A look into the mirror (II)

The quintic

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Topics in Algebraic Geometry Seminar

Outline

Our main character

$$Q \subset \mathbb{P}^4$$

is the zero set of a generic degree 5 homogeneous polynomial in five variables.

Facts:

- By adjunction, Q is a CY threefold.
- $H^2(Q, \mathbb{Z}) \cong Pic(Q) = \mathbb{Z} = \langle H \rangle$.
- $H_2(Q, \mathbb{Z}) = \mathbb{Z} = \langle \ell \rangle$.
- $\dim(H^1(TQ)) = 101$.

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A-model Yukawa coupling

Recall:
For $D_1, D_2, D_3 \in H^2(X, \mathbb{Z})$, define:

$$< D_1, D_2, D_3 > := D_1 \cdot D_2 \cdot D_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} < D_1, D_2, D_3 >_{\beta}^{g=0} q^{\beta},$$

where

$$< D_1, D_2, D_3 >_{\beta}^{g=0} = \int_{[\overline{M}_{0,3}(X,\beta)]^{vir}} ev_1^*(D_1) \cdot ev_2^*(D_2) \cdot ev_3^*(D_3)$$

is a three pointed Gromov-Witten invariant for X.

A-model Yukawa coupling

In this case:

$$< H, H, H >= 5 + \sum_{d>0} < H, H, H >_{d\ell}^{g=0} q^d.$$

Divisor equation:

$$< H, H, H >_{d\ell} = d^3 <>_{d\ell} .$$

Multiple covers:

$$<>_{d\ell} = n_d + \sum_{k|d} \frac{1}{(d/k)^3} n_k,$$

where n_d is the number of rational curves of degree d on the quintic.

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Final form

If we regroup our generating function by collecting n_d's, we obtain:

$$< H, H, H >= 5 + \sum_{d>0} d^3 n_d \left(q^d + q^{2d} + q^{3d} + \dots \right).$$

Adding up the geometric series:

$$< H, H, H >= 5 + \sum_{d>0} d^3 n_d \, \frac{q^d}{1 - q^d}.$$

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B-model and GKZ

On the B-model side things are quite a bit more involved. We must:

1. identify a mirror family.
2. identify a large complex structure (LC) limit point in the family.
3. compute the periods near the LC point to obtain canonical coordinates.
4. compute the Yukawa coupling.

GKZ will lead us through the first three steps of this program.

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The mirror family

Consider the short exact sequence of lattices:

$$0 \to \mathbb{Z} \xrightarrow{R} \mathbb{Z}^6 \xrightarrow{A} \mathbb{Z}^5 \to 0,$$

where

$$R = \begin{bmatrix} -5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The mirror family

Construct a family of hypersurfaces in $(\mathbb{C}^*)^5/\mathbb{C}^*$ from the matrix *A* using the following recipe:

- Associate a coordinate x_i of $(\mathbb{C}^*)^5$ to each row.
- Associate a family parameter u_i to each column.
- Think of the entries of the matrix as the exponents of the x_i's.

(This will all be clear in a second with the explicit example)

The mirror family

In practice:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ | & | & | & | & | & | \end{array} \\
\begin{array}{c} x_1 \ - \\ x_2 \ - \\ x_3 \ - \\ x_4 \ - \\ x_5 \ - \\ x_6 \ - \end{array} &
\left[\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{array}\right]
\end{array}
$$

$$
x_1\left(u_1 + u_2x_2 + u_3x_3 + u_4x_4 + u_5x_5 + \frac{u_6}{x_2x_3x_4x_5}\right).
$$

Homogeneity in $x_1 \Rightarrow$ this family can be viewed in $(\mathbb{C}^*)^4$.

The mirror family

Now we can compactify to a family of quintics in $\mathbb{P}^4$ by homogenizing:

$$\left(u_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 + u_5 x_5 + \frac{u_6}{x_2 x_3 x_4 x_5}\right).$$

$$\Downarrow$$

$$P(X) = \Big(u_1 X_1 X_2 X_3 X_4 X_5 + u_2 X_2^2 X_3 X_4 X_5 + u_3 X_2 X_3^2 X_4 X_5 +$$

$$+ u_4 X_2 X_3 X_4^2 X_5 + u_5 X_2 X_3 X_4 X_5^2 + u_6 X_1^5\Big).$$

$P(X)$ “is” the mirror family to the general quintic $Q \subset \mathbb{P}^4$.

Remarks:

1. The first presentation (in COGP) of the mirror family was different: it was the quotient of the one-parameter family

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi X_1 X_2 X_3 X_4 X_5$$

by a specific action of the cyclic group $(\mathbb{Z}_5)^3$.

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LC point

The LC point for our family is at $u_1 = \infty$ (or, if you prefer, to the other coordinates $= 0$).

It corresponds to a singular quintic (the union of the five coordinate hyperplanes); we will discover that the periods have logarithmic monodromy going around this point.

Calabi-Yau form

We define a never vanishing $(3,0)$ form on the fibers of $P(X)$ in local coordinates $x_1, \ldots, x_4$ by:

$$\Omega(x) = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \frac{1}{\partial P/\partial x_4}.$$

(This is indeed regular and never vanishing on the (smooth) fibers of a small neighborhood of the LC point).

Periods

We would like, for any closed 3-cycle Υ, to compute:

$$I(u) = \int_{\Upsilon} \Omega.$$

Our first step will be to find one period. Trick: we can reduce the computation to an integral over the 4-torus $T^4 = \{|x_i| = 1\}$:

$$I(u) = \int_{T^4} \frac{1}{P} \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \wedge \frac{dx_4}{x_4}.$$

Note: close to the LC point the hypersurface is "close to" the arrangement of hyperplanes and hence does not intersect T^4 - which makes the above formula valid.

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GKZ differential equations:

The period $I(u)$ is a solution of a GKZ system of differential equations, corresponding to the matrices R and A written above and to the complex vector $\beta = [-1, 0, 0, 0, 0]$.

mixed partials:

$$\frac{\partial^5}{\partial u_1^5} = \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3}\frac{\partial}{\partial u_4}\frac{\partial}{\partial u_5}\frac{\partial}{\partial u_6}$$

homogeneity 1:

$$u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4} + u_5\frac{\partial}{\partial u_5} + u_6\frac{\partial}{\partial u_6} = -1$$

homogeneity 2: for $2 \leq i \leq 5$,

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homogeneity 2: for $2 \leq i \leq 5$,

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One solution

GKZ tell us that one formal solution for this system can be given as a power series involving Γ functions. In this particular case the answer is:

$$I_0(u) = \frac{1}{u_1} \sum_{n \geq 0} (-1)^n \frac{(5n)!}{(n!)^5} z^n,$$

where

$$z = \frac{u_2 u_3 u_4 u_5 u_6}{u_1^5}.$$

Remarks:

- this solution has trivial monodromy.
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The other periods

GKZ hands us a method to compute all the other periods (H_3 is 4-dimensional), by taking a deformation of this function over a special artinian ring constructed from the GKZ combinatorial data.

In this case,

$$\overline{\mathcal{R}} = \frac{\mathbb{C}[\varepsilon]}{\varepsilon^4}.$$

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The other periods

Claim: the function

$$I^{\varepsilon}(u) = \frac{1}{u_1} \sum_{n \geq 0} (-1)^n \frac{(5n+\varepsilon)!}{((n+\varepsilon)!)^5} z^{n+\varepsilon},$$

where we define

$$(n+\varepsilon)! := (n+\varepsilon)(n-1+\varepsilon)\dots(1+\varepsilon),$$

satisfies our GKZ system of differential equations over the ring $\overline{\mathcal{R}}$.

The other periods

Punchline: expanding in ε

$$I^{\varepsilon}(z) = I_0 + I_1\varepsilon + I_2\varepsilon^2 + I_3\varepsilon^3,$$

one gets 4 independent solutions to our GKZ system!

Remark: the logarithmic monodromy comes from expanding the term

$$z^{\varepsilon} := e^{\varepsilon \log(z)} = 1 + \varepsilon \log(z) + \frac{(\varepsilon \log(z))^2}{2!} + \frac{(\varepsilon \log(z))^3}{3!}$$

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Canonical coordinates

Finally, we can define the canonical coordinates

$$w := \frac{I_1}{I_0}$$

and

$$q := e^{2\pi i w}.$$

Sketch:

Most of the remaining work is now simply tedious computations and a few tricks. We quickly outline how these computations go. Mark Gross's notes are detailed and clear.

It is not too hard to see that the Yukawa coupling is:

$$\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \frac{c_1}{z^3(5^5 z - 1) I_0^2},$$

for some constant c_1 to be determined.

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Sketch:

By the chain rule one can write:

$$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \left(\frac{\partial z}{\partial w} \right)^3 \langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle,$$

and after some laborious substitution and series manipulation one can expand the above expression in terms of q to get:

$$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \sum \frac{c_1}{c_2} \frac{h_j(0)}{j!} q^j,$$

where

- c_1, c_2 are constants to be determined;
- $h_j(z)$ is defined inductively. (in the next slide)

Sketch:

$$h_0(z) := \frac{1}{(5^5 z - 1) I_0^2 (1 + z\frac{dw}{dz})^3}$$

$$h_j(z) := \frac{1}{(1 + z\frac{dw}{dz}) e^w} \frac{dh_{j-1}}{dz}$$

At the end of the day...

Putting everything toghether, one can finally expand both Yukawa couplings in q and match coefficients.

$H \cdot H \cdot H = 5$ and $n_1 = 2875$ are needed as initial conditions to determine c_1 and c_2. Then all other numbers are predicted:

$$n_2 = 609250$$

$$n_3 = 317206375$$

$$n_4 = 242467530000$$

et cetera