Math 281

Vector Subspaces and Linear Independence Worksheet

The goal of this worksheet is to explore the concept of linear independence of vectors. We first introduce the concept informally, with the goal of arrriving to a formal definition.

Recall

A vector space is a set V with two operations:

addition $(V, +)$ is a group with a commutative operation;

scalar multiplication i.e. we can multiply vectors with real numbers to obtain vectors. This corresponds to the intuitive notion of "stretching" a vector.

Such operations must satisfy the following properties:

1. for any vector v
$$1v = v$$
2. for any real number r and for any two vectors v, w:
$$r(v + w) = rv + rw$$
3. for any vector v and any two real numbers r, s:
$$(r + s)v = rv + sv$$

1 Vector Subspaces

Given a vector space V, we define a vector subspace to be a subset $W \subseteq V$ that is itself a vector space using the operations of V. In other words:

1. for any two vectors $w_1, w_2 \in W$ the vector $w_1 + w_2$ is in W.
2. for any $w \in W$, $r \in \mathbb{R}$, the vector rw is in W.

Problem 1. *If W is a vector subspace (and yes, it is not an empty set Jon), then show that W must contain the vector 0. Also, show that if a vector $w \in W$, then the additive inverse of w belongs to W.*

Problem 2. *What are the silly vector subspaces?*

Problem 3. *Which of the following subsets are vector subspaces of $\mathbb{R}^2$?*

- $W_1 = \{(x, y) \text{ such that } x = y\}$
- $W_2 = \{(x, y) \text{ such that } x = y + 2\}$
- $W_3 = \{(x, 2) \text{ such that } x \in \mathbb{R}\}$
- $W_4 = \{(x, y) \text{ such that } 4x = 7y\}$
- $W_5 = \{(x, y) \text{ such that } x = y^2\}$
- $W_6 = \{(x, y) \text{ such that } x^2 = y^2\}$

Problem 4. *Which of the following subsets are vector subspaces of* $\mathbb{R}[x]$*?*

- W_1= *all polynomials of degree* 0.
- W_2= *all polynomials of degree* 2.
- W_3= *all polynomials of degree* 0 *or* 2.
- W_4= *all polynomials of degree* $0, 1,$*or* 2.
- W_5= *all polynomials of degree* 0 *or* 2 *with no monomial of degree* 1.

Problem 5. *Give an example of a vector subspace and of a not vector subspace for the vector space of continuous functions* $V = \mathcal{C}[0, 1]$ *and for the vector space of* 2×2 *matrices.*

2 Subspaces generated by a set of vectors

Now we explore the concept of a vector subspace generated by a set of vectors. Here is a formal (but maybe not so useful) definition.
Given a vector space V and a set of vectors S, the **vector subspace generated by** S, denoted $\langle S\rangle$, is the smallest vector subspace of V that contains all the vectors in S. Just to throw some more dictionary in, the vectors in S are called generators for $\langle S\rangle$. S is also called a generating set.

Problem 6. *Make sense of this definition. Get yourself a familiar vector space, some small sets of vectors, and find such subspaces. Try to find a way to describe all elements in the vector subspace generated by* S *in terms of the vectors in* S.

You may want to make use of the following notation: if $v_1, \ldots, v_n$ are vectors in V, and $\alpha_1, \ldots, \alpha_n$ are real numbers, then the vector

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

is called a **linear combination** of $v_1, \ldots, v_n$.

3 Linear Independence

Consider the vector space $\mathbb{R}^3$, which is isomorphic to (coordinatizing) three dimensional euclidean space.

Problem 7. *Take the vector* $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *What are all the points in space that you can "reach" by stretching* v_1 *and adding* v_1 *to itself?*

In other words, describe the vector subspace generated by the vector v_1. Now, we haven't defined the notion of dimension rigorously yet (we will soon), but we have an intuitive notion of dimension. What is the dimension of this vector subspace $\langle v_1 \rangle$?

Problem 8. *The vector* $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *is just the same as before. Now we introduce a new vector* v_2. *For each possibility below, what are all the points in the plane that you can "reach" by stretching and adding* v_1 *and* v_2*?*

1. $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2. $v_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$
3. $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

In other words, what is the intuitive dimension of $\langle v_1, v_2 \rangle$?

Problem 9. *Now let we bring in a third vector. Let* $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, *and answer the same question as above for the following choices of* v_3.

1. $v_3 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$
2. $v_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

3. $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ok, now let us make a stretch of fantasy here. If you were to define a mathematical concept called **linear independence** to assign to collection of vectors, based on the example above, which collections do you think would be linearly independent? Can you actually make up some sort of a definition for this concept?

Problem 10. *Still thinking about three dimensional space as your ambient vector space, can you have* 4 *linearly independent vectors?*

OK, so now we have a decent intuition for what we are looking for, but we need to make things precise. For once, we actually do not have a mathematical definition of dimension yet, so we can't really use such a concept. Secondly, we want to be able to apply our reasoning to "funky" vector spaces, where we can't really draw pictures or so. So what we want to hunt for now is some sort of definition that is equivalent to this one in the case of $\mathbb{R}^3$, but that relies ONLY on those features that are common to all vector spaces: the two operations.

Problem 11. *Look back at all examples above. Suppose that you want to take a stroll in your vector space (aka a linear combination), starting and ending at the zero vector. Remember you are allowed to use each vector in your generating set only once. In the various examples above, how many different such strolls can you take? Can you find a difference between the examples where you "want" the vectors to be linearly independent, and the examples where you "want" the vectors to be dependent?*

Finally, let's do it.

Problem 12. *Find a definition for when a set of vectors is linearly independent that relies only on the concept of linear combinations.*

Problem 13. *Give a few example of sets of vectors that are and of sets that aren't linearly independent in some interesting vector spaces (e.g.* $\mathbb{R}[x]$*, matrices, functions...)*