Worksheet on Symmetric Polynomials

Renzo's Math 281

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A polynomial in several variables $x_1, \ldots, x_n$ is called **symmetric** if it does not change when you permute the variables. For example:

1. any polynomial in just one variable is symmetric because there is nothing to permute...
2. the polynomial $x_1 + x_2 + 5x_1^2 + 5x_2^2 - 8$ is a symmetric polynomial in two variables.
3. the polynomial $x_1 x_2 x_3$ is a symmetric polynomial in three variables.
4. the polynomial $x_1^2 + x_1 x_2$ is NOT a symmetric polynomial. Permuting x_1 and x_2 you obtain $x_2^2 + x_1 x_2$ which is different!

Problem 1. *Familiarize yourself with symmetric polynomials. Write down some examples of symmetric polynomials and of polynomials which are not symmetric.*

Problem 2. *Describe all possible symmetric polynomials:*

- *in two variables of degree one;*
- *in three variables of degree one;*
- *in four variables of degree one;*
- *in two variables of degree two;*
- *in three variables of degree two;*
- *in four variables of degree two;*
- *in two variables of degree three;*
- *in three variables of degree three;*
- *in four variables of degree three.*

Now observe the set of symmetric polynomials in n variables. It is a vector subspace of the vector space of all possible polynomials. (Why is it true?) In particular we want to think about $\mathbb{Q}$-vector subspaces...this simply means that we only allow coefficients to be rational numbers, and that we allow ourselves to only stretch by rational numbers. This is VERY IMPORTANT for our purposes, but in practice it doesn't change anything in terms of how you should go about these probems. Let us do a bit of dimension counting. Let us introduce first a little more notation: a polynomial is called **homogeneous** if every monomial appearing has the same degree.

Problem 3. *What is the dimension of the vector subspace of:*

- *symmetric polynomials in two variables homogeneous of degree one;*
- *symmetric polynomials in three variables homogeneous of degree one;*
- *symmetric polynomials in n variables homogeneous of degree one;*
- *symmetric polynomials in two variables homogeneous of degree two;*
- *symmetric polynomials in three variables homogeneous of degree two;*
- *symmetric polynomials in n variables homogeneous of degree two;*
- *symmetric polynomials in two variables homogeneous of degree three;*
- *symmetric polynomials in three variables homogeneous of degree three;*
- *symmetric polynomials in n variables homogeneous of degree three.*

For each of these vector subspaces write down a "natural" basis.

OK, so by now you should have gained a fair amount of experience with this. Let us try to systematize it a bit.

Problem 4. *Find a natural indexing set for a basis of the vector space of symmetric polynomials in n variables homogeneous of degree d. What I am looking for is NOT any sort of an exact formula, but a prescription of the sort: "elements in a basis for the above vector space correspond (bijectively) to ways of writing blah as blah blah blah. This correspondence works as follows: ..."*

Now we concentrate our attention to some symmetric polynomials which will play a special role in our story.

Problem 5. *Expand the expression:*

$$(x - a_1)(x - a_2) \ldots (x - a_n)$$

Start by doing concrete examples with small values of n, and then try to obtain a general statement. Observe that the coefficient of x^{n-i} is a symmetric polynomial in $a_1, \ldots, a_n$. Try to "describe" this symmetric polynomial (what is its degree? Is it homogeneous? How does it "look like"?).

These symmetric polynomials play a very important role in our story. So much so that they deserve a name of their own: the coefficient of x^{n-i} in the expansion of $(x-a_1)(x-a_2)\dots(x-a_n)$ is called the **i-th elementary symmetric polynomial** in n variables, and it is denoted σ_i.

Observe that what you just described is the following powerful mathematical statement: the coefficients of a polynomial are the elementary symmetric functon of its roots.

Problem 6. *Take a minute to meditate on the previous statement and to make sure it makes sense to you.*

The reason that we care so much about elementary symmetric polynomials is that any symmetric polynomial can be obtained as a polynomial with rational coefficients in the elementary symmetric polynomials. Before we try to prove this let us try to understand it through some examples.

Problem 7. *Write down $x_1^2 + x_2^2$ as a polynomial in σ_1 and σ_2. Or, which is the same thing, as a linear combination of σ_1^2 and σ_2.*

Problem 8. *Write down $x_1^3 + x_2^3 + x_3^3 + 4x_1x_2x_3$ as a polynomial in $\sigma_1, \sigma_2, \sigma_3$.*

At this point you guys maybe are starting to believe the above statements, but also to appreciate that it is quite hard, in practical terms, to figure out concretely how to get a polynomial in the σ_i's from a symmetric polynomial...so let us take a different route instead. Assume the following fact (which is not really hard, but it is kind of technical and messy): different monomials in the σ_i's are linearly independent from each others.

Problem 9. *Count the dimension of the following vector subspaces of the vector space of symmetric polynomials.*

- *the vector space of homogeneous symmetric degree one polynomials that are polynomials in σ_1.*
- *the vector space of homogeneous symmetric degree two polynomials that are polynomials in σ_1, σ_2.*
- *the vector space of homogeneous symmetric degree three polynomials that are polynomials in $\sigma_1, \sigma_2, \sigma_3$.*
- *the vector space of homogeneous symmetric degree four polynomials that are polynomials in $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.*

Compare this with the results in problem 3. What do you notice?

Problem 10. *Can you prove what you observed above? In light of exercise 4, you can try and show that the two corresponding vector subspaces have bases elements corresponding to the same indexing set. Or you could try to directly construct a bijection between basis elements.*