LECTURES for the

SCHOOL on BIRATIONAL GEOMETRY and MODULI SPACES

University of Utah, June 1-4, 2010

Renzo Cavalieri, (C.S.U.)

AN EXPLORATION of THE MODULI SPACE of CURVES

LECTURE 1

§1. A quick and dirty introduction to moduli spaces

Informally, a moduli space for (equivalence classes of) geometric objects of a given type consists of:

(1) a set $\mathcal{M}$ whose points are in bijective correspondence with the objects we wish to parameterize.

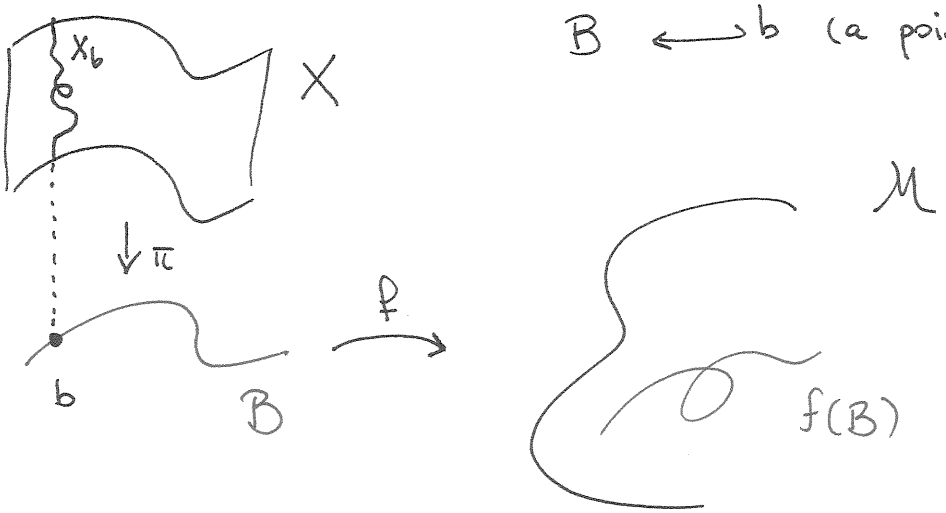
(2) the notion of "good" functions to $\mathcal{M}$, described in terms of families of objects:

For any space B,

$$\mathrm{Hom}(B,\mathcal{M}) \longleftrightarrow \begin{tikzcd} X \arrow[d, "\pi"] & X_b \arrow[l, hook'] \arrow[d] \\ B & b \arrow[l, hook'] \end{tikzcd}$$

(the fiber is an object of the type we want)

(a point)



Remarks:

(A) In modern language we are describing the (scheme) structure of $\mathcal{M}$ via its functor of points. Often a moduli problem is described as a functor

$$\mathcal{F}_{\mathcal{M}} : \{Sch\} \to \{Sets\}$$
$$B \longmapsto \left\{\begin{array}{c}\text{families of objects}\\ \text{over } B\end{array}\right\}$$

and a fine moduli space $\mathcal{M}$ is a scheme that represents the functor, i.e. a scheme whose functor of points is $\mathcal{F}_{\mathcal{M}}$.

(B) A fine moduli space comes with a universal family

$$\mathcal{U} \xrightarrow{\pi} \mathcal{M}$$

i.e. a family such that the fiber $\pi^{-1}(m)$ is the object parameterized by m.

(C) Fine moduli spaces HARDLY EVER exist! When the objects we parameterize have automorphisms, it is easy to cook up non-isomorphic families that give rise to the same function to $\mathcal{M}$.

Examples/Exercises:

1. Every scheme X is a moduli space. What is the functor?
2. What distinguished function to $\mathcal{M}$ does the universal family correspond to?

(3) What set parameterizes unit segments in $\mathbb{R}^2$ up to rigid motion? Show that such set cannot become a fine moduli space by exhibiting two non-isomorphic families giving rise to the same map to $\mathcal{M}$

(4) How about isomorphism classes of 1-dimensional vector spaces? What familiar mathematical creatures prove the non-existence of a fine moduli space for this moduli problem?

(5) Produce a family of elliptic curves which is ISOTRIVIAL but not TRIVIAL: i.e. each point of the base gives isomorphic fibers, but the family is not $\cong$ to a product $B \times E$.

Try to do it in 2 ways:

(1) by giving an equation for such family

(2) by thinking of elliptic curves as complex tori.

— o — o — o —

How do we rescue a theory that seems hopeless from the very beginning? There are three possible directions I want to briefly discuss:

(a) coarse moduli spaces;

(b) adding structure to the moduli problem; (rigidifying the problem)

(c) stacks/orbifolds.

From a philosophical point of view, (a) means accepting that life sucks and we can't have as beautiful a theory as we wish. (b) that life can still be good so long as you readjust your goals. (c) that we can deal with the suckiness of life by just accepting it ... more mathematically though ...

A <u>coarse moduli space</u> is a scheme M whose closed points are in bijection with the objects we wish to parameterize, plus the universal property that if M' is such that there is a function

$$\left\{ \begin{array}{c} \text{families of} \\ \text{objects over } B \end{array} \right\} \xrightarrow{\varphi_{M'}} \mathrm{Hom}(B, M'),$$

then there is a unique morphism $M \xrightarrow{f} M'$ such that

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{families of} \\ \text{objects over } B \end{array} \right\} \arrow[rr, "\varphi_{M'}"] \arrow[dr, "\varphi_M"'] \arrow[rr, phantom, shift right=6ex, "\square"] & & \mathrm{Hom}(B, M') \\ & \mathrm{Hom}(B, M) \arrow[ur, "f_*"'] & \end{tikzcd}$$

Coarse moduli spaces are typically singular, and don't allow for a completely satisfactory dictionary between the geometry of a moduli space and the geometry of families of objects. On the other hand for a long time we just thought that's the best we could do ...

<u>Examples</u>:

① The point is a coarse moduli space for examples ③ and ④ in the previous page.

② $\mathbb{P}^1$ (the j-line) is a coarse moduli space for elliptic curves.

Since automorphisms of the objects are what causes trouble for the existence of moduli spaces, one idea is to decorate the objects with some extra structure so that only the identity automorphism preserves the extra structure. Then we can form a fine moduli space. In fact this strategy is important even if we want to construct ultimately moduli spaces of objects with automorphisms: first you obtain a fine moduli space for a rigidified problem, then you show that you can obtain the moduli space for the problem you started with by taking a quotient via an appropriate group action.

Examples:

1. If in problem 3 on page (3) we "color" the enpoints of the segments two different colors, we obtain a fine moduli space $= \{pt\}$.

2. In problem 4 we could add the point "1" $\in \mathbb{C}$ as a datum of the problem to rigidify it. More formally, instead of parameterizing families of 1-dimensional vector spaces we would parameterize families with a neverzero section.

3. One way to construct the moduli space of curves is to first parameterize curves toghether with a multicanonical embedding. This realizes the rigidified problem inside an appropriate Hilbert scheme, and then one takes a quotient by the general linear group parameterizing changes of bases in $H^0(C, nK_C)$.

Stacks/Orbifolds are the modern approach to moduli space theory, where we surrender the idea that a moduli space should be a scheme. If the automorphisms of the objects parameterized are the crux to the existence of a moduli space, then a moduli space should ~~a~~ be a more "intelligent" creature, whose points $\boxed{\text{Know}}$ about the automorphisms of the objects they parameterize. From a very intuitive and imprecise point of view, an orbifold is then a space whose points have ~~a~~ groups associated to them. These groups are called *isotropy groups*.

I will <u>NOT</u> define formally what a stack is. First, I wouldn't be able to do it justice. Second, I don't think that is the appropriate initial approach to moduli theory. The right question is not what a stack <u>IS</u> but what it <u>DOES FOR YOU</u>!

And what it does is to recover that dictionary between families of objects and functions to the moduli space. In other words, we hardwire our moduli space to be fine. But in doing so we have to accept that the space is now some nasty categorical construction, and we have to work with that. The key point is though that we often can work on the stack by working on families of objects. We will see how in many instances throughout this mini-course.

Examples

① <u>Quotient Stacks</u>: $[X/G]$ is called the stack quotient of X by G.

Functions $B \xrightarrow{\varphi} [X/G]$ correspond to:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "p"] & X \circlearrowleft G \arrow[d, "\pi"] \\ B \arrow[r, "\varphi"] & {[X/G]} \end{tikzcd}$$

- $P \xrightarrow{p} B$ a principal G-bundle over B
- ϕ a G-equivariant map.

② <u>Our favorite stack</u>: $BG := [pt/G]$; in the case of $X = pt$, the datum of the G-equivariant map is vacuous, and BG is the classifying stack for principal G-bundles.

Exercises:

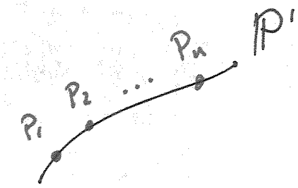
① Understand the moduli space for plane unit segments up to rigid motion as $B\mathbb{Z}_2$

② Understand the moduli space for 1-dim'l $\mathbb{C}$-vector spaces as $B\mathbb{C}^*$.

LECTURE 2

§1. Rational pointed curves

We denote $M_{0,n}$ the moduli space of isomorphism classes of configurations of n-marked points on $\mathbb{P}^1$. The marked points are required to be distinct.



Objects: $(\mathbb{P}^1, P_1, \dots, P_n)$ $\quad P_i \in \mathbb{P}^1$

$\qquad\qquad\qquad\qquad\qquad\quad P_i \neq P_j$ if $i \neq j$

Equivalence Relation: $(\mathbb{P}^1, P_1, \dots, P_n) \sim (\mathbb{P}^1, Q_1, \dots, Q_n)$ if there is an automorphism of $\mathbb{P}^1$ $\phi \in \mathbb{P}GL(2)$ such that $\phi(P_i) = Q_i$ for $i = 1, \dots, n$

Functor:

$$B \longmapsto \left\{ \begin{array}{c} B \times \mathbb{P}^1 \\ \big\downarrow \pi_1 \quad \big\uparrow \sigma_1 \cdots \big\uparrow \sigma_n \\ B \end{array} \right\} \qquad \sigma_i \text{ distinct sections}$$

Ex 1: Formulate the equivalence relation for families.

From the well known fact that there is a unique automorphism of $\mathbb{P}^1$ that sends any 3 points to $0, 1, \infty$, the fine moduli spaces $M_{0,n}$ are readily described:

- $M_{0,3} = \{pt\}$
- $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
- $M_{0,n} = (\mathbb{P}^1)^{n-3} \setminus \{\text{all diagonals}\}$

Ex 2: Describe the universal families for the various $M_{0,n}$'s.

$M_{0,n}$ is NOT compact. It is useful to find a good compactification for our moduli space.

Requirements for a compactification $\underbrace{}_{}^{X_n}$ to be "good":

(1) $M_{0,n} \subseteq X_n$ as a dense open set

(2) X_n be a "nice" space (e.g. smooth, projective)

(3) X_n represents a modular functor, extending $M_{0,n}$ to "natural degenerations" of configurations of points on $\mathbb{P}^1$.

(4) Boundary be modular: irreducible components of the boundary can be described in terms of products, quotients etc of X_m's for $m \leq n$.

(5) Boundary divisors have simple normal crossings.

All of the above are satisfied by the moduli space $\overline{M}_{0,n}$ of rational stable pointed curves.

Objects $(C, P_1, \dots, P_n)$
- C a nodal rational (connected) curve
- $P_1, \dots, P_n$ distinct points in the smooth locus of C
- $|\mathrm{Aut}(C, P_1, \dots, P_n)| < \infty$ (stability)

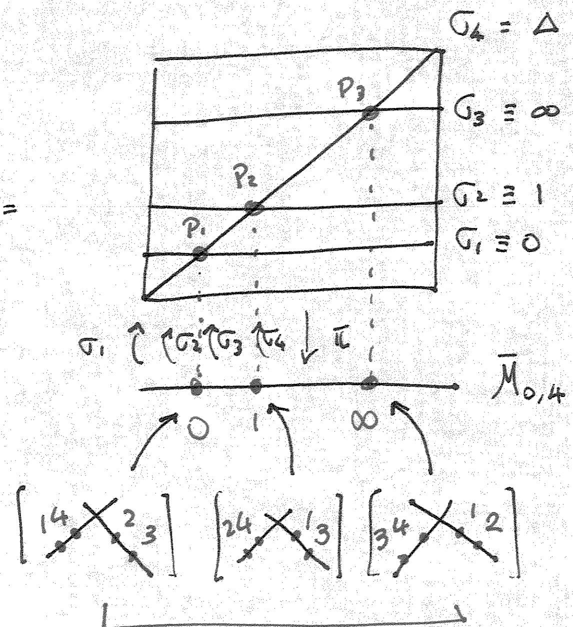
Ex 3: define the functor $\overline{M}_{0,n}$. Check that stability is equivalent to each component of C having at least 3 special points, where a special point is either a mark or a node.

Examples:

(1) $\overline{M}_{0,3} \cong \{pt\}$ $\quad \mathbb{P}^1 = U_3 \longrightarrow \overline{M}_{0,3}$ (sections 0, 1, ∞)

(2) $\overline{M}_{0,4} \cong \mathbb{P}^1$

$$U_{0,4} \cong \mathrm{Bl}_{(0,0),(1,1),(\infty,\infty)}\, \mathbb{P}^1 \times \mathbb{P}^1 =$$



$\mathbb{P}^1 \times \mathbb{P}^1$ must be blown up at P_1, P_2, P_3 to prevent sections from intersecting.

{ Note that the boundary of $\overline{M}_{0,4}$ consists of points $\cong \overline{M}_{0,3} \times \overline{M}_{0,3}$ }

Theorem (Knudsen): for $n \geq 3$, $\overline{M}_{0,n}$ is a smooth proj. variety representing the functor "configuration of n distinct points on the smooth locus of a rational curve" + stability.

In what follows, we give a rough sketch of a couple ways one can construct $\overline{M}_{0,n}$ and its universal family, thus proving the thm above.

① Knudsen's recursive construction: the key point here is that the universal family $U_{0,n} \longrightarrow \overline{M}_{0,n}$ can be identified with the forgetful morphism $\pi_{n+1} : \overline{M}_{0,n+1} \longrightarrow \overline{M}_{0,n}$.

Ex 4: (IMPORTANT!!!) Become best friends with the above statement!!

With these facts, one constructs the fiber product

$$\begin{tikzcd}
\overline{M}_{0,n+2} \cong U_{0,n+1} \arrow[d] & \text{is a minimal desingularization of } \bigstar \text{ that separates the intersection of the sections} \\
\bigstar \arrow[r] \arrow[d] & U_{0,n} \ (\cong \overline{M}_{0,n+1}) \arrow[d] \\
\overline{M}_{0,n+1} \arrow[r] \arrow[u, bend left, "\sigma_1 \cdots \sigma_n"] \arrow[u, bend right, "\Delta"'] & \overline{M}_{0,n} \arrow[u, bend right, "\sigma_1 \cdots \sigma_n"']
\end{tikzcd}$$

Exercise 5: reinterpret the construction of $U_{0,4} \cong \overline{M}_{0,5}$ in this light.

If you feel brave, construct $\overline{M}_{0,6}$.

② Kapranov's construction: interprets $\overline{M}_{0,n}$ as the family of (nodal degenerations of) rational normal curves in $\mathbb{P}^{n-2}$ through n points in general position. $\overline{M}_{0,n}$ is constructed via a sequence of blow-ups starting from $\mathbb{P}^{n-3}$, thus making smoothness and projectivity obvious.

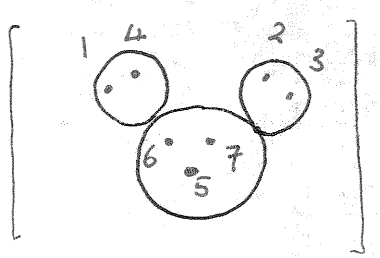
Exercise 6: exhibit $\overline{M}_{0,4}$ as the pencil of conics through 4 points in $\mathbb{P}^2$, and $U_{0,4}$ ($\cong \overline{M}_{0,5}$) as the total space of such family.

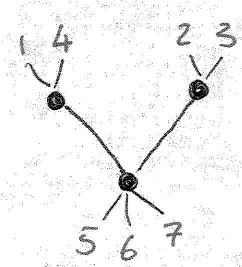
Boundary Stratification of $\overline{M}_{0,n}$

1) Irreducible boundary strata parameterize rational pointed curves that share the same dual graph : $V \longleftrightarrow$ components of C

$E \longleftrightarrow$ nodes of C

half edges $\leftrightarrow$ HE $\longleftrightarrow$ marks

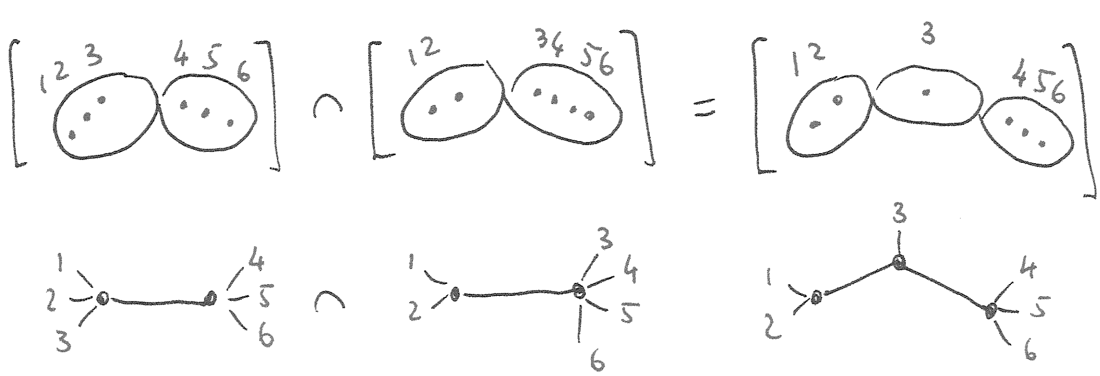
E.g.: 

has dual graph 

(2) Codimension of a stratum = # nodes = # edges in the dual graph.

(3) Boundary strata are $\cong$ to products of $\overline{M}_{0,m}$ for $m \lneq n$ (boundary is modular and recursive)

(4) Set theoretic intersections of strata correspond to common degenerations of the dual graphs:



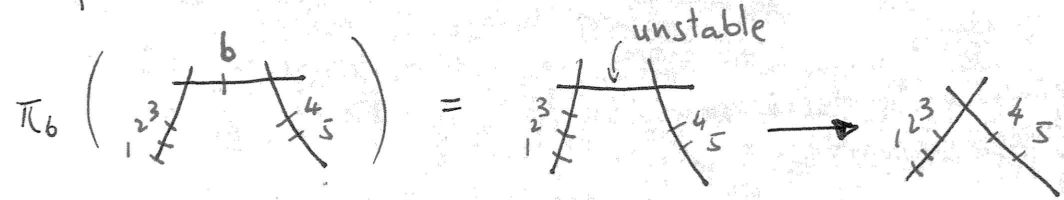
Ex 7: Find combinatorial description to describe when strata $\cap$ is $\begin{cases} \emptyset \\ \text{transversal} \end{cases}$

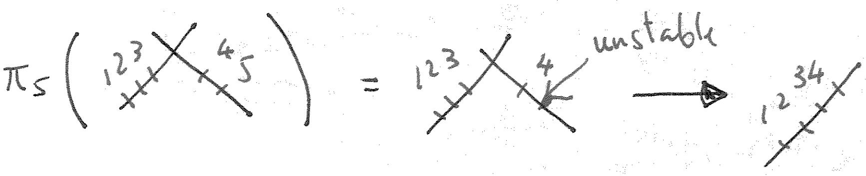
Natural Morphisms

(1) Forgetful morphisms: $\overline{M}_{0,N} \xrightarrow{\pi_I} \overline{M}_{0,n}$

Note: after you forget points, you need to "stabylize", i.e. contract components that have become unstable.

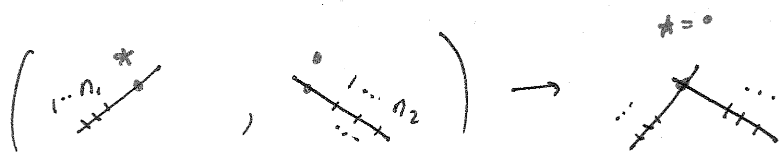
E.g.:





(2) Gluing morphisms:

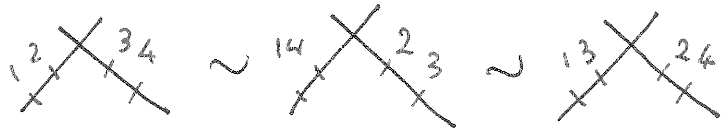
$$gl : \overline{M}_{0,n_1+\star} \times \overline{M}_{0,n_2+\circ} \longrightarrow \overline{M}_{0,n_1+n_2}$$



Note: Boundary strata are images of gluing morphisms.

WDVV Relations: Any two points are $\left\{\begin{array}{l}\text{rationally}\\ \text{numerically}\\ \text{cohomologically}\end{array}\right.$ equivalent in $\mathbb{P}^1$

$\Rightarrow$



$\in \overline{M}_{0,4}$

Pulling back this equivalences via forgetful morphisms (and pushing forward via gluing morphisms) one obtains relations in $A^*(\overline{M}_{0,n})$ (or $H^*(\overline{M}_{0,n})$) called WDVV relations.

Ex 8 : Make friends with WDVV relations
Describe all WDVV relations for $\overline{M}_{0,5}$ (careful!)

Ex 9: Prove that all 0-dimensional boundary strata in $\overline{M}_{0,n}$ are Chow-equivalent. (note this is a delicate question in general, as opposed to cohomology equivalence, which is obvious since all such strata are $\sim$ [pt.].

Intersection theory on $\overline{M}_{0,n}$ is completely determined by what we know so far (with the caveat we haven't yet talked about how to intersect non-transversal boundary strata)

Theorem (S. Keel): $A^*(\overline{M}_{0,n})$ is generated by boundary divisors. The only relations are given by:

(1) set theoretically empty intersections.

(2) WDVV relations.

ψ classes

We now explore how to use the dictionary "geometry of families" - "geometry of the moduli space" to define interesting and useful cohomology/Chow classes on $\overline{M}_{0,n}$.

A line bundle on a moduli space consists of the assignment of a line bundle on the base of each family, subject to the natural compatibility conditions.

Here goes a general statement. (7)

(Semi)Stable Reduction (Mumford(?))

Let B be a smooth curve an $B^\circ = B \setminus \{pt\}$.

$X^\circ \to B^\circ$ a flat family of (semi) stabbe curves ($g \geq 2$)

Then

$$\begin{tikzcd} X \arrow[rr] \arrow[d] & & X^\circ \arrow[d] \\ B' \arrow[r] & B & B^\circ \arrow[l, hook'] \end{tikzcd}$$

branched cover totally ram. @ pt

X is a flat family of (semi) stable curves extending $X^\circ \times_{B^\circ} B'$.

If we are in the semi-stable case we can assume X smooth.

All fibers of X have stable models obtained by contracting rational components with excess automorphisms.

Procedure:

① Complete $\begin{matrix} X^\circ \\ \downarrow \\ B^\circ \end{matrix}$ to $\begin{matrix} X \\ \downarrow \\ B \end{matrix}$ in whatever way.

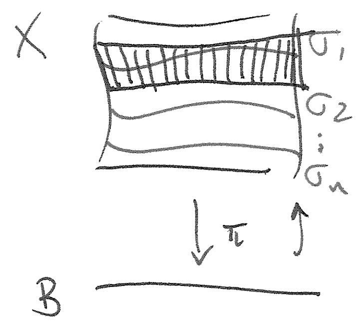
② Blow the heck out of the central fiber to resolve nasty singularities

③ Base change/normalize repeatedly to replace non-reduced components by reduced covers of them.

④ Contract (−1)-curves that you might have produced in ③.

To ensure that the compatibility conditions are satisfied, one often constructs the line bundles in a "canonical way" from the geometry of the family.

Given a family of rational, pointed, stable curves, the



i-th tangent line bundle sticks over each $b \in B$ the tangent line of X_b at the i-th mark.

$$\boxed{\mathbb{T}_{i_\pi} := \sigma_i^*(T_X/\pi^*T_B)}$$

Such an assignement to each family determines a line bundle on $\overline{M}_{0,n}$, called $\mathbb{T}_i$.

The i-th cotangent line bundle, $\mathbb{L}_i$, has fiber over a moduli point corresponding to the cotangent line at the i-th marked point. (Note $\mathbb{T}_i = \mathbb{L}_i^\vee$)

<u>Note</u>: a more global description of $\mathbb{L}_i$ is

$$\boxed{\mathbb{L}_i = \sigma_i^*(\omega_\pi)}$$

ω_π ↳ relative dualizing sheaf on $\mathcal{U}_{0,n} \to \overline{M}_{0,n}$

Now we can define the i-th ψ class:

$$\boxed{\psi_i := c_1(\mathbb{L}_i)}$$

After Keel's theorem, why should we be excited about ψ classes? After all, they __must__ be linear combinations of boundary divisors! There are several reasons in fact:

($\bullet$) they are geometrically meaningful, as they play a key role in intersecting non-transversal strata.

($\bullet\bullet$) they are computable and combinatorially interesting.

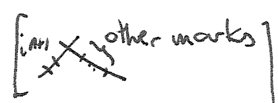
($\bullet\bullet\bullet$) they generalize in higher genus to non-boundary classes, giving examples of interesting classes on the open part of the moduli space.

__Exercises to grow to know and love ψ classes__

ψ1. __Pullback relation__

$$\begin{array}{c} \overline{M}_{0,n+1} \\ \downarrow \pi \\ \overline{M}_{0,n} \end{array} \qquad \boxed{\psi_i = \pi^*\psi_i + \underset{\text{image of the i-th section.}}{D_{i,n+1}}} \quad (\star)$$

$D_{i,n+1}$ "



__Sketch__ : (1) Show that $\mathbb{L}_i$ and $\pi^*\mathbb{L}_i$ are naturally isomorphic outside the image of σ_i

$$\Rightarrow \qquad \mathbb{L}_i = \pi^*\mathbb{L}_i \otimes \mathcal{O}(k D_{i,n+1}) \quad (\bullet)$$

(2) Determine k by using an intersection computation

e.g. :
* intersect ($\bullet$) with $D_{i,n+1}$
* intersect ($\bullet$) with an appropriate 1-dim'l stratum in $\overline{M}_{0,n}$.

(then compute ψ explicitely on $\overline{M}_{0,4} = \mathbb{P}^1$)

ψ2. Note that ex. ψ1. together with the trivial information $\psi_1 = \psi_2 = \psi_3 = 0$ on $\overline{M}_{0,3}$ gives (non-unique) boundary descriptions of ψ classes on arbitrary $\overline{M}_{0,n}$.

Find a combinatorial description of ψ_i on $\overline{M}_{0,n}$.

ψ3. <u>String Equation</u>

Let $\psi^I = \psi_1^{i_1} \psi_2^{i_2} \cdots \psi_n^{i_n}$ be a monomial of degree $n-2$ on $\overline{M}_{0,n+1}$. Consider

$$\begin{tikzcd} \overline{M}_{0,n+1} \arrow[d, "\pi_{n+1}"] \arrow[dr, "p"] & \\ \overline{M}_{0,n} \arrow[r, "q"'] & pt \end{tikzcd}$$

Then

$$p_*(\psi^I) = q_*\left(\sum_{j=1}^{n} \psi_1^{i_1} \cdots \psi_j^{i_j - 1} \cdots \psi_n^{i_n} \right) \quad \left(\text{with the convention } \psi^{-1} = 0\right)$$

<u>Hint</u>: Rewrite ψ^I using the pullback relation ($\star$), then use the projection formula to push forward to $\overline{M}_{0,n}$.

ψ4. Note that the string equation determines a formula for top intersections of ψ classes:

$$\boxed{\int_{\overline{M}_{0,n}} \psi^I = \binom{n-3}{I}}$$

multinomial coefficient.

LECTURE 3

Higher genus Curves

oday our main characters are

$$\mathcal{M}_g, \mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{moduli spaces of} \\ \text{smooth genus } g \text{ curves} \\ \text{(eventually with marks)} \end{array} \right\}$$

$$\cap \qquad \cap$$

$$\overline{\mathcal{M}}_g \quad \overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{Deligne-Mumford compactification} \\ \text{to stable curves} \end{array} \right\}$$

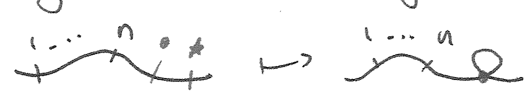
↳ NODAL

↳ FINITELY MANY AUTO'S $\iff$ 3 special points on each rational component

Some quick and dirty facts, some of which we will come back to:

- Not representable by schemes, but...
- Smooth orbifolds
- Dimension $3g-3+n$
- Connected, Irreducible
- $\overline{\mathcal{M}}_{g,n}$ compact
- Boundary is modular, like $\overline{\mathcal{M}}_{0,n}$. Boundary strata are products <u>and quotients</u> of smaller moduli space
- Combinatorial boundary stratification. Dual graphs can now have genus, corresponding to cycles of curves or curves with self nodes.
- Natural forgetful and gluing morphisms. In particular

$$g_{irr} : \overline{\mathcal{M}}_{g-1, n+\bullet+\star} \longrightarrow \overline{\mathcal{M}}_{g,n}$$



- Universal family is again $\overline{\mathcal{M}}_{g,n+1} \xrightarrow{\pi_{n+1}} \overline{\mathcal{M}}_{g,n}$

Some even quicker and dirtier facts about how these spaces can be constructed.

(•) topologists consider a pair of pants decomposition of genus g surfaces to realize $\mathcal{M}_g = B^{6g-6} / \Gamma$ → an open ball; Γ → mapping class group – a finite group

(which gives $\mathcal{M}_g$ for free as a smooth orbifold)

(••) alg. geometers first rigidify the problem:

$$\mathcal{M}_{rig} = \left\{ \begin{array}{l} \text{curves of genus } g \\ \text{plus a } \phi_{3K_C} : C \to \mathbb{P}^N \end{array} \right\}$$

$$\mathcal{M}_{rig} \overset{\text{locally closed}}{\subseteq} \text{Hilb}$$

$$\overline{\mathcal{M}}_g = \mathcal{M}_{rig} \,\overset{\text{G.I.T. quotient}}{/\!/}\, PGL(N+1, \mathbb{C})$$

G.I.T. quotient in fact also tells us how to compactify $\mathcal{M}_g$.

OK. Enough of this. The point being that, once we know that these moduli space exist, we can do a lot of good geometry on them without even knowing how they are constructed. We will:

(1) Compute the dimension of $\overline{\mathcal{M}}_{g,n}$

(2) Define interesting Chow classes on $\overline{\mathcal{M}}_{g,n}$

(3) Learn how to intersect them

(4) Compute $K_{\overline{\mathcal{M}}_{g,n}}$ in terms of these classes.

Before we get started, recall two fundamental theorems in curve theory:

<u>Riemann-Roch</u>: $L \to C$ a line bundle

$$\boxed{\chi(L) = h^0(L) - h^1(L) = c_1(L) + 1 - g}$$

<u>Riemann-Hurwitz</u>: $C^g \longrightarrow D^h$ a degree d map of smooth curves

$$\boxed{2g-2 = d(2h-2) + \underbrace{D}_{\text{ramification}}}$$

<u>Dimension of M_g / take 1</u>

Let $d \gg 0$. We count the dimension of the space of degree d covers ~~on~~ of $\mathbb{P}^1$ in 2 ways and deduce $\dim M_g$.

For a given C^g:
- there is a g-dimensional family of line bundles of deg d
- each such L has $h^1 = 0$
- $\Rightarrow$ by R-R $h^0(L) = d+1-g$

$\Rightarrow$ a map to $\mathbb{P}^1$ of deg d is given by a choice of 2 sections of a deg d line bundle, up to rescaling of both sections.

$$\Rightarrow \dim H_{g,d} = \dim M_g + g + 2(d+1-g) - 1$$
$$\| \qquad\qquad = \dim M_g + 2d - g + 1$$

using R-H formula

$$2g + 2d - 2$$

$$\Longrightarrow \boxed{\dim M_g = 3g-3}$$

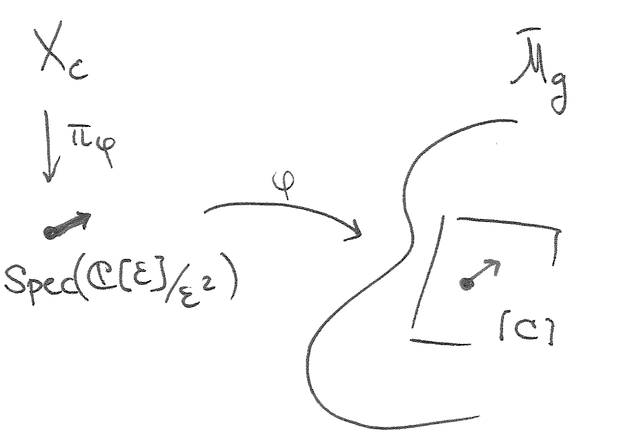
Dimension of M_g / take 2

We compute $\dim M_g$ by computing $\dim T_C M_g$. This gives us a chance to explore the dictionary we've so often talked about.

(1) A tangent vector $\in T_C M_g$ =

$= $ a map $\varphi : \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \longrightarrow M_g$ with $\varphi(\text{closed pt}) = [C]$

$=$ first order deformation of C



(2) First order deformations of C are parameterized by $H^1(C, T_C)$

<u>Heuristic</u>: $H^0(C, T_C)$ param. vector fields, aka <u>trivial</u> 1st order deformations. $H^1(C, T_C)$ gives local vector fields that fail to be global aka non-trivial deformations.

<u>Exercise</u>: turn the heuristic into a rigorous argument by showing that the data required to patch a local trivialization of a deformation is a 1-cocycle of derivations.

(3) At this point $h^1(C, T_C)$ is given by R-R:

$$\overset{0 \text{ if } g>0}{h^0(\cancel{C}, T_C)} - h^1(C, T_C) = \deg T_C + 1 - g$$
$$= 2 - 2g + 1 - g$$

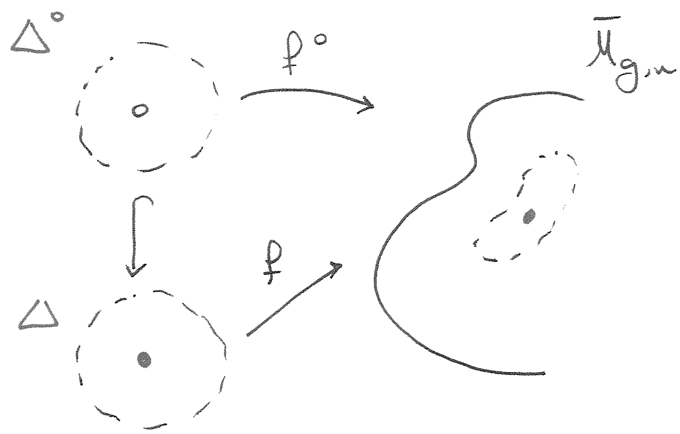
$$\Rightarrow \boxed{h^1(C, T_C) = 3g - 3}$$

An aside on Stable/Semistable Reduction

Compactness of $\overline{M}_{g,n}$ can be shown using our dictionary, via the valuative criterion for properness.

This means:

- every map from the punctured disc can be extended to a map from the disc.



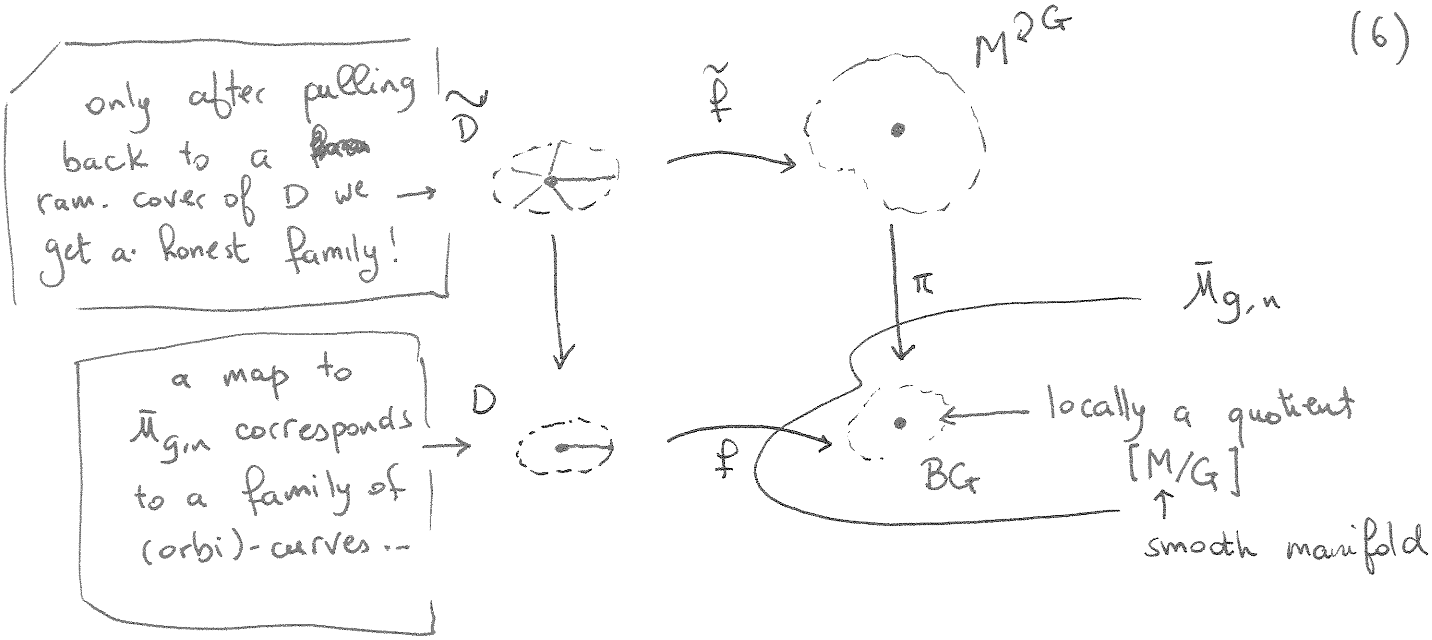
In our dictionary, this should translate to:

- every family of (stable) curves} over the punctured disc should extend to a family over the disc where the central fiber is also stable.

Two remarks:

(1) This should come as a bit of a surprise, for we all know how to degenerate honest smooth curves to pretty much any monstruosity we want. The idea is that such limits are the wrong ones and can be substituted in a reasonable way by good limits. This surgery process is called $\boxed{\text{stable reduction}}$.

(2) The statement above is really only true up to a cyclic base change of the base of the family. This has to do with the fact that $\overline{M}_{g,n}$ is really an orbifold.



<u>Example</u> Let $F = \{y^2 + y = x^3 + tx\}$ be a family of elliptic curves. Note that @ $t=0$ the curve is the unique elliptic curve with $\mathbb{Z}/6\mathbb{Z}$ automorphism group.

$$\begin{array}{c} F = \{y^2+y = x^3+tx\} \\ \downarrow \\ \tilde{D} = \{|t| \leq 1\} \end{array}$$

Let $G = \mathbb{Z}/3\mathbb{Z}$ act on F by

$$\omega(x,y,t) = (\omega x, y, \bar{\omega} t)$$

And consider

$$\begin{array}{c} F/G = F' \\ \downarrow \\ \tilde{D}/G \cong D \end{array}$$

$$\boxed{\left\{ \begin{array}{l} u = t^3 \\ v = x^3 \\ w = tx \end{array} \right.}$$

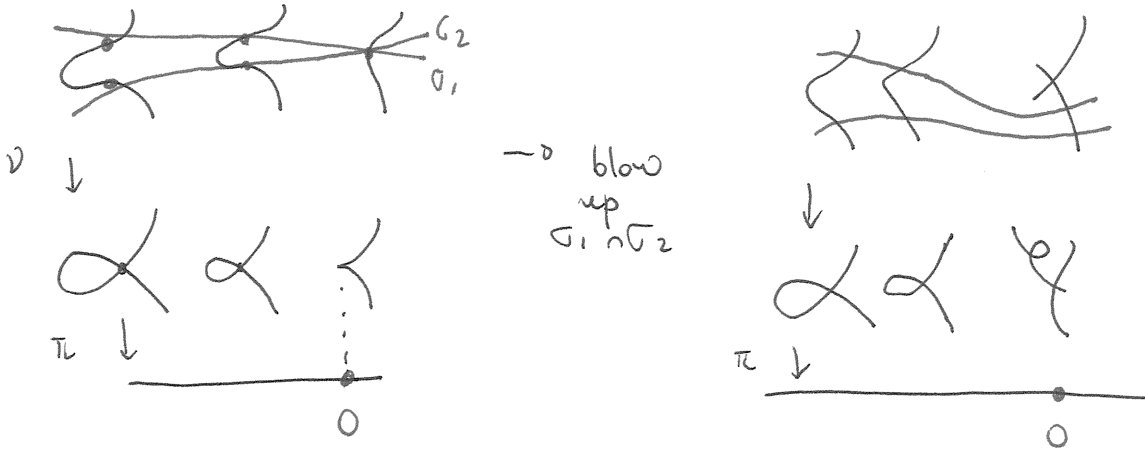
$$\uparrow$$

$$\boxed{\left\{ \begin{array}{l} y^2 + y = v + w \\ w^3 = uv \end{array} \right.}$$

Note that F' gives a good family of ell. curves over $D \setminus \{0\}$, but the central fiber is the $\mathbb{Z}/3\mathbb{Z}$ quotient of the previous central fiber, which is a rational curve.

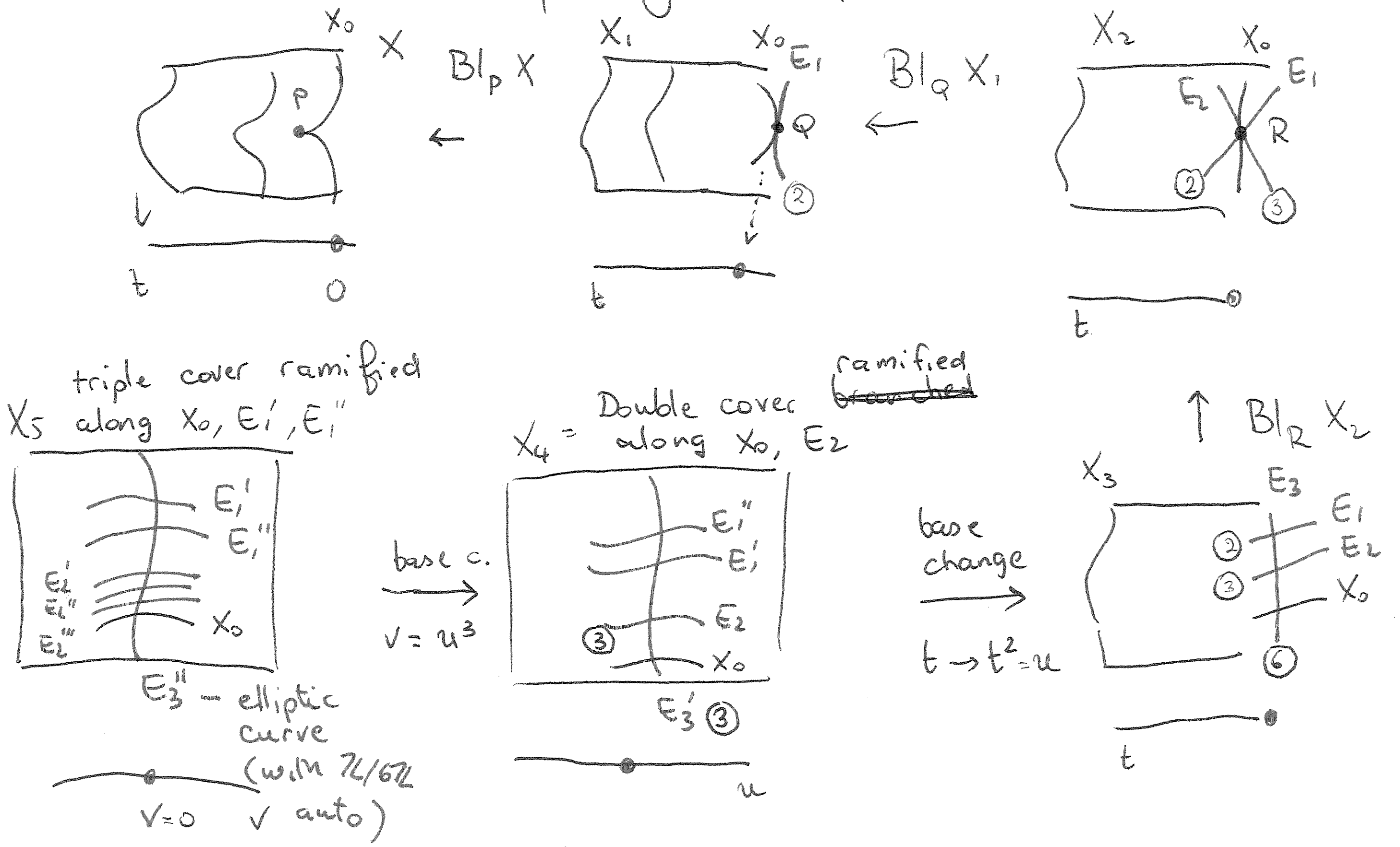
Examples / Exercises:

1) Nodal curves acquiring a cusp:

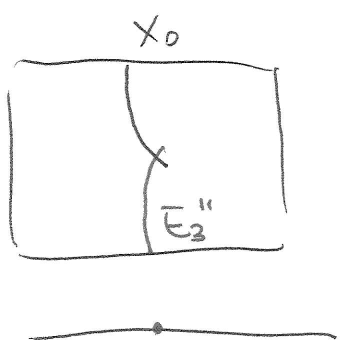


stable limit has replaced cusp with a "pigtail"
$j = \infty$

2) Smooth curves acquiring a cusp:



Contract ↪ −1 curves



stable limit has replaced cusp with elliptic curve with $\mathbb{Z}/6\mathbb{Z}$ autom. group $\quad j = 0$

3 Plane quartics specializing to a double conic

Let $Q(x,y,z)=0$ define a conic in $\mathbb{P}^2$

$F(x,y,z)=0$ ——— quartic in $\mathbb{P}^2$

Consider the pencil of quartics:

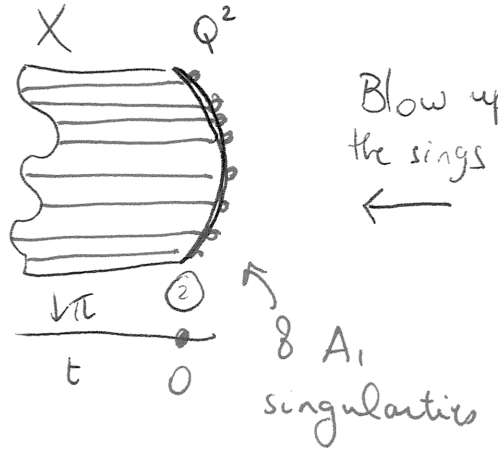
$$X=\{Q^2+tF=0\}\subseteq \mathbb{P}^2\times \mathbb{A}^1$$

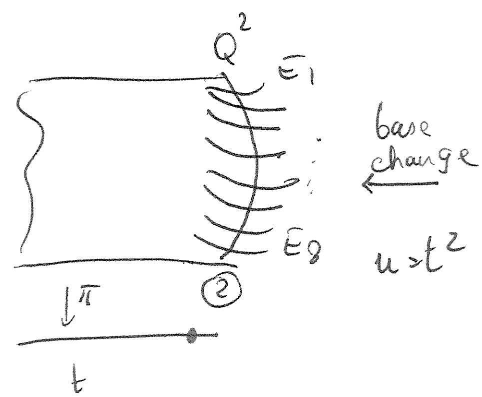
Assume X_t is smoot for $t\neq 0 \Rightarrow X_t$ is a genus 2 curve and $X\setminus X_0$ define a map to $\mathcal{M}_2$.

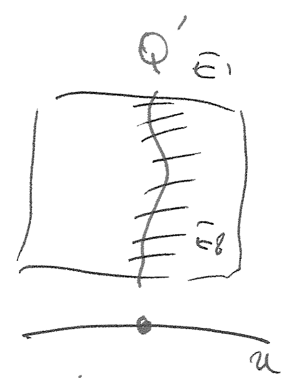
The central fiber X_0 is non reduced of mult. 2.

Let $\{P_1,\dots,P_8\} = Q\cap F$

Local equation of X at each of the points $(P_i, 0)$ is of the form $X^2+t\varphi \Rightarrow X$ has 8 A_1 singularities.

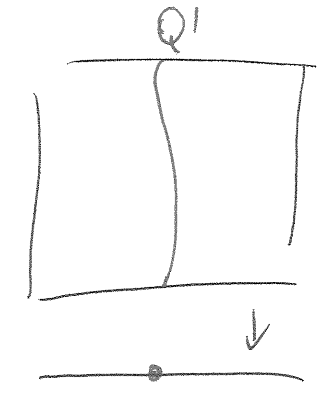






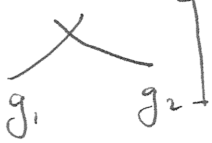
Q' = double cover of Q^2 ramified at 8 points $\Rightarrow$ $g=2$ curve

blow down $E_1,\dots E_8$



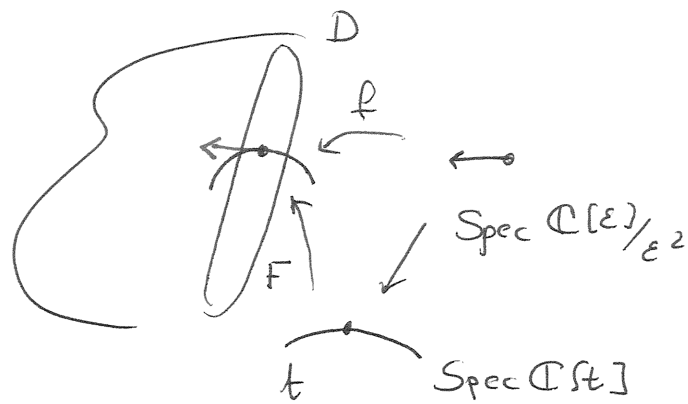
double conic is replaced by a hyperelliptic curve!!

Normal bundle to a boundary divisor

Consider $D = \Big[$  $\Big] \subseteq \bar{M}_g$

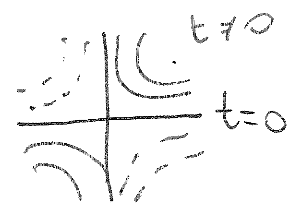
Our goal is to use our dictionary to describe the normal bundle $N_{D/\bar{M}_g}$ in terms of the curves parameterized.

$$\boxed{\text{Normal directions to } D \text{ correspond to smoothing the node!!}}$$

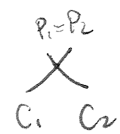


Since we are interested only in infinitesimal info, we can look at the local analitic expression of a 1 parameter family F at the node inside the central fiber. this is

$$\{xy = t\} \subseteq \mathbb{A}^2 \times \mathbb{A}^1$$
$$\downarrow$$
$$(\mathbb{A}^1, t)$$



t gives me a local section of $N_{D/\bar{M}_g}$. x and y are local sections of the tangent spaces ~~of~~ at the shadows of the normalization of the nodal curve.

In other words, the fiber over $C =$  of the normal bundle $N_{D/\bar{M}_g}$ is canonically isomorphic to $T_{p_1}C_1 \otimes T_{p_2}C_2$ (11)

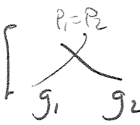
$$\Rightarrow \quad \boxed{N_{D/\bar{M}_g} \cong \Pi'_{p_1} \boxtimes \Pi'_{p_2}}$$

LECTURE 4

Tautological Classes on $\overline{\mathcal{M}}_{g,n}$

A natural way to construct Chow classes on $\overline{\mathcal{M}}_{g,n}$ is to produce vector spaces canonically attached to the geometry of a pointed curve; this defines a vector bundle on $\overline{\mathcal{M}}_{g,n}$ and we can subsequently take its chern classes. We have defined ψ classes on $\overline{\mathcal{M}}_{0,n}$ this way. ψ classes on $\overline{\mathcal{M}}_{g,n}$ are defined analogously.

Exercise 1: Use ψ classes to compute non transverse intersections of boundary strata in $\overline{\mathcal{M}}_{g,n}$

(In particular, if $D_{g_1 g_2} = \Big[$  $\Big]$, and

$$D_{g_1 g_2} \cong \overline{\mathcal{M}}_{g_1,1} \times \overline{\mathcal{M}}_{g_2,1}$$

$$\overline{\mathcal{M}}_{g_1,1} \xleftarrow{\pi_1} \qquad \xrightarrow{\pi_2} \overline{\mathcal{M}}_{g_2,1}$$

then

$$D^2_{g_1 g_2} = \left(-\pi_1^*\psi_1 - \pi_2^*\psi_1\right) \in A^*(D_{g_1,g_2})$$

The Hodge Bundle

$\mathbb{E}_{g,n}$ is a rank g vector bundle on $\overline{\mathcal{M}}_{g,n}$, whose fibers over a moduli point are (canonically identified with):

- $\Omega^1(C)$ (holomorphic differentials) $\Big\}$ C smooth
- $H^0(C, K_C)$
- $H^0(C, \omega_C)$ arbitrary C

ω_C is a sheaf (the relative dualizing sheaf) that substitutes the canonical bundle for nodal curves (and makes Serre duality hold). The sections of ω_C are meromorphic differentials, with at worst 1-poles at the nodes and cancelling residues at the shadows of a node.

The chern classes of $\mathbb{E}_{g,n}$ are called λ classes by Mumford:

$$\boxed{\lambda_i := c_i(\mathbb{E}_{g,n})}$$

Nice Properties of λ classes and of $\mathbb{E}$:

① $\mathbb{E}_g|_{D_{g_1,g_2}} \cong \mathbb{E}_{g_1} \boxplus \mathbb{E}_{g_2}$

② $\mathbb{E}_g|_{D_{irr}=[\alpha^{g-1}]} \cong \mathbb{E}_{g-1} \boxplus \mathcal{O}$

③ $c_{tot}(\mathbb{E}_g \oplus \mathbb{E}_g^\vee) = 1$ (Mumford Relation)

Exercise 2: Use the above properties to show that

(i) $\lambda_g^2 = 0$ if $g > 0$

(ii) $\lambda_g \lambda_{g-1} \equiv 0$ on the boundary of $\overline{M}_g$

(iii) $\lambda_g|_{\text{curves NOT of compact type}} = 0$

(iv) $\lambda_2 = \lambda_1^2/2$

Exercise 3: Show $\psi_1 = \lambda_1$ on $\overline{M}_{1,1}$

Exercise 4: Compute $\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}$

Sketch
1. Consider a general pencil of plane cubics ~~___~~ to get a map $\varphi : \mathbb{P}^1 \longrightarrow \overline{M}_{1,1}$
2. By computing χ of the total space of the family ~~of~~ in two ways argue that there are 12 nodal fibers
3. Keeping in mind the hyperelliptic involution, deduce $\deg \varphi = 24$
4. Explicitly compute $\int_{\mathbb{P}^1} \varphi^*(\psi_1) = 1$

GROTHENDIECK RIEMANN ROCH $\quad X \xrightarrow{f} Y$ a proper morphism,

$$\Rightarrow \quad \begin{array}{ccc} K(X) & \xrightarrow{R^\bullet\pi_*} & K(Y) \\ {\scriptstyle ch\_.td(X)}\downarrow & & \downarrow{\scriptstyle ch\_.td\,Y} \\ H^*(X) & \xrightarrow{\pi_*} & H^*(Y) \end{array}$$

$$ch(R^\bullet\pi_* E).td\,Y = \pi_*(ch(E).td(X))$$

or, using the projection formula

$$\boxed{ch(R^\bullet\pi_* E) = \pi_*(ch(E).td(X/Y))}$$

Recall, if α_i are the Chern roots of $E \Rightarrow$ $\quad ch(E) = \sum e^{\alpha_i}$

$td(E) = \prod \alpha_i / 1 - e^{-\alpha_i}$

Exercise: show that if $\varphi = pt$ GRR specializes to Hirzebruch RR:

$$\chi(E) = ch(E)\, td(X)$$

And if X is a curve, we get baby Riemann-Roch.

GRR in ACTION

① Relation between λ_1 and κ_1 in $\mathcal{M}_g$ – $\mathcal{U}_g$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \shortparallel$

First off, define $\kappa_1 := \pi_*(\psi_1^2)$ $\quad(\pi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g)$

We apply GRR to the universal family π.

Let $E = \mathcal{O}_{\mathcal{U}_g}$:

$$ch(R^{\bullet}\pi_* \mathcal{O}_{\mathcal{U}_g}) = \pi_*\left(ch(\mathcal{O}_{\mathcal{U}_g}) . td(\omega_\pi^\vee)\right)$$

$$ch(\mathcal{O}_{\mathcal{M}_g} - \mathbb{E}^\vee) = \pi_*\left(1 . \left(1 - \frac{\psi}{2} + \frac{\psi^2}{12} - \ldots\right)\right)$$

$$(1-g) + \lambda_1 + \ldots = -\frac{(2g-2)}{2} + \frac{\kappa_1}{12}$$

$$\Rightarrow \quad \boxed{\lambda_1 = \frac{\kappa_1}{12}}$$

Exercise: compute the same relation using $E = \omega_\pi$

② Computation of $K_{\mathcal{M}_g}$

Recall $\quad T_C \mathcal{M}_g \cong H^1(C, T_C) \quad \Rightarrow \quad T\mathcal{M}_g \cong R^1\pi_*(\omega_\pi^\vee)$

$$\Rightarrow \quad T_C^* \mathcal{M}_g \cong H^0(C, 2K_C) \quad \Rightarrow \quad \Omega^1 \mathcal{M}_g \cong \pi_*(\omega_\pi^{\otimes 2})$$

Since $K_{\mathcal{M}_g} = c_1(\Omega^1 \mathcal{M}_g)$, we see that we can tackle this computation with GRR choosing $E = \omega_\pi^{\otimes 2}$

$$ch(R^\bullet\pi_*(\omega_\pi^{\otimes 2})) = \pi_*\left(ch(\omega_\pi^{\otimes 2}) \cdot td(\omega_\pi^\vee)\right)$$

$$ch(\Omega^1 \mathcal{M}_g) = \pi_*\left((1 + 2\psi + 2\psi^2)(1 - \frac{\psi}{2} + \frac{\psi^2}{12})\right)$$

$$(3g-3) + K_{\mathcal{M}_g} + \dots = \frac{3}{2}(2g-2) + \frac{13}{12}K_1 + \dots$$

$$\Rightarrow \quad \boxed{K_{\mathcal{M}_g} = \frac{13}{12} K_1 = 13\lambda_1}$$