

Cayley's Theorem

M366

Fall 2013

This homework will guide you to understanding and proving one of the highlights of our course: Cayley's theorem. I am going to look at this homework very carefully, so please take great care in doing it!

Cayley's Theorem. *Every finite group G is isomorphic to a subgroup of some symmetric group.*

Philosophically, this is quite amazing. This is saying that symmetric groups contain all finite groups inside themselves. Before we embark in proving the theorem, let us make sure we understand what it means, by doing some examples.

Problem 1. *Construct an injective group homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to S_n .*

Problem 2. *Construct an injective group homomorphism from $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to S_6 .*

Now we break down the proof into a couple simple steps.

Problem 3. *Given a finite group G of order n , for any fixed $g \in G$ we construct a function:*

$$r_g : G \rightarrow G$$

defined as follows: for any $x \in G$, $r_g(x) = xg$.

1. *Is r_g a group homomorphism?*
2. *Prove that r_g is a bijection.*

Now, having established that r_g is a bijection between G and itself - and remembering that G is in particular a set with n elements, then r_g is an element of the symmetric group S_n (we just arbitrarily choose a labeling $1, \dots, n$ of the elements of G). This leads us to the next step.

Problem 4. *Now consider the function:*

$$\phi : G \rightarrow S_n$$

defined as follows: for any $g \in G$, $\phi(g) = r_g$.

1. *Prove that ϕ is a group homomorphism.*
2. *Prove that ϕ is injective by showing $\text{Ker}(\phi) = e$.*

All together, G is isomorphic to $Im(\phi)$ which is a subgroup of S_n . This proves Cayley's theorem.

We finally note that this proof shows that for every finite group G of order n , G is isomorphic to a subgroup of S_n . Sometimes n is not the smallest integer for which it is possible to find a subgroup isomorphic to G in the symmetric group. For example, in problem 2 you constructed an isomorphism between a group of order 8 and a subgroup of S_6 . Familiarize yourself with Cayley's algorithm by doing another inefficient thing.

Problem 5. *Realize S_3 as a subgroup of S_6 using the algorithm in the proof of Cayley's theorem.*