours.
- (d) I neither ski nor skate.
- (e) I ski and I don't skate.

1:

$$(A \cup B)^c$$

2:

$$A^c \subseteq B$$

3:

$$A \subseteq B$$

4:

$$\phi \subseteq A$$

5:

$$A \cap B^c$$

5. (10 pts.) For any natural number $n \geq 3$, consider the expression

$$(x_1 + x_2 + x_3 + \dots + x_n)^3.$$

Prove that the largest coefficient of a monomial in the expansion of the above expression is 6.

Hint: recall the groupwork proof of Newton's theorem.

6. (10 pts.) Prove that the set of irrational numbers (real numbers which are not rational) is uncountable.

Exam #2

M235

Spring 2018

Name: _____ Score: _____

Read the problems. Read them again. Make sure to express your answers in a *clear and logical* way. *Don't rush!*

1. (20 pts.) TRUE-FALSE: no explanation is needed for this problem.

Let $f : X \rightarrow Y$ be a function.

- (a) If $x_1 = x_2$, then $f(x_1) = f(x_2)$.
- (b) If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- (c) If f is injective and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- (d) If f is surjective and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- (e) If $x \in f^{-1}(\{y\})$, then $f(x) = y$.
- (f) If $x_1 \in f^{-1}(\{y\})$ and $x_2 \in f^{-1}(\{y\})$, then $f(x_1) = f(x_2)$.
- (g) If $x_1 \in f^{-1}(\{y\})$ and $x_2 \in f^{-1}(\{y\})$, then $x_1 = x_2$.
- (h) If f is injective, $x_1 \in f^{-1}(\{y\})$ and $x_2 \in f^{-1}(\{y\})$, then $x_1 = x_2$.
- (i) If f is injective, then $f(X) = Y$.
- (j) If f is surjective, then $f(X) = Y$.

2. (20 pts.) Let $X = \{N, E, S, W\}$ and $Y = \{0, \pi/2, \pi, 3\pi/2\}$.

- (a) Give an example of an invertible function $f : X \rightarrow Y$.
- (b) Write down the inverse function f^{-1} .
- (c) Give an example of an invertible function $g : Y \rightarrow X$ which is not the inverse of f .
- (d) Compute the compositions $g \circ f$ and $f \circ g$.

3. (20 pts.) Let X be the set of all possible orderings (or permutations) of the natural numbers $1, 2, \dots, 7$, and Y be the set of all possible orderings of the natural numbers $1, 2, \dots, 6$. Define the function $p : X \rightarrow Y$ as follows: given an ordering of the first 7 integers, obtain an ordering of the first 6 integers simply by deleting the number 7.

- (a) What is $p(1, 3, 5, 7, 2, 4, 6)$?
- (b) Prove that p is a surjective function.
- (c) Write all elements of $p^{-1}(1, 2, 4, 6, 5, 3)$.

Define an equivalence relation by setting two orderings of $1, 2, \dots, 7$ to be equivalent if and only if they take the same value under the function p .

(If you prefer symbols, for $\sigma_1, \sigma_2 \in X$: $\sigma_1 \sim \sigma_2 \iff p(\sigma_1) = p(\sigma_2)$).

- (d) How many elements are in each equivalence class?
- (e) How many elements does the quotient set $X / \sim$ have?

4. (20 pts.) Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **continuous** if it is *continuous at every point* $x_0 \in \mathbb{R}$.
- (a) Write the definition of $f : \mathbb{R} \rightarrow \mathbb{R}$ being continuous at a point $x_0 \in \mathbb{R}$.
 - (b) Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, then the composition $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

5. (20 pts.) Let X and Y be two finite sets with the same number of elements ($|X| = |Y|$), and $f : X \rightarrow Y$ a function.
- (a) Prove that if f is injective, then f is surjective.
 - (b) If X and Y are countable sets, does the statement in (a) still hold? (explain why it holds or give a counterexample if it doesn't).

Final Exam

M235

Spring 2018

Name: _____ Score: _____

Read the problems. Read them again. Make sure to express your answers in a *clear and logical* way. *Don't rush!*

1. (25 pts.) TRUE OR FALSE:

- (a) If $f : X \rightarrow Y$ is a function, the inverse function always exists.
- (b) If $f : X \rightarrow Y$ is a function and $U \subseteq Y$, the inverse image of U always exists.
- (c) If $f : X \rightarrow Y$, then we can think of f^{-1} as a function from the power set of X to the power set of Y .
- (d) If $f : X \rightarrow Y$, then we can think of f^{-1} as a function from the power set of Y to the power set of X .
- (e) If $f : X \rightarrow Y$ and the inverse function exists, then f is bijective.
- (f) If $f : X \rightarrow Y$ is injective and $y \in Y$, then $f^{-1}(\{y\})$ contains one element.
- (g) If $f : X \rightarrow Y$ is injective and $y \in Y$, then $f^{-1}(\{y\})$ contains at most one element.
- (h) If $f : X \rightarrow Y$ is injective and $y \in Y$, then $f^{-1}(\{y\})$ contains at least one element.
- (i) If $f : X \rightarrow Y$ is surjective and $y \in Y$, then $f^{-1}(\{y\})$ contains one element.
- (j) If $f : X \rightarrow Y$ is surjective and $y \in Y$, then $f^{-1}(\{y\})$ contains at most one element.
- (k) If $f : X \rightarrow Y$ is surjective and $y \in Y$, then $f^{-1}(\{y\})$ contains at least one element.
- (l) If $f : X \rightarrow Y$ is surjective and $|U| = 3$ has three elements, then it is possible that $|f^{-1}(U)| = 2$.

2. (20 pts.) Exhibit an explicit bijection between the set of integers $\mathbb{Z}$ and the set $X = \mathbb{Z} \cup \{\star\}$ consisting of the integers with one distinct additional element called $\star$.

3. (20 pts.) Let X be the set of all possible orderings of the natural numbers 1, 2, 3, 4, 5, 6, and Y be the set of all possible orderings of the natural numbers 1, 2, 3, 4. Define the function $p : X \rightarrow Y$ as follows: given an ordering of the first 6 positive integers, obtain an ordering of the first 4 positive integers by deleting the numbers 5 and 6.

(a) What is $p(1, 3, 5, 2, 4, 6)$?

(b) Prove that p is a surjective function.

(c) What is the number of elements in $p^{-1}(1, 3, 4, 2)$?

Define an equivalence relation by setting two orderings of 1, 2, 3, 4, 5, 6 to be equivalent if and only if they take the same value under the function p .

(If you prefer symbols, for $\sigma_1, \sigma_2 \in X$: $\sigma_1 \sim \sigma_2 \iff p(\sigma_1) = p(\sigma_2)$).

(d) How many elements are in each equivalence class?

(e) How many elements does the quotient set $X/\sim$ have?

4. (20 pts.) Let X and Y be two finite sets with the same number of elements ($|X| = |Y|$), and $f : X \rightarrow Y$ a function.
- (a) Prove that if f is surjective, then f is injective.
 - (b) If X and Y are countable sets, does the statement in (a) still hold? (explain why it holds or give a counterexample if it doesn't).

5. (15 pts.) Consider the following definitions.

Definition 1. A function $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is continuous at the point x_0 if, **for every** $\varepsilon > 0$, **there exists** δ (depending on ε), such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. We say f is a **continuous function** if it is continuous at every point x_0 .

Definition 2. A subset $U \subseteq \mathbb{R}$ is **open** if for every point $u \in U$ there exists a $\delta > 0$ (depending on u) such that the open interval $(u - \delta, u + \delta)$ is entirely contained in U . In symbols, $(u - \delta, u + \delta) \subseteq U$.

- (a) Is the empty set an open set?
- (b) Prove the following theorem.

Theorem 1. If $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$ is a continuous function and $V = \{y \text{ such that } a < y < b\} = (a, b)$ is an open interval in the y -axes, then $f^{-1}(V)$ is an open set.

6. (10 pts. extra-credit) Give an outline of the proof of the following theorem.

Theorem 2. *An integer n is divisible by 9 if and only if the sum of its digits is divisible by 9.*

Note: an outline of the proof should not prove all details of every statement that you are using (for example, if you are claiming that a particular equivalence relation is compatible with addition and multiplication, you don't need to write the proof of that statement); however it must provide a clear strategy, and be extremely clear about all logical steps that are used.