The GAUSS-BONNET THM

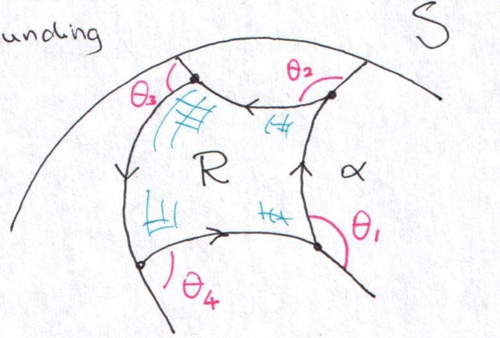
Thm: Let S be a cpt, boundaryless surface (regular). Then

$$\boxed{\iint_S K \, d\sigma = 2\pi \chi(S)}$$

GAUSSIAN CURVATURE

TOPOLOGICAL EULER CHARACTERISTIC.

LOCAL GAUSS-BONNET THM: let α be a piecewise regular curve bounding a region $R \subseteq S$, with R homeomorphic to a disc. Then:



$$\boxed{\oint_\alpha \kappa_g \, ds + \iint_R K \, d\sigma + \sum \theta_i = 2\pi}$$

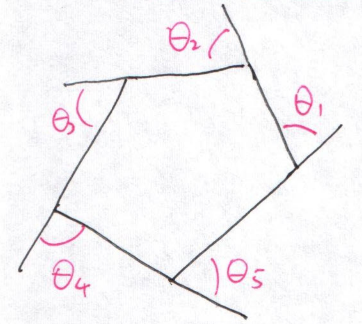
THIS IS ONLY MEANT TO DENOTE A SCALAR PIECEWISE INTEGRAL [illegible] [illegible] ON RANGE ST. α TRACES A FULL CONTOUR OF THE CLOSED CURVE

GEODESIC CURVATURE of α

OUTSIDE ANGLES

EVIDENCE // SPECIAL CASES:

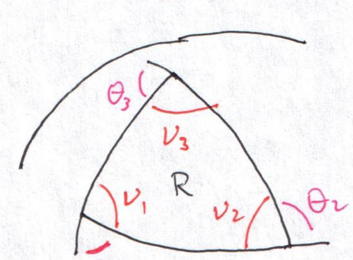
① PLANE POLYGONS



$$\boxed{\sum \theta_i = 2\pi}$$

CAN YOU PROVE THIS FOR AN ARBITRARY POLYGON

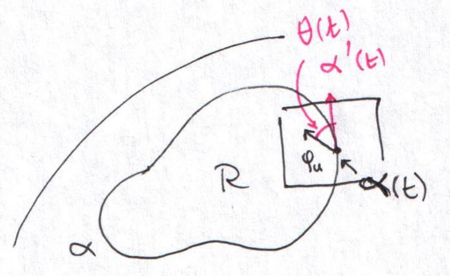
② GEODESIC TRIANGLES (Gauss)



$$\boxed{\sum \nu_i = \pi + \iint_R K \, d\sigma}$$

CHECK THIS on THE SPHERE.

③ REGULAR CURVES



φ_u NORMALIZED TO HAVE LENGTH 1

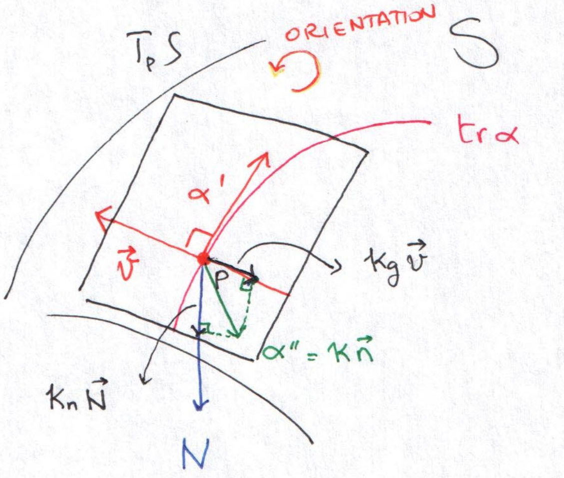
$$\boxed{\kappa_g = \left| \frac{d \tilde{\varphi}_u}{dt} \right| + \frac{d}{dt} \theta(t)}$$

INTEGRATING AND USING STOKES' THM

$$\boxed{\oint \kappa_g \, ds + \iint_R K \, d\sigma = 2\pi}$$

INGREDIENTS

① Geodesic Curvature



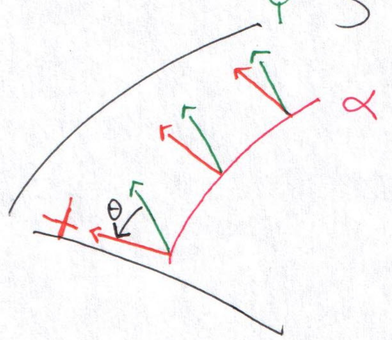
Let α be a reg curve parameterized by arclength with $tr\alpha \subseteq S$, an oriented surface. For every point $\underset{\alpha(t)}{\underset{\shortparallel}{P}} \in S$, the orientation of S defines a unique unit vector $\vec{v} \in T_pS$ with $\boxed{\vec{v} \perp \alpha'}$. Then the geodesic curvature can be defined as:

① $\kappa\vec{n} = \kappa_n\vec{N} + \kappa_g\vec{v}$

② $\kappa_g = D_{\alpha'}\alpha' \cdot \vec{v}$

→ NOTE THAT κ_g CAN BE <0, AND IN GENERAL IT SWITCHES SIGN IF YOU CHANGE THE ORIENTATION OF S.

↳ Note that $\kappa_g = \pm|D_{\alpha'}\alpha'|$ depending on Ω of S.



LEMMA: α as before, X a unit vector field, Y a parallel unit vector field, θ = angle between X and Y. Then

$$|D_{\alpha'}X| = \pm\theta'$$

→ depending on the orientation of S

NOTE:
- θ is measured from Y to X.
- the lemma can be strengthened as follows: If $\vec{v}$ is the unit vector ⊥ to X determined by the orientation, then

$$\boxed{D_{\alpha'}X \cdot \vec{v} = \theta'} \leftarrow$$

EXERCISE FOR YOU!

PROOF: shortcut: denote $D_{\alpha'}\bullet = \bullet'$.

$\boxed{X \text{ unit} \Rightarrow X' \perp X}$

$$X\cdot Y = \cos\theta$$

$$\Rightarrow X'\cdot Y + \underbrace{\cancel{XY'}}_{0 - Y \text{ PARALLEL}} = -\sin\theta\,\theta' \qquad |X'|\cos\left(\tfrac{\pi}{2}\pm\theta\right) = -\sin\theta\,\theta'$$

$$\Downarrow$$

$$\boxed{|X'| = |\theta'|}$$

· **COROLLARY:** Let X, Y be $\boxed{\text{UNIT}}$ vector fields along a curve α.

Denote $[X'] = X' \cdot \vec{v}$, where $\vec{v}$ is the vector orthogonal to X determined by the orientation.

Then:

$$\boxed{[Y'] - [X'] = \theta'} \rightarrow \boxed{\text{DERIVATIVE of THE ORIENTED ANGLE FUNCTION FROM } X \text{ to } Y.}$$

COROLLARY^2

$$\boxed{K_g = \left[\left(\frac{\varphi_u}{|\varphi_u|}\right)'\right] + \theta'}$$

STEP 1: what K_g has to do with angles!

GAUSSIAN CURVATURE

Assume for simplicity that φ is an orthogonal parameterization. $(F=0)$

Let $\tilde{\varphi}_\bullet$ be a vector field in $\mathbb{R}^3$, differentiated in $\mathbb{R}^3$ and then projected onto T_pS.

Let φ be a vector field on S, differentiated covariantly in S!

in $\mathbb{R}^3$ mixed partials agree

$$\begin{aligned} \tilde{\varphi}_{uuv} &= \varphi_{uuv} + e N_v \\ \tilde{\varphi}_{uvu} &= \varphi_{uvu} + f N_u \end{aligned} \Longrightarrow \boxed{\varphi_{uuv} - \varphi_{uvu}} = \varphi_u (f a_1^1 - e a_2^1) + \varphi_v (f a_1^2 - e a_2^2) = \boxed{-EK\varphi_v}$$

The last equality is obtained by using:

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

$$K = \det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

CRASH COURSE on STOKES:

Definitions

1-form: $\omega(u,v) = f(u,v)\,du + g(u,v)\,dv$

2-form: $\Omega(u,v) = H(u,v)\,du\wedge dv$

d : $d\omega = (f_v - g_u)\,du\wedge dv$

Line integration $\alpha(t) = (u(t), v(t))$ a regular curve par by arclength:

$$\Rightarrow \oint_\alpha \omega := \int_{t_0}^{t_1} (f u' + g v')\,dt \qquad (\star)$$

Stokes:

$$\boxed{\oint_\alpha \omega = \iint_R d\omega}$$

LEMMA Let $\bar{\varphi}_u = \dfrac{\varphi_u}{|\varphi_u|}$, $\bar{\varphi}_v = \dfrac{\varphi_v}{|\varphi_v|}$ be the normalized φ_u, φ_v vector fields. Keep assuming φ is an orthogonal parameterization ($F=0$, $\varphi_u \perp \varphi_v$).
$\begin{pmatrix} |\varphi_u| = \sqrt{E} \\ |\varphi_v| = \sqrt{G} \end{pmatrix}$

$$\bar{\varphi}_u' \cdot \bar{\varphi}_v = \left(u'\Big[\overbrace{\Big(\frac{1}{\sqrt{E}}\Big)_u \varphi_u + \frac{\varphi_{uu}}{\sqrt{E}}}^{\text{☺}}\Big] + v'\Big[\overbrace{\Big(\frac{1}{\sqrt{E}}\Big)_v \varphi_u + \frac{\varphi_{uv}}{\sqrt{E}}}^{\text{☹}}\Big]\right) \cdot \bar{\varphi}_v$$

$$= \left(u'(\text{☺} \cdot \bar{\varphi}_v) + v'(\text{☹} \cdot \bar{\varphi}_v) \right)$$

$$\Rightarrow \int_{t_0}^{t_1} \bar{\varphi}_u' \cdot \bar{\varphi}_v \, ds \overset{(\star)}{=} \oint_\alpha (\text{☺} \cdot \bar{\varphi}_v)\,du + (\text{☹} \cdot \bar{\varphi}_v)\,dv$$

$$\overset{(\text{Stokes})}{=} \iint_R \left[(\text{☺} \cdot \bar{\varphi}_v)_v - (\text{☹} \cdot \bar{\varphi}_v)_u \right] du\wedge dv = (\text{◉})$$

CAREFUL: all of this is happening in $\mathbb{R}^2$, the local coordinate patch. any differentiation I am doing is covariant differentiation!!!

$\varphi_u \perp \overline{\varphi}_v$

$$\left(\text{☺}\cdot\overline{\varphi}_v\right)_v = \left[\cancel{\left(\frac{1}{\sqrt{E}}\right)_{uv}\varphi_u\cdot\overline{\varphi}_v} + \cancel{\left(\frac{1}{\sqrt{E}}\right)_u\varphi_{uv}\cdot\overline{\varphi}_v} + \cancel{\left(\frac{1}{\sqrt{E}}\right)_v\varphi_{uu}\cdot\overline{\varphi}_v} + \left(\frac{1}{\sqrt{E}}\right)\varphi_{uuv}\cdot\overline{\varphi}_v + \cancel{\text{☺}\cdot\left(\overline{\varphi}_v\right)_v}\right]$$

☺ is in the φ_v direction

$(\overline{\varphi}_v)_v$ is $\perp$ to φ_v

$$-\left(\text{☹}\cdot\overline{\varphi}_v\right)_u = -\left[\cancel{\left(\frac{1}{\sqrt{E}}\right)_v\varphi_{uu}\cdot\overline{\varphi}_v} + \cancel{\left(\frac{1}{\sqrt{E}}\right)_u\varphi_{uv}\cdot\overline{\varphi}_v} + \left(\frac{1}{\sqrt{E}}\right)\varphi_{uvu}\cdot\overline{\varphi}_v\right]$$

$$\Longrightarrow \quad (\bullet) = \iint_R \frac{1}{\sqrt{E}}\left[\left(\varphi_{uuv}-\varphi_{uvu}\right)\cdot\overline{\varphi}_v\right]\,du\wedge dv =$$

$$= \iint_R \frac{1}{\sqrt{E}}\left(-EK\right)\varphi_v\cdot\overline{\varphi}_v\,du\wedge dv =$$

$$= -\iint_R \sqrt{EG}\,K\,du\wedge dv = -\iint_R K\,d\sigma$$

LOCAL GB :

From COROLLARY 2

$$\int_{s_0}^{s_1} k_g \, ds = \int_{s_0}^{s_1} \bar{\varphi}_u{}' \cdot \bar{\varphi}_v \, ds + \int_{s_0}^{s_1} \theta' \, ds$$

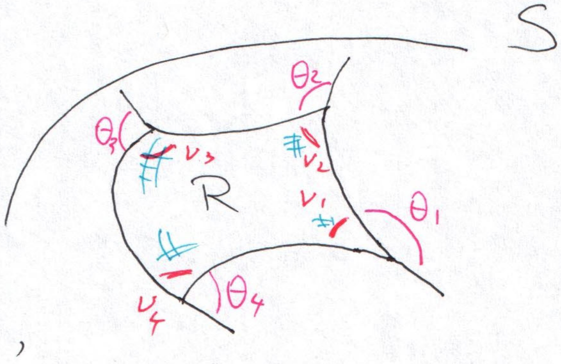
$$= -\iint_R K \, d\sigma + \underbrace{\theta \Big|_{s_0}^{s_1}}_{\boxed{2\pi - \sum \theta_i}}$$

the "change in angle" along a smooth simple closed curve is 2π. At every corner, you lose θ_i

Reorganizing the terms, you get the formula in the local GB theorem.

LOCAL to GLOBAL

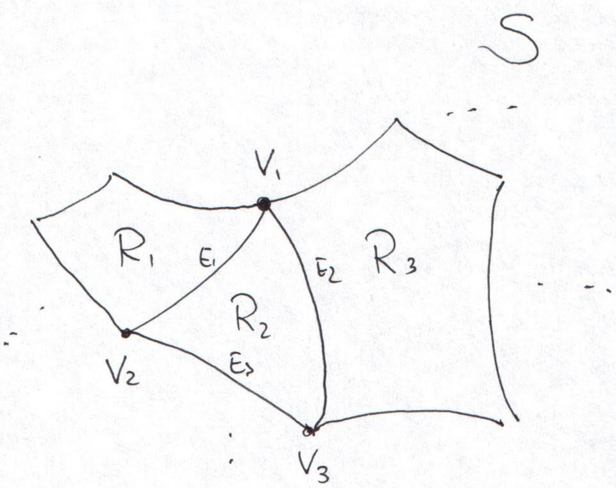
① Rewrite local GB as



$$\oint_\alpha k_g \, ds + \iint_R K \, d\sigma = 2\pi + \sum (\nu_i - \pi),$$

where ν_i are the internal angles.

② Decompose the surface into regions R_i s.t.
- → each R_i is homeo to a disc
- → each R_i is inside a coord patch.



Summing local G.B. over all R_i's, we get

$$\boxed{\iint_S K \, d\sigma = 2\pi \left(\# R_i\text{'s} - \# E + \# V \right) = 2\pi \, \chi(S)}$$

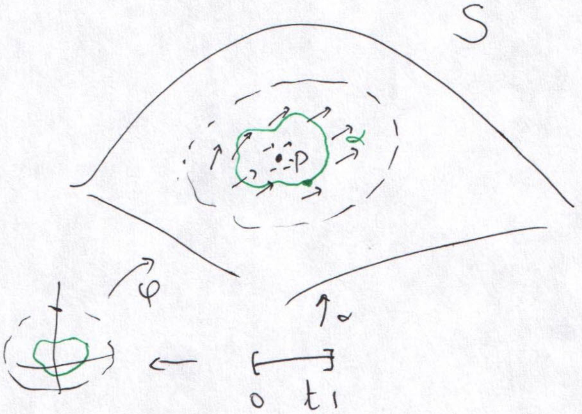
THE INDEX THEOREM

Let S be a $\overset{\text{COMPACT}}{\vee}$ surface, and X a vector $\overbrace{\text{field}}^{\text{with a discrete} \Rightarrow \text{finite } \# \text{ of } 0\text{'s}}$ on S.

For $p \in S$, let $\varphi : U \longrightarrow S$ be a chart around p, and let $\theta(t)$ be the angle between φ_u and X along a simple "small" regular curve around p.

↓

SMALL HERE MEANS THAT INSIDE the TRACE of α THERE IS NO ZERO of X OTHER THAN (POSSIBLY) p.



Def:

$$\boxed{\text{Ind}_p X = \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt}$$

Note: ① The index @ p does NOT depend on the parameterization φ.

→ Let Z be an auxiliary parallel vector field.

→ Let $\bar{X} = \frac{\{X\}}{|X|}$

$\Rightarrow \theta(t) = \text{angle}_{Z \to \bar{X}} - \text{angle}_{Z \to \varphi_u}$

$$\boxed{\int \theta'(t)\,dt = \int (\text{angle}_{Z \to X})'\,dt + \iint_R K\,d\sigma} \quad (\circ)$$

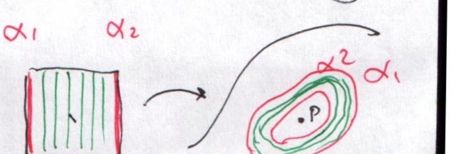
INDEPENDENT of PARAMETERIZATION

② The index is an integer (by FUND. THM of CALCULUS)

③ The index @ p is independent of choice of α ~~[illegible]~~

$\alpha_1 \sim \alpha_2$ if there is a map

$A : I \times I \longrightarrow X$ s.t. $A(0,t) = \alpha_1$ $X(A(s,t)) \neq 0$
$A(1,t) = \alpha_2$



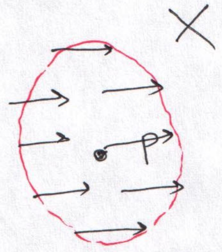
Then $\text{ind}: [0,1] \longrightarrow \mathbb{Z}$

$$s \longmapsto \frac{1}{2\pi}\int_0^1 \theta_s'(t)\,dt$$

is a continuous function from a connected to a discrete set => constant.

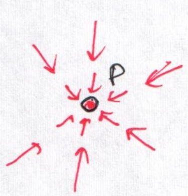
④ If $X(p) \neq 0 \Rightarrow \text{Ind}_p(X) = 0$

ZOOMING IN

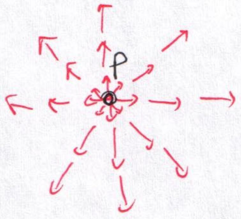


Examples

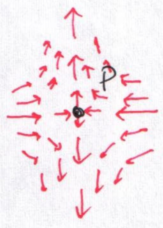
SINK $\text{ind}_p = 1$



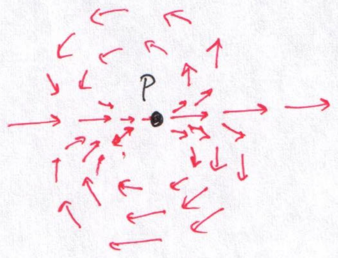
SOURCE $\text{ind}_p = 1$



SADDLE $\text{ind}_p = -1$



DIPOLE $\text{ind}_p = 2$



THEOREM: S a compact surface, X a vector field with discrete (=> finite) zeroes, =>

$$\boxed{\sum_{p \in S} \text{ind}_p S = 2\pi \chi(S)}$$

PF: Subdivide S as in the picture and sum equation (*) over all R_i's: get that the middle term, which is independent of parameterizations, cancels for segments traversed in opposite directions.