

Projects: the classification theorem for closed compact surfaces.

Renzo's math 590

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The goal of this set of projects is to prove and fully understand the following amazing theorem.

Classification of closed compact surfaces. *Any closed compact surface is homeomorphic to a surface in the following list:*

- S^2 : a sphere;
- T_g : a connected sum of g tori;
- a connected sum of m projective planes.

Further, no two surfaces in this list are homeomorphic to each other.

FACT: existence of a finite triangulation

Assume that any closed compact surface admits a finite triangulation. This seems sort of obvious, but it is actually quite tricky to prove formally. If any of you feel up for the challenge, go for it! You want to exploit compactness to get a finite open cover where every open set is homeomorphic to a disc in the plane...but then you will have to worry that such discs don't intersect in bizarre ways. So you will have to refine your cover appropriately.

GROUP A: from surfaces to polygons

Given a surface with a finite triangulation, show that you can associate an identification polygon to it.

At this point we have shown that any surface can be represented by a polygon. Now we need to show that any polygon represents a surface in our "list". We start by proving an apparently slightly weaker statement:

Any identification polygon is either a sphere or the connected sum of tori and projective planes

We proceed by induction. We have shown that any polygon with 2 or 4 sides is “good”.

GROUP B: adjacent edges

Show that if there are two adjacent edges with the same label, your polygon is (inductively) good.

GROUP C: the non orientable case

Show that if your polygon represents a non-orientable surface, then it is good because it can be seen as a connected sum of a $\mathbb{P}^2$ and something inductively good.

GROUP A: the orientable case

Show that if your polygon represents a orientable surface, then it is good because it can be seen as a connected sum of a torus and something inductively good.

GROUP B: Wrapping the theorem up

Now you have that any surface is a connected sum of tori or projective planes. Show that you can restrict your attention to only connected sums of tori or connected sums of projective planes. Show that any two such surfaces are not homeomorphic to each other.

GROUP C: Recognizing a polygon

Devise an algorithm to easily recognize what surface does an identification polygon represent.