Prep Exercises for Exam #2

M366 Spring 2012

1. Define the *gcd* of two positive integers.
2. TRUE-FALSE:
   (a) Suppose that $gcd(a, b) = 24$ Can 20 be a divisor of a?
   (b) Suppose that $gcd(a, b) = 24$ Can 20 be a divisor of both a and b?
   (c) Assume there exist two integers r and s such that
   $$ar + bs = 8.$$
   Can you conclude that a and b are both even?
   (d) Suppose you are given integers a and b,such that for every pair of integers r, s, we have that
   $$ar + bs > 0$$
   implies
   $$ar + bs \geq 8.$$
   Further, there exist two integers r and s such that
   $$ar + bs = 8.$$
   Then:
      i. a and b are both even.
      ii. a and b are both divisible by 8.
      iii. a and b can both be divisible by 16.
   (e) The *gcd* of two odd numbers can be even.
3. If p is a prime number, what can the $gcd(a, p)$ be?
4. Give an example of:
   (a) A field.
   (b) A commutative ring.
   (c) A non-commutative ring.
   (d) A ring homomorphism from $\mathbb{Z}/25\mathbb{Z} \rightarrow \mathbb{Z}/5\mathbb{Z}$.
   (e) A ring homomorphism from $\mathbb{Z}/5\mathbb{Z} \rightarrow \mathbb{Z}/25\mathbb{Z}$(What do you notice?).
   (f) A ring isomorphism.
5. What is
   $$\sum_{k=0}^{n} \binom{n}{k}?$$
   Show that this computation gives yet another proof of the fact that the power set of a set X with n elements has cardinality 2^n.

6. What is
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}?$$

7. What is
$$\sum_{k_1+k_2+k_3=n}\binom{n}{k_1,k_2,k_3}?$$

8. Prove that for $n \geq 4$
$$n! \geq 2^n$$

9. Let X be a set with n elements and $a_1,\ldots,a_k$ be k natural numbers such that $\sum a_i = n$. Prove that the number of possible ways of partitioning X into k subsets $A_1,\ldots,A_k$ with $|A_i| = a_i$ is
$$\binom{n}{a_1,\ldots,a_k} := \frac{n!}{a_1!a_2!\ldots a_k!}$$

10. Prove that $\mathbb{R}[x]$ is not a field.

11. Consider the ring $\mathbb{Z}/12\mathbb{Z}$. What elements have a multiplicative inverse? What elements are zero divisors? Can you make a statement for the general case of $\mathbb{Z}/n\mathbb{Z}$? (You may want to do more than one example before you tackle the general case!)

12. Write the definition of an ideal of a ring.

13. Prove that the set of all numbers that are divisible by 6 is an ideal in $\mathbb{Z}$. What is the quotient ring isomorphic to? What is the image of the number 27 under the projection map?

14. Consider the ring $R = \mathbb{R}[x]$ and the subset I of polynomials such that $p(0) = 0$.
    (a) Is I an ideal?
    (b) What is the quotient ring R/I isomorphic to?
    (c) What element does the polynomial $x^7 + x^5 - 12$ correspond to under this natural isomorphism?

15. Consider the ring $R = \mathbb{R}[x]$ and the subset $I = \{p(x)\ \textit{divisible by}\ (x^2+5)\}$.
    (a) Is I an ideal?
    (b) Describe the quotient ring R/I.
    (c) What element does the polynomial x^3 correspond to under this natural isomorphism?