

Projects

Renzo's math 590

September 11, 2007

Instructions

Below you find a list of projects, whose goal is to explore and expand the concepts presented in class in the first week or so of lectures. You will be working in groups. Each group is assigned two projects.

1 Group A: On homeomorphisms

Prove that the sphere minus one point is homeomorphic to the whole plane. Prove that the plane is homeomorphic to the open disc of radius 1.

Here, all spaces are assumed to have the euclidean topology. You can (and should) prove this statements in as many ways as possible. Can you construct homeomorphisms via geometric constructions? Give formulae? One suggestion, in order to get some inspiration, is to start from showing the analogous result in dimension 1. The open interval $(0, 1)$ is homeomorphic to all of $\mathbb{R}$.

2 Group B: More on homeomorphisms

Subdivide the letters of the alphabet into homeomorphism classes.

Assume the letters being given the topology induced from the euclidean topology on $\mathbb{R}^2$. (Also assume that the letters are really one-dimensional gadgets). Use the following figure as the rule for how to write the letters. Pay especially close attention and care in motivating when two letters are NOT homeomorphic. “Come on, can’t you tell?” and “Geez, we tried real hard and couldn’t find any homeomorphisms” are not acceptable explanations, whereas a little bit of handwaving and conjecturing of topological properties is allowed.

A B C D E F G H I J
 K L M N O P Q R S
 T U V W X Y Z

3 Group C: Vocabulary

The goal of this project is to build up a little more topology vocabulary. Below, I give you a list of concepts explained in the setting of metric spaces. The first step will be for you to generalize to definitions that only involve topological concepts.

Interior: the interior of a set A (denoted A°) is the set of all points $a \in A$ such that there is a little ball centered at a and entirely contained in A .

Closure: the closure of a set A (denoted $\overline{A}$) is the set A union all of its limit points.

Frontier: the frontier of a set A (denoted ∂A) is the set of limit points of A that are not interior points of A .

Problem 1. *Prove that the interior of a set is open. Prove that the closure and the frontier of a set are closed.*

Problem 2. *Prove:*

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
2. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
3. $\overline{\overline{A}} = \overline{A}$.
4. $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.
5. $(A \cap B)^\circ = A^\circ \cap B^\circ$.
6. $A^{\circ\circ} = A^\circ$.

Show, that equality need not hold in 2 or 4.

Problem 3. *Let $A = \mathbb{Q} \subset \mathbb{R}$, and assume $\mathbb{R}$ has the euclidean topology. What are A° , $\overline{A}$ and ∂A ?*

Problem 4. The *characteristic function* of a set $A \subseteq X$ is the real valued function on X which assigns the value 1 to points of A , 0 to points of $X \setminus A$. Describe the frontier of A in terms of this function.

Problem 5. How do interior, closure and frontier work for subsets of spaces with the discrete or with the stupid topology?

4 Group C: Base for a Topology

Consider a topological space (X, τ) . A **base for a topology** is a subset $\beta \subseteq \tau$ such that any open set in τ can be obtained as the union of sets in β .

Problem 1. Show that alternatively, one can define a base of a topology by requiring that for any point $x \in X$ and any open set $O \ni x$, there exists a set $B \in \beta$ that contains x and is contained in O .

Problem 2. Show that open balls of any radius form a basis for the euclidean topology on $\mathbb{R}^n$. How about open cubes?

Problem 3. Show that if X has a countable base for its topology, then X contains a countable dense subset. Refer to project 6 for the definition of a dense set.

5 Group B: On the concepts of Fineness and on Separating Stuff

Assume you have two topologies τ, τ' on a set X . We say that τ is **finer** than τ' if every set in τ' is also in τ (equivalently, we say that τ' is **coarser** than τ).

Problem 1. Given a set X , which are the finest and the coarsest topologies that you can define on X ? If you know that the identity $Id : (X, \tau) \longrightarrow (X, \nu)$ is a continuous function, what can you say about the topologies τ and ν ?

What functions $f : X \longrightarrow Y$ are continuous if:

- X has the stupid topology and Y does NOT have the stupid topology?
- X has the discrete topology?
- Y has the stupid topology?
- Y has the discrete topology (careful, here the answer is a little bit more sophisticated than in the previous cases)?

A topological space X is called **Hausdorff** if, for any pair of points $x, y \in X$, there are two open sets O_x, O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$.

Problem 2. Show that if a space X has a **metrizable** topology, then it is Hausdorff.

A topological space is called **T1** if for any pair of points $x, y \in X$, there is an open set O_x such that $x \in O_x$, $y \notin O_x$.

Problem 3. Show that being T1 is equivalent to the fact that points are closed sets.

Problem 4. Can you think of a topological space that is T1 but not Hausdorff? After you've thought about it for a while, you may want to seek inspiration by reading project 6.

6 Group A: Dense Sets

Let X be a topological space. A set A is called **dense** if its closure is X (see project 3 for the definition of closure).

Problem 1. Give some example of some dense sets in some familiar topologies. What sets are dense in the discrete topology?

Consider the **finite complement topology** on a set X , where the open sets are $\emptyset$, X and those sets such that their complement is finite.

Problem 2. What sets are dense in the finite complement topology? Make sure to discuss both the case where X is a finite or an infinite set.

Problem 3 (The Zariski Topology). Consider the following topology on the plane: a set is defined to be closed if it is the zero set of a system of polynomials in two variables, i.e. $C = \{(x, y) \text{ such that } f_1(x, y) = f_2(x, y) = \dots = 0\}$.

1. Verify that this is indeed a topology on the plane.
2. Check that all open sets that are not $\emptyset$ are dense.
3. Define in the appropriate way the Zariski topology on the line. What topology is it? (i.e. have you seen it before?)