

FINAL PROJECTS

Renzo's math 590

November 26, 2007

The following list of individualized projects will replace a final exam in this class. Here are the rules:

- 1. You are given two hours in class to work on your problem. You are encouraged to discuss with each other - work in groups - ask me questions etc...*
- 2. Starting friday, each person is given 20 minutes to present the work done. The oral presentation will be formally evaluated (worth 40 %).*
- 3. Everyone must write a question or a comment on every presentation. These will be collected at the end of each hour, and be part of your evaluation (20 % of grade).*
- 4. Everyone must write a L^AT_EX document of their work, and turn it in on Monday, Dec 10th. These documents will be evaluated (40 %), and finally compiled into one document that will be returned to everybody.*

NOTE: assume all spaces are path connected, unless otherwise specified!

1 Fundamental Groups

Joel

Definition: The **wedge product** of two spaces X and Y is the identification space obtained by choosing a point $x \in X$, a point $y \in Y$ and then identifying the two points.

$$X \wedge Y = X \bigcup_{x=y} Y$$

Draw a picture of what this means.

Problem 1 (Does the choice of the points matter?). *Show that the wedge product is not well defined at the level of homeomorphism classes, but it is at the level of homotopy equivalence classes. This means that by choosing different x and y to identify you may obtain spaces that are NOT homeomorphic, but are homotopy equivalent.*

Problem 2. Assume you know the fundamental groups of X and Y . What is

$$\pi_1(X \wedge Y)?$$

Problem 3. The fundamental group of a product space is isomorphic to the product of the fundamental groups of the factors.

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

Hints:

1. use the projections to construct a function $f : \pi_1(X \times Y) \rightarrow \pi_1(X) \times \pi_1(Y)$.
2. show that this function is a group homomorphism.
3. show that it is onto.
4. show that it is one to one.

Yu-Sin

Problem 1. Use the Seifert-Van Kampen theorem to calculate the fundamental group of any compact closed surface X .

Problem 2. Use your knowledge of identification polygons for representing surfaces to show that a presentation of a group is NOT unique.

Give some simple examples of the above phenomenon and try to explain what is happening!

Keilor

Problem 1. Prove that the fundamental group of a bouquet of n circles is the free group F_n .

Problem 2. Construct the universal cover of the bouquet of n circles. Do the cases $n = 1, 2$ very explicitly.

This tree (why is it a tree?) is called the **Cayley graph** of the free group F_n (denoted C_{F_n}).

Problem 3. Show that the free group F_n acts on its Cayley graph, and that the quotient space is the bouquet of n circles

Problem 4 (Hard challenge!). Take some specific subgroup H of a free group F_n (start as easy as you can). Try to observe what happens when you take the quotient of the Cayley graph C_{F_n} by the action of the subgroup H . Try to make a conjecture describing the situation in general.

Kimmy

Definition: a (finite) **CW-complex** is a topological space X obtained from the following procedure:

1. start from a finite set of points X_0 . (this is called the 0-skeleton of the complex).
2. attach a (finite) bunch of segments to these points. You can attach them so that the two endpoints are attached to different points or to the same point. All endpoints must be attached. This graph is called the 1-skeleton of the complex.
3. attach a (finite) number of discs to the 1-skeleton. The discs are attached along their boundary circle. In reality these maps are allowed to be pretty wild, (so long as they are continuous) but just not to get ourselves a headache let us assume that these maps consist of winding the circle around the 1-skeleton "reasonably". By this I mean the following. Any continuous map from a circle to the 1-skeleton determines an element α in the fundamental group of the 1-skeleton: however there are some functions that "naturally" represent α . Allow yourself to use these functions to do the gluing. This is called the 2-skeleton.
4. got the trick? Now if you want, attach three dimensional discs to the 2-skeleton to obtain the 3-skeleton and so on and so forth. For our purposes now we can stop at things that don't go beyond the 2-skeleton, but this is just the beginning of a rich and interesting theory.

Problem 1. *Give some examples of CW complexes. Show that all of our surfaces are CW-complexes, but that not all CW complexes (going up to the 2-skeleton) are surfaces.*

Problem 2. *Given a group G with a finite presentation, construct a topological space whose fundamental group is G .*

2 Covering Spaces

Mike H.

Let $f : Y \rightarrow X$ be a covering map.

Problem 1. *Show that for any $x \in X$ the number of preimages doesn't change.*

Call the number of preimages of any point of the cover the **degree** of the covering map.

Problem 2. *If X and Y are surfaces, how are the Euler characteristics of X and Y related?*

Problem 3. *Show that any non-orientable surface admits an orientable double cover (i.e. a cover of degree 2).*

Gyemin

We saw in class that a covering map $f : Y \rightarrow X$ realizes $\pi_1(Y)$ as a subgroup of $\pi_1(X)$, i.e. that the map $\pi_1(f)$ is an injective group homomorphism.

Problem 1. *Deduce that you can't have a covering space from a connected sum of g tori to the sphere.*

Problem 2. *Use Mike's work on Euler characteristics to deduce even more constraints between surfaces in order for degree d covers to exist. What surfaces can cover the torus?*

Problem 3. *Find all coverings of a torus by a torus. Start by describing subgroups of the fundamental group of the torus. Now for a given subgroup H , look for a way to cover the torus by a torus in such a way that the image of the fundamental group via the induced homomorphism is precisely H .*

Sung Jin

Definition: the **symmetric group** on d letters (S_d) is the group of permutations of the elements of a set with d elements.

Problem 1. *Show that this is a group of order $d!$. Illustrate how this group works with some examples. Show that in general S_d is not an abelian group.*

Let $f : Y \rightarrow X$ be a covering map of degree d . We construct a group homomorphism

$$\phi : \pi_1(X) \longrightarrow S_d$$

called the **monodromy representation** of the cover f .

Pick a point $x_0 \in X$ and label its preimages $y_1, \dots, y_d$. A loop α based at x_0 can be lifted to the cover and any y_i can be chosen as a basepoint. Let $\tilde{\alpha}_i$ be the lift of α such that $\tilde{\alpha}_i(0) = y_i$. Then:

$$y_i \rightarrow \tilde{\alpha}_i(1)$$

determines a permutation of the set $y_1, \dots, y_d$, i.e. an element of S_d .

Problem 2. *Give some concrete examples of this construction. Start with really simple covers, and work your way up. Give some justification (without entering too much detail) of why this construction indeed gives you a group homomorphism.*

Problem 3. *What happens to ϕ if you change the labels of your set $y_1, \dots, y_d$?*

This is the beginning of another cool story. In fact not only does a cover determine (up to the ambiguity discussed above) a monodromy representation, but it can be shown that the representation is sufficient information to recover the cover. So we are building an interesting equivalence between the geometry of covers and the representation theory of the symmetric group. But this is another story...

3 Some Cool Theorems

Mike B.

Problem 1 (Brower fixed point theorem). *Any continuous function $f : D^2 \rightarrow D^2$ has a fixed point, i.e. a point $x \in D^2$ such that $f(x) = x$.*

You go about this by contradiction. Assume the existence of a map without fixed points, and use this to make a geometric construction that ...

Problem 2 (Very Baby Borsuk-Ulam). *Let*

$$f : S^2 \rightarrow \mathbb{R}^2$$

be a map such that:

- *the image of the equator is a simple closed curve;*
- *f restricted to the northern hemisphere is one to one;*
- *f restricted to the southern hemisphere is one to one.*

Show that f must identify two antipodal points.

Brian

The end-goal of this project is to prove the full fledged Borsuk-Ulam theorem:

Problem 1 (Borsuk-Ulam theorem). *For any continuous map $f : S^2 \rightarrow \mathbb{R}^2$, there is a point x such that*

$$f(x) = f(-x).$$

Start by showing that, if instead of $\mathbb{R}^2$ you ask the function to land in $\mathbb{R}$, then it is just a matter of intermediate value theorem problem (so not too exciting). How about if you ask for a function to $\mathbb{R}^3$?

You now may follow the following steps.

Definition: a function from S^n to S^m is **antipode preserving** if, for any point x ,

$$f(-x) = -f(x)$$

Problem 2. *Show that any antipode preserving map:*

$$f : S^1 \rightarrow S^1$$

is not null-homotopic.

This is pretty tricky. Start by noticing that an antipode preserving map descends to a map between $\mathbb{Z}_2$ quotients of the circles (which are still circles). Get a commutative diagram of functions. Show the bottom arrow gives rise to a nontrivial homomorphism on the fundamental groups and deduce that the top arrow must too!

Problem 3. *There is no continuous antipode preserving map:*

$$g : S^2 \rightarrow S^1.$$

Think of what happens when you restrict such a map to the upper hemisphere.

Zhao

Problem 1 (Fundamental theorem of algebra). *Any complex polynomial of degree d has at least one root. (from which it follows that, counting multiplicities, it has exactly d roots)*

Restrict the map $f(z) = z^d$ to a circle of radius r centered at the origin. What element does it determine in the fundamental group of $\mathbb{C} \setminus 0$?

Now, the idea that you want to make precise is the following. If you have a polynomial of degree d , and let r grow very large, then the leading term "dominates" over the others and the restriction to the function to a circle of radius r will have the same homotopy type of z^d . And now: what happens if the polynomial has no zeroes inside the circle of radius r ?