

Projects

Renzo's math 590

October 5, 2007

Instructions

Some more projects! The protocol will be the same as before:

- 1. Each group should nominate a representative who, over the course of the next two lectures, will have 15 minutes to present some of your work. **Notice:** it is not necessary that the representative presents ALL that you have done. And it shouldn't be like... "go, go, go" until you run out of time: the whole group should help the representative select topics to design an effective 15 minute delivery of the essence of your work.*
- 2. Each group will write up the work done (all of it). This document will be graded and compiled into a booklet that will be made available to everybody!*

Group A

Project 1

Prove the following theorem:

Theorem 1. *$X \times Y$ is compact if and only if both X and Y are compact.*

One direction of this theorem is pretty tricky. Here's a hint. Starting from an arbitrary open cover of **basic open sets** $U \times V$, first get yourself a finite subcover of every slice $\{x\} \times Y$. Then argue that you can use only finitely many of these covers of slices to cover all of $X \times Y$. **It is very important that you draw a picture of the situation.**

Project 2

We have seen in class that a torus (i.e. the glaze of a doughnut) is homeomorphic to the product space of a circle times a circle. Here are another couple ways to have the torus appear.

Problem 1. Start from the square $\{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Define the following equivalence relation on the square:

- interior points are only equivalent to themselves.
- boundary points are identified to their “pac-man” partners. I.e., for any x and for any y :

$$- (x, 0) \sim (x, 1);$$

$$- (0, y) \sim (1, y).$$

Show that the identification space obtained by considering the set of equivalence classes and inducing a topology via the natural function onto it is homeomorphic to the torus.

Problem 2. Consider the function:

$$\begin{aligned} \mathbb{Z}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (m, n), (x, y) &\mapsto (x + m, y + n) \end{aligned}$$

Show that this function is a group action on the plane, and the quotient space is homeomorphic to the torus.

Group B

Project 3

Prove the following theorems:

Theorem 2. $X \times Y$ is path connected if and only if both X and Y are path connected.

Theorem 3. $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Project 4

The projective plane is a space parameterizing all lines through the origin in $\mathbb{R}^3$. The following are some useful ways to visualize the set of points in this space.

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\{(X, Y, Z) = (\lambda X, \lambda Y, \lambda Z)\}} = \frac{\text{Sphere}}{\{P = -P\}}$$

There are three natural functions from the plane to $\mathbb{P}^2$:

$$\begin{aligned} \varphi_z : \quad \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (x, y) &\mapsto (x : y : 1) \end{aligned}$$

$$\begin{aligned}\varphi_y : \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (x, z) &\mapsto (x : 1 : z)\end{aligned}$$

$$\begin{aligned}\varphi_x : \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (y, z) &\mapsto (1 : y : z)\end{aligned}$$

Problem 1. Define a topology on $\mathbb{P}^2$ as follows: a set $U \subset \mathbb{P}^2$ is open if all of its preimages $\varphi_x^{-1}(U)$, $\varphi_y^{-1}(U)$, $\varphi_z^{-1}(U)$ are open sets of the plane with the euclidean topology. Show that this is indeed a topology, that it makes the three φ maps continuous, and that the images $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$, $\varphi_z(\mathbb{R}^2)$ become open dense sets of $\mathbb{P}^2$

Problem 2. Define a natural map from the sphere to $\mathbb{P}^2$. Check that this map is continuous. Use this to show that $\mathbb{P}^2$ is compact.

Group C

Compact spaces are very nice for many different reasons, most of which will only become evident much further along in your mathematical journey. Given a non compact space X , one would sometimes like to construct a homeomorphism from X to an open dense set of a compact space $\tilde{X}$. (Informally, this means, adding some points so that X "becomes compact".) $\tilde{X}$ is called a **compactification** of X .

Assume throughout that X is:

- Hausdorff.
- **Locally compact:** this simply means that every point of x is contained in a compact neighborhood.

Define the **one-point compactification** $\tilde{X}$ of X :

as a set: $\tilde{X}$ has precisely one more point than the points of X . We suggestively call this point **infinity**.

$$\tilde{X} = X \cup \{\infty\}$$

topology: we discuss open sets according to whether they contain ∞ or not:

- if $\infty \notin U$, then U is open in $\tilde{X}$ if and only if it is open in X .

- if $\infty \in U$, then U is open in $\tilde{X}$ if and only if its complement is compact.

Problem 1. *If (each connected component of) X is not compact, then prove that:*

- X is homeomorphic to an open dense set in $\tilde{X}$.
- $\tilde{X}$ is compact.
- $\tilde{X}$ is connected.
- $\tilde{X}$ is Hausdorff.

What happens if you start with an X that is already compact?

Problem 2. *Show that if X and Y are homeomorphic, so are their one point compactifications. Show that the converse is not true: find an (or even better, some) example(s) of two spaces X and Y that are NOT homeomorphic, but such that their one point compactifications are homeomorphic.*

Problem 3. *Show that the one point compactification of the plane (with euclidean topology) is (homeomorphic to) the sphere.*

Problem 4. *What is the one point compactification of:*

- an open disc in the plane.
- a closed disc in the plane minus a point.
- an open interval of the real line.
- a bunch of open intervals of the real line.