Math 281

Homework

Sep. 26th

These exercises build upon and extend some of the work you did with the in-class projects. Each exercise requires a proof!

1 Subsets of a finite set

Let X be a set with n elements. What is the total number of subsets of X?

Note: Remember to count the empty set and X itself as subsets of X.

2 Newton's binomial theorem

Prove that for a positive integer n:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Can you give two different proofs of this theorem?

3 Pascal's triangle

Construct a "triangle of numbers in the following way: I'll tell you how to build horizontal rows.

1. In the n-th row you have n numbers.
2. You start in the first row with the number 1.
3. Each entry is obtained by summing the two numbers directly to the left and right above it (if there is no number to the left or right, just consider it a 0).

Here are the first few rows of this triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Now for the following questions:

1. What does this triangle have to do with exercise 2? Why?

2. What is the sum of the numbers in each row? Can you prove this in two ways: one using exercise 2, another using exercise 1?

3. Prove that the alternating sum of the numbers in each row (except the very first one) gives 0. (See, for example 1-4+6-4+1=0). Again, you might want to rely on exercise 2.

4 On the existence of n-th roots

Use exercise 2 to prove that for k a positive integer, and $x \geq 0$, the function $y = x^k$ is increasing. I.e. that if $x_1 < x_2$, then $x_1^k < x_2^k$.

Now use this fact and the existence of supremums in $\mathbb{R}$ to show that any positive real number admits a positive real k-th root.