Affine Geometry and Discrete Legendre Transform

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April 24, 2008

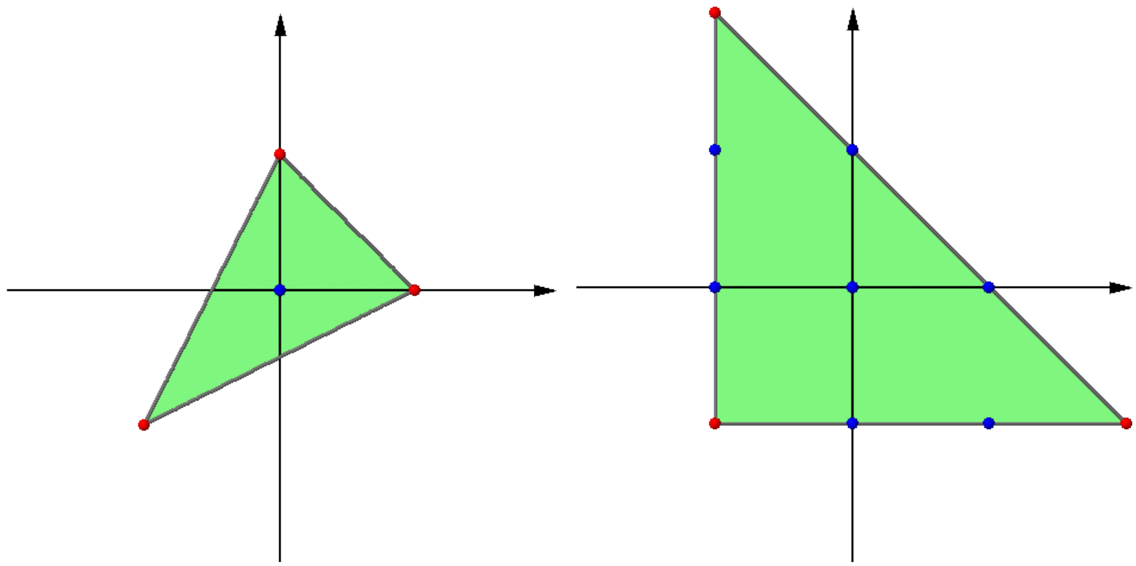
Abstract

The combinatorial duality used in Gross-Siebert approach to mirror symmetry is the discrete Legendre transform of a polarized integral tropical manifold (B, P, φ), where B is an affine manifold with singularities, P is its polyhedral decomposition, and φ is a multi-valued piecewise linear function on it. In this talk we will give precise definitions of these objects and the discrete Legendre transform, accompanied by a detailed treatment of an example coming from reflexive polytopes. The exposition is based on [GS1, GS2].

1 Combinatorial Constructions in Mirror Symmetry

First of all, let us (very) briefly recall some combinatorial constructions used in mirror symmetry.

- Hypersurfaces in toric varieties coming from reflexive polytopes. Duality: $\Delta \leftrightarrow \Delta^\circ$. For example, the following reflexive polygons are polar to each other (and will be used as an example of discrete Legendre transform later):



- Complete intersections in toric varieties coming from NEF-partitions of reflexive polytopes. Duality: $\{\Delta_1, \dots, \Delta_r\} \leftrightarrow \{\nabla_1, \dots, \nabla_r\}$, such that

$$\Delta = \operatorname{Conv}\{\Delta_1, \dots, \Delta_r\}, \qquad \Delta^\circ = \nabla_1 + \dots + \nabla_r,$$
$$\nabla = \operatorname{Conv}\{\nabla_1, \dots, \nabla_r\}, \qquad \nabla^\circ = \Delta_1 + \dots + \Delta_r.$$

- Calabi-Yau manifolds in Grassmannians and (partial) flags. Duality is obtained using degeneration to toric varieties.

- Gross and Siebert propose another degeneration approach. Duality: discrete Legendre transform of a polarized positive integral tropical manifold $(B, \mathcal{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathcal{P}}, \check{\varphi})$.

2 Integral Tropical Manifold

A *convex polyhedron* σ is the (possibly unbounded) intersection of finitely many closed affine half-spaces in $\mathbb{R}^n$ with at least one vertex. It is *rational* if functions defining these half-spaces can be taken with rational coefficients. It is *integral* or *lattice*, if all vertices are integral.

Let **LPoly** be the category of integral convex polyhedra with integral affine isomorphisms onto faces as morphisms. An *integral polyhedral complex* is a category $\mathcal{P}$ and a functor $F : \mathcal{P} \to \mathbf{LPoly}$ such that if $\sigma \in F(\mathcal{P})$ and τ is a face of σ, then $\tau \in F(\mathcal{P})$. To avoid self-intersections, we require that there is at most one morphism between any two objects of $\mathcal{P}$. (This requirement is not essential.) The topological space B associated to $\mathcal{P}$ is the quotient of $\coprod_{\sigma \in \mathcal{P}} F(\sigma)$ by the equivalence relation of face inclusion. From now on we will denote $F(\sigma)$ just by σ and call them cells.

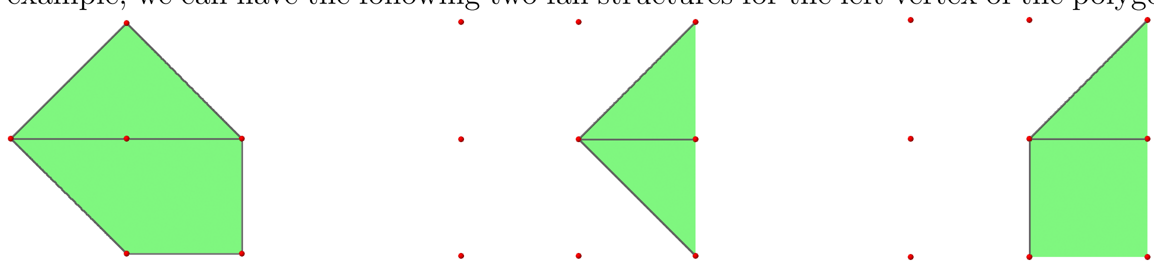
We would like to put an affine structure on B, i.e. to cover it by charts with affine transition functions. Interiors of top dimensional cells of $\mathcal{P}$ give some of these charts. In order to get a chart in a neighborhood of a lower dimensional cell, we need to specify an affine structure in normal directions.

Let $\Lambda_\sigma \simeq \mathbb{Z}^{\dim \sigma}$ be the free abelian group of integral vector fields along σ. For any $y \in \operatorname{Int} \sigma$ there is a canonical injection $\Lambda_\sigma \to T_{\sigma,y}$, inducing the isomorphism $T_{\sigma,y} \simeq \Lambda_{\sigma,\mathbb{R}} = \Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. Using the exponential map, we can also identify σ as a polytope $\widetilde{\sigma} \subset T_{\sigma,y}$. Different choices of y will correspond to translations of $\widetilde{\sigma}$.

Let N_τ be a lattice of rank k and $N_{\tau,\mathbb{R}} = N_\tau \otimes_{\mathbb{Z}} \mathbb{R}$. A *fan structure* along $\tau \in \mathcal{P}$ is a continuous map $S_\tau : U_\tau \to N_{\tau,\mathbb{R}}$, where U_τ is the open star of τ, such that

- $S_\tau^{-1}(0) = \operatorname{Int} \tau$;
- for each $e : \tau \to \sigma$ the restriction $S_\tau|_{\operatorname{Int} \sigma}$ is an integral affine submersion onto its image (i.e. is induced by an epimorphism $\Lambda_\sigma \to W \cap N_\tau$ for some vector subspace $W \subset N_{\tau,\mathbb{R}}$);
- the collection of cones $K_e := \mathbb{R}_{\geqslant 0} \cdot S_\tau(\sigma \cap U_\tau)$, $e : \tau \to \sigma$, defines a finite fan Σ_τ in $N_{\tau,\mathbb{R}}$.

For example, we can have the following two fan structures for the left vertex of the polygons:



If $\sigma \supset \tau$ then $U_\sigma \subset U_\tau$. The *fan structure along σ induced by S_τ* is the composition

$$U_\sigma \to U_\tau \xrightarrow{S_\tau} N_{\tau,\mathbb{R}} \to N_{\tau,\mathbb{R}}/L_\sigma = N_{\sigma,\mathbb{R}},$$

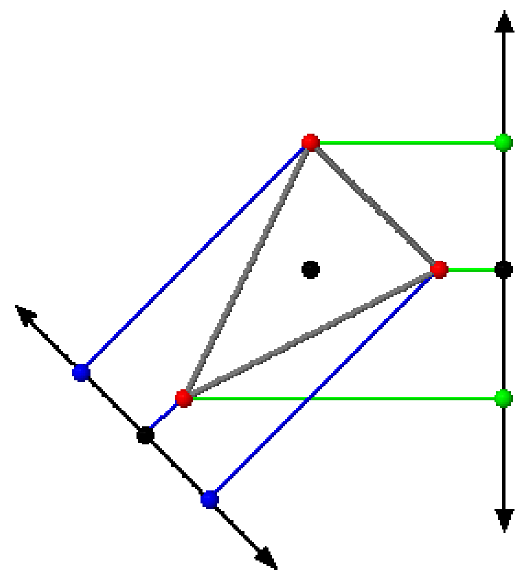
where $L_\sigma \subset N_{\mathbb{R}}$ is the linear span of $S_\tau(\operatorname{Int} \sigma)$. This is well-defined up to equivalence (S_τ is equivalent to a fan structure $S'_\tau : U_\tau \to N'_{\tau,\mathbb{R}}$ if they differ by an integral linear transformation). In the example above the induced fan structure along the common edge of the polygons is the complete 1-dimensional fan.

An *integral tropical manifold* of dimension n is an integral polyhedral complex $\mathcal{P}$, which we assume countable, with a fan structure $S_v : U_v \to N_{v,\mathbb{R}} \simeq \mathbb{R}^n$ at each vertex $v \in \mathcal{P}$ such that

- for any vertex v the support $|\Sigma_v| = \bigcup_{C \in \Sigma_v} C$ is (non-strictly) convex with nonempty interior (hence is an n-dimensional topological manifold with boundary);

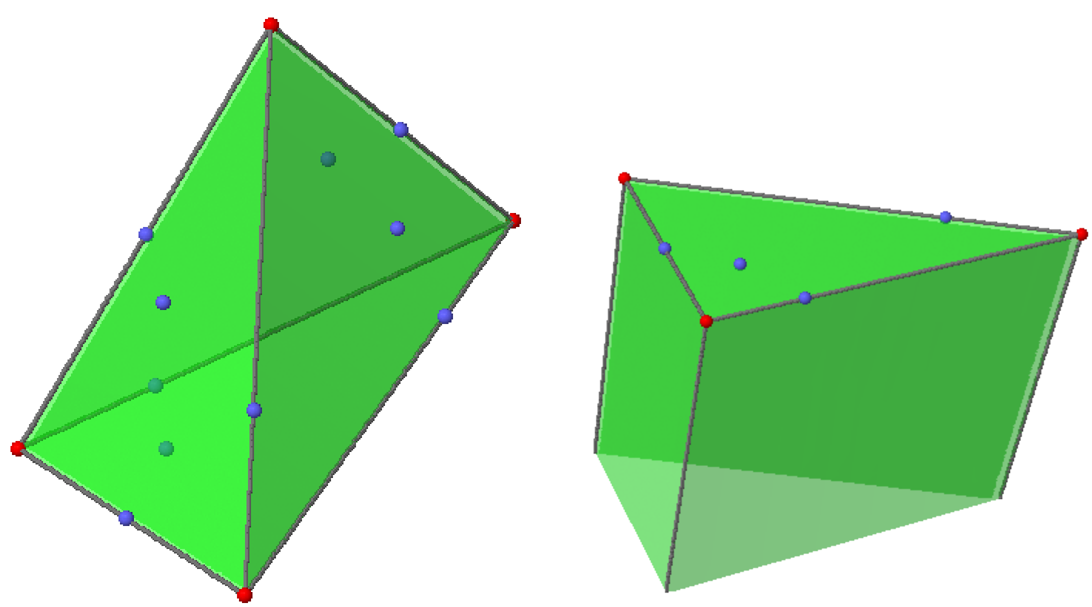
- if v and w are vertices of τ, then the fan structures along τ induced from S_v and S_w are equivalent.

Example. *It is possible to obtain an integral tropical manifold from the boundary of a reflexive polytope Δ in the following way. The polyhedral complex is given by the facial structure of $\partial\Delta$. The fan structure at each vertex is given by the projection along this vertex (i.e. along the vector from this vertex to the origin). For the 2-dimensional reflexive simplex which we have seen above, this construction gives the following:*



3 Discriminant Locus

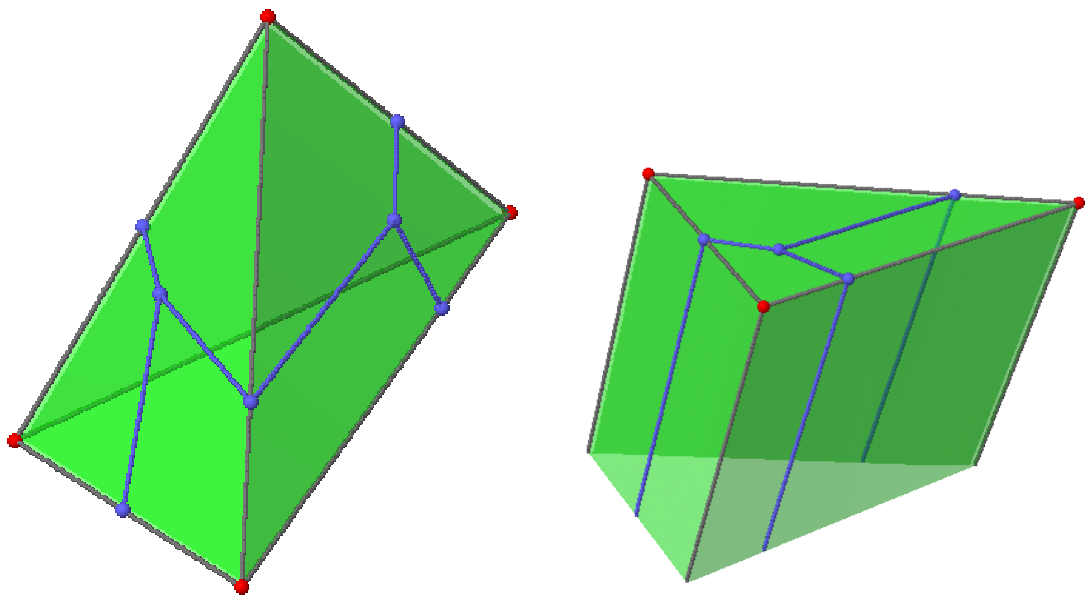
The fan structures at vertices allow us to define an affine structure on B away from a closed subset of codimension two Δ, called the *discriminant locus*, which can be constructed in the following way. For each bounded $\tau \in \mathcal{P}$, except for vertices and top-dimensional cells, choose $a_\tau \in \operatorname{Int} \tau$:



For each unbounded τ, except for top-dimensional cells, choose a non-zero $a_\tau \in \Lambda_{\tau,\mathbb{R}}$ such that $a_\tau + \tau \subset \tau$. For each chain $\tau_1 \subset \cdots \subset \tau_{n-1}$ with $\dim \tau_i = i$ and τ_i bounded for $i \leqslant r$, where $r \geqslant 1$, let

$$\Delta_{\tau_1 \ldots \tau_{n-1}} = \operatorname{Conv}\{a_{\tau_i} : 1 \leqslant i \leqslant r\} + \sum_{i>r} \mathbb{R}_{\geqslant 0} \cdot a_{\tau_i} \subset \tau_{n-1}$$

and let Δ be the union of such polyhedra (only "visible" half of Δ is shown for the bounded case):



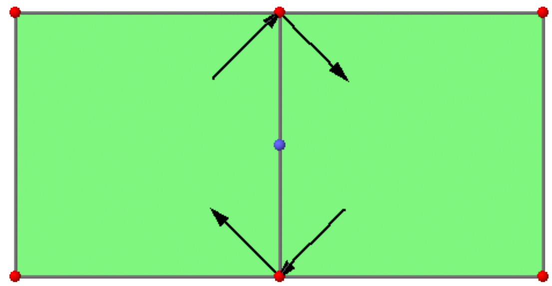
Observe, that if $\varrho \in \mathcal{P}$ is a cell of codimension 1, the connected components of $\varrho \setminus \Delta$ are in one-to-one correspondence with the vertices of ϱ. Thus we may use interiors of top dimensional cells and fan structures at vertices to define an affine structure on $B \setminus \Delta$ (fan structures give charts via the exponential maps). In particular, this defines an affine connection on $T_{B\setminus\Delta}$ which we may use for parallel transport of tangent vectors along paths.

There is some flexibility in constructing Δ. We may use barycenters to get a canonical choice, but for certain purposes it is better to take a "generic" discriminant locus, such that coordinates of a_τ are "as algebraically independent as possible" and Δ does not contain rational points.

4 Local Monodromy

Let $\omega \in \mathcal{P}$ be a bounded edge (i.e. a one-dimensional cell) with vertices $v^\pm$ and $\varrho \in \mathcal{P}$ be a face of codimension one, such that $\omega \subset \varrho$ (possibly $\omega = \varrho$), $\varrho \not\subset \partial B$, and $\varrho \subset \sigma^\pm$ for two top-dimensional cells.

Follow the change of affine charts given by the fan structure at v^+, the polyhedral structure of σ^+, the fan structure at v^-, the polyhedral structure of σ^-, and back to the fan structure at v^+:



We obtain a transformation $T_{\omega\varrho} \in \mathrm{SL}(\Lambda_{v^+})$, where Λ_{v^+} is the lattice of integral tangent vectors to B at the point v^+. (Not the integral vector fields along the zero-dimensional face v^+!) This transformation has the form

$$T_{\omega\varrho}(m) = m + \varkappa_{\omega\varrho}\langle m, \check{d}_\varrho\rangle d_\omega,$$

where $d_\omega \in \Lambda_\omega \subset \Lambda_{v^+}$ is the primitive integral vector from v^+ to v^- and $\check{d}_\varrho \in \Lambda_\varrho^\perp \subset \Lambda_{v^+}^*$ is the primitive integral vector evaluating positively on σ^+. In particular, if $m \in \Lambda_\varrho$, then $T_{\omega\varrho}(m) = m$, and for any m we have $T_{\omega\varrho}(m) - m \in \Lambda_\varrho$. The constant $\varkappa_{\omega\varrho}$ is independent on the ordering of $v^\pm$ and $\sigma^\pm$. It is non-negative for "geometrically meaningful manifolds". If all these constants are non-negative, B is called *positive*.

The monodromy is the obstruction to extending the affine structure from $B \setminus \Delta$ to B. It extends to a neighborhood of $\tau \in \mathcal{P}$ if and only if $\varkappa_{\omega\varrho} = 0$ for all ω and ϱ as above and such that $\omega \subset \tau \subset \varrho$.

5 Functions on Tropical Manifolds

An *affine function* on an open set $U \subset B$ is a continuous map $U \to \mathbb{R}$ that is affine on $U \setminus \Delta$.

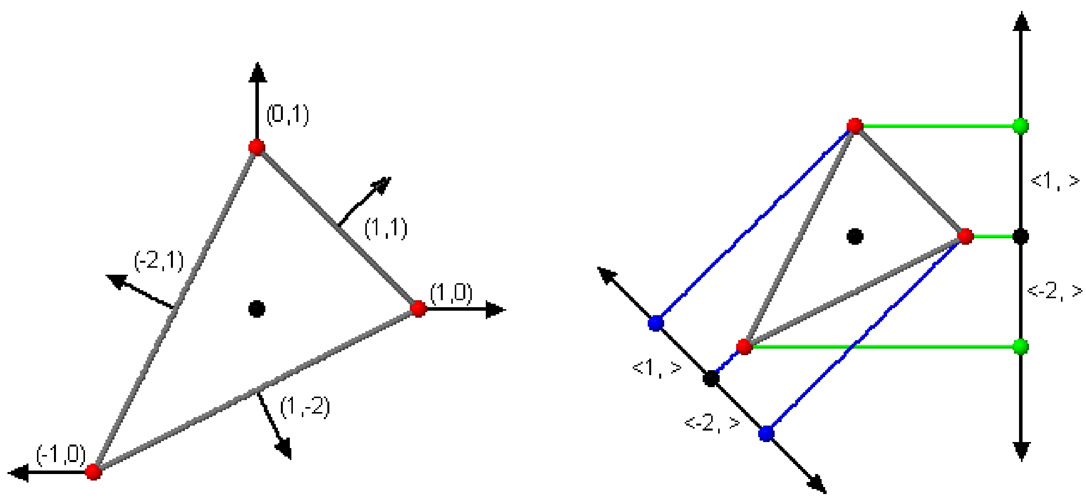
A *piecewise-linear (PL) function* on U is a continuous map $\varphi : U \to \mathbb{R}$ such that if $S_\tau : U_\tau \to N_{\tau,\mathbb{R}}$ is the fan structure along $\tau \in \mathcal{P}$, then $\varphi|_{U \cap U_\tau} = \lambda + S_\tau^*(\varphi_\tau)$ for some affine function $\lambda : U_\tau \to \mathbb{R}$ and a function $\varphi_\tau : N_{\tau,\mathbb{R}} \to \mathbb{R}$ which is piecewise-linear with respect to the fan Σ_τ. (This definition ensures that φ is "good enough" near the discriminant locus.)

A *multivalued piecewise-linear (MPL) function* φ on U is a collection of PL functions $\{\varphi_i\}$ on some open cover $\{U_i\}$ of U, such that φ_i's differ by affine functions on overlaps. We see that an MPL function φ can be given by specifying maps $\varphi_\tau : N_{\tau,\mathbb{R}} \to \mathbb{R}$ as above.

All of the above definitions can be restricted to integral functions in the obvious way.

If all local PL representatives of an integral MPL function φ are strictly convex, we call φ a *polarization* of $(B, \mathcal{P})$ and $(B, \mathcal{P}, \varphi)$ a *polarized integral tropical manifold*. (If $\partial B \neq \varnothing$ we also require that $|\Sigma_\tau|$ is convex for every $\tau \in \mathcal{P}$.)

Example. *For a manifold coming from a reflexive polytope Δ as described above, a polarization can be obtained in the following way. Let ψ be a piecewise linear function on the fan generated by Δ, such that $\psi|_{\partial\Delta} \equiv 1$. For each vertex v choose an integral affine function ψ_v such that $\psi_v(v) = 1$. Let $\varphi_v = \psi - \psi_v$ on U_v. Taking the manifold constructed before, ψ is given on top dimensional cones by pairing with $(1, 1)$, $(-2, 1)$, and $(1, -2)$, while for ψ_v we can choose $(1, 0)$, $(0, 1)$, and $(-1, 0)$. (This choice is not unique, but different choices will change φ_v by an integral affine function, thus specifying the same MPL function.) Then φ_v on each fan is the pairing with -2 on one of the cones and 1 on the other.*



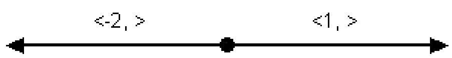
6 Discrete Legendre Transform

The discrete Legendre transform is a duality transformation of the set of polarized integral tropical manifolds $(B, \mathcal{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathcal{P}}, \check{\varphi})$.

As a category, $\check{\mathcal{P}}$ is the opposite of $\mathcal{P}$. The functor $\check{F} : \check{\mathcal{P}} \to \mathbf{LPoly}$ is given by $\check{F}(\check{\tau}) = \mathrm{Newton}(\varphi_\tau)$, where $\check{\tau} = \tau$ as objects and $\mathrm{Newton}(\varphi_\tau)$ is the Newton polyhedron of φ_τ:

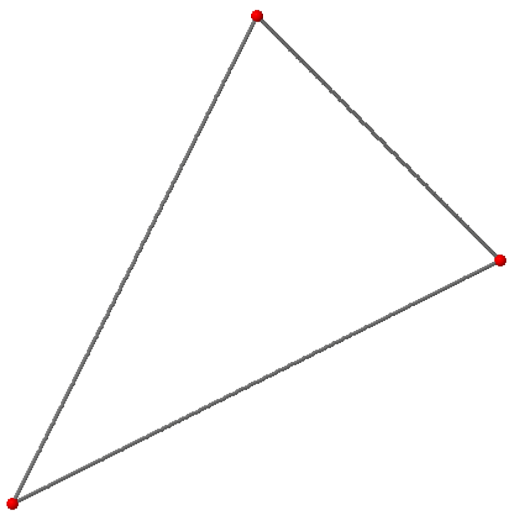
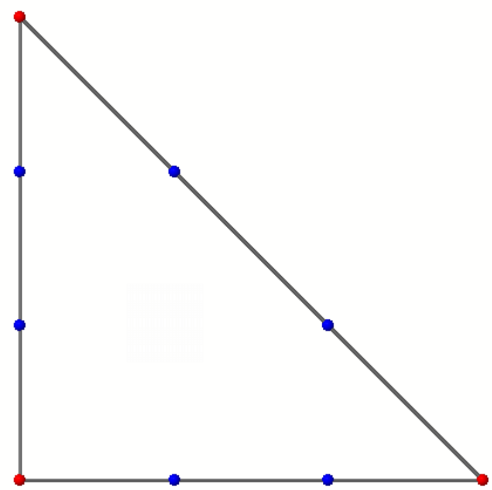
$$\mathrm{Newton}(\varphi_\tau) = \left\{ x \in N^*_{\tau,\mathbb{R}} : \varphi_\tau + x \geqslant 0 \right\}.$$

Example. *For the fans from our example we obtain a line segment of length three:*



These data are enough to construct $\check{B}$ as a topological manifold. (It also can be obtained as the dual cell complex of $(B, \mathcal{P})$ using the barycentric subdivision, which gives a homeomorphism between B and $\check{B}$, however, this dual cell complex is not a polyhedral complex.)

Example. *Three line segments of the length three give us the boundary of the polar reflexive polygon:*

We still need to give fan structures at vertices and PL functions on them — these data come from the polyhedral structure of the top-dimensional cells of $\mathcal{P}$. Namely, we will produce a fan at each vertex, a PL-function on this fan, and give integral affine correspondence between cones of this fan and (parts of) faces containing this vertex.

Let $\sigma \in \mathcal{P}$ be a top-dimensional cell. Then $\check{\sigma}$ is a vertex. Let $\Sigma_{\check{\sigma}}$ be the normal fan of σ in $\Lambda^*_{\sigma,\mathbb{R}}$ (i.e. the fan generated by inward normals to facets of σ). Using the parallel transport, we can identify $\Lambda^*_{\sigma,\mathbb{R}}$ with $\Lambda^*_{v,\mathbb{R}} = T^*_{B,v}$ for a vertex v of σ. Let $\widetilde{\sigma} \subset T_{B,v}$ be the inverse image of σ under the exponential map. Let $\check{\varphi}_{\check{\sigma}} : |\Sigma_{\check{\sigma}}| \to \mathbb{R}$ be given by $\check{\varphi}_{\check{\sigma}}(m) = -\inf(m(\widetilde{\sigma}))$. If $\Sigma_{\check{\sigma}}$ is incomplete, extend $\check{\varphi}_{\check{\sigma}}$ to $\Lambda^*_{\tau,\mathbb{R}}$ by infinity. Note, that a different choice of v corresponds to the translation of $\widetilde{\sigma}$ by an integral vector, thus to the change of $\check{\varphi}_{\check{\sigma}}$ by an integral affine function, which will lead us to the same MPL function.

Example. *For the "top dimensional cells" from our example we obtain MPL functions represented by pairing with* 0 *on one of the cones of the normal fan and* -1 *on the other one (same as if we repeated construction with vectors):*

Finally, consider the cones

$$\{m \in \Lambda^*_{v,\mathbb{R}} : m(\widetilde{\sigma}) \geqslant 0\} \in \Sigma_{\check{\sigma}}, \qquad \{m \in N^*_{v,\mathbb{R}} : m(dS_v(\widetilde{\sigma})) \geqslant 0\} \in \Sigma^*_v$$

and note that dS_v is an integral affine identification of cones corresponding to cells containing v ("tangent wedges") in $T_{B,v} \simeq \Lambda_{v,\mathbb{R}}$ and $N_{v,\mathbb{R}}$. Therefore, the above cones can be identified via $dS^*_v : N^*_{v,\mathbb{R}} \to \Lambda^*_{v,\mathbb{R}}$ and these maps induce the fan structure at $\check{\sigma}$. Thus we obtain the structure of a polarized integral tropical manifold.

In general, we may use the homeomorphism mentioned above to take $\Delta = \check{\Delta}$. It also turns out, that the Legendre transform of a positive manifolds is positive.

References

[GS1] M. Gross, B. Siebert: *Mirror symmetry via logarithmic degeneration data I*, J. Differential Geom. **72** (2006), 169–338.

[GS2] M. Gross, B. Siebert: *From real affine geometry to complex geometry*, arXiv:math/0703822v2 [math.AG] (2007).