

## Final Exam

M235

Spring 2018

Name: \_\_\_\_\_ Score: \_\_\_\_\_

**Read the problems. Read them again. Make sure to express your answers in a *clear and logical* way. *Don't rush!***

1. (25 pts.) TRUE OR FALSE:

- (a) If  $f : X \rightarrow Y$  is a function, the inverse function always exists.
- (b) If  $f : X \rightarrow Y$  is a function and  $U \subseteq Y$ , the inverse image of  $U$  always exists.
- (c) If  $f : X \rightarrow Y$ , then we can think of  $f^{-1}$  as a function from the power set of  $X$  to the power set of  $Y$ .
- (d) If  $f : X \rightarrow Y$ , then we can think of  $f^{-1}$  as a function from the power set of  $Y$  to the power set of  $X$ .
- (e) If  $f : X \rightarrow Y$  and the inverse function exists, then  $f$  is bijective.
- (f) If  $f : X \rightarrow Y$  is injective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains one element.
- (g) If  $f : X \rightarrow Y$  is injective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains at most one element.
- (h) If  $f : X \rightarrow Y$  is injective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains at least one element.
- (i) If  $f : X \rightarrow Y$  is surjective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains one element.
- (j) If  $f : X \rightarrow Y$  is surjective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains at most one element.
- (k) If  $f : X \rightarrow Y$  is surjective and  $y \in Y$ , then  $f^{-1}(\{y\})$  contains at least one element.
- (l) If  $f : X \rightarrow Y$  is surjective and  $|U| = 3$  has three elements, then it is possible that  $|f^{-1}(U)| = 2$ .

2. (20 pts.) Exhibit an explicit bijection between the set of integers  $\mathbb{Z}$  and the set  $X = \mathbb{Z} \cup \{\star\}$  consisting of the integers with one distinct additional element called  $\star$ .

3. (20 pts.) Let  $X$  be the set of all possible orderings of the natural numbers 1, 2, 3, 4, 5, 6, and  $Y$  be the set of all possible orderings of the natural numbers 1, 2, 3, 4. Define the function  $p : X \rightarrow Y$  as follows: given an ordering of the first 6 positive integers, obtain an ordering of the first 4 positive integers by deleting the numbers 5 and 6.

(a) What is  $p(1, 3, 5, 2, 4, 6)$ ?

(b) Prove that  $p$  is a surjective function.

(c) What is the number of elements in  $p^{-1}(1, 3, 4, 2)$ ?

Define an equivalence relation by setting two orderings of 1, 2, 3, 4, 5, 6 to be equivalent if and only if they take the same value under the function  $p$ .

(If you prefer symbols, for  $\sigma_1, \sigma_2 \in X$ :  $\sigma_1 \sim \sigma_2 \iff p(\sigma_1) = p(\sigma_2)$ ).

(d) How many elements are in each equivalence class?

(e) How many elements does the quotient set  $X/\sim$  have?

4. (20 pts.) Let  $X$  and  $Y$  be two finite sets with the same number of elements ( $|X| = |Y|$ ), and  $f : X \rightarrow Y$  a function.
- (a) Prove that if  $f$  is surjective, then  $f$  is injective.
  - (b) If  $X$  and  $Y$  are countable sets, does the statement in (a) still hold? (explain why it holds or give a counterexample if it doesn't).

5. (15 pts.) Consider the following definitions.

**Definition 1.** A function  $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$  is continuous at the point  $x_0$  if, **for every**  $\varepsilon > 0$ , **there exists**  $\delta$  (depending on  $\varepsilon$ ), such that, if  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \varepsilon$ . We say  $f$  is a **continuous function** if it is continuous at every point  $x_0$ .

**Definition 2.** A subset  $U \subseteq \mathbb{R}$  is **open** if for every point  $u \in U$  there exists a  $\delta > 0$  (depending on  $u$ ) such that the open interval  $(u - \delta, u + \delta)$  is entirely contained in  $U$ . In symbols,  $(u - \delta, u + \delta) \subseteq U$ .

- (a) Is the empty set an open set?
- (b) Prove the following theorem.

**Theorem 1.** If  $f : (\mathbb{R}, x) \rightarrow (\mathbb{R}, y)$  is a continuous function and  $V = \{y \text{ such that } a < y < b\} = (a, b)$  is an open interval in the  $y$ -axes, then  $f^{-1}(V)$  is an open set.

6. (10 pts. extra-credit) Give an outline of the proof of the following theorem.

**Theorem 2.** *An integer  $n$  is divisible by 9 if and only if the sum of its digits is divisible by 9.*

**Note:** an outline of the proof should not prove all details of every statement that you are using (for example, if you are claiming that a particular equivalence relation is compatible with addition and multiplication, you don't need to write the proof of that statement); however it must provide a clear strategy, and be extremely clear about all logical steps that are used.