Evaluating tautological classes using only Hurwitz numbers

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Algebraic Geometry Seminar, Columbia University

A talk answering a question posed by Ravi Vakil, to whom goes my gratitude for multiple reasons.

Outline

Motivation 1

Faber Conjecture

$R^*(\mathcal{M}_g)$ is a Poincaré duality ring with socle in degree $g-2$.

The class $\lambda_g\lambda_{g-1}$ vanishes on the boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, and hence is an evaluation class for $R^*(M_g)$.

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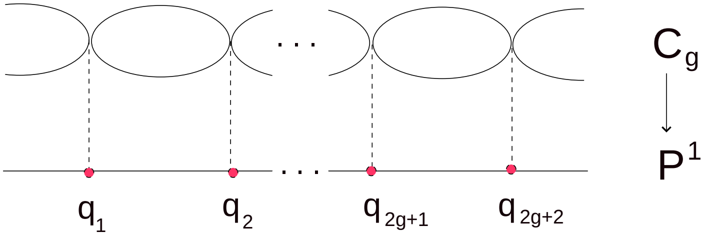

The hyperelliptic locus H_g is a $2g-1$ dimensional tautological class. Our computation shows in particular that it is a non-trivial class in $R^{g-2}(\mathcal{M}_g)$.

Motivation 2

The evaluation of $\lambda_g\lambda_{g-1}$ on the hyperelliptic locus determines completely the degree 2 (level $(0,0)$) *local Gromov-Witten theory of curves* of Bryan and Pandharipande.

Can show this in two steps:

1. Local invariants of curves can be organized to be the structure constants of a Topological Quantum Field Theory.
2. The generators of the TQFT can be reduced via localization to the evaluation of $\lambda_g\lambda_{g-1}$ on H_g.

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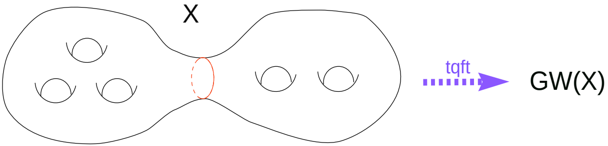

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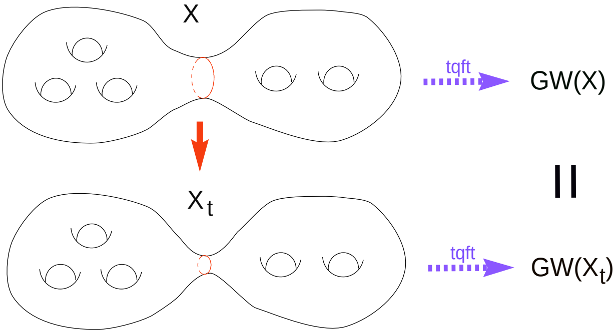

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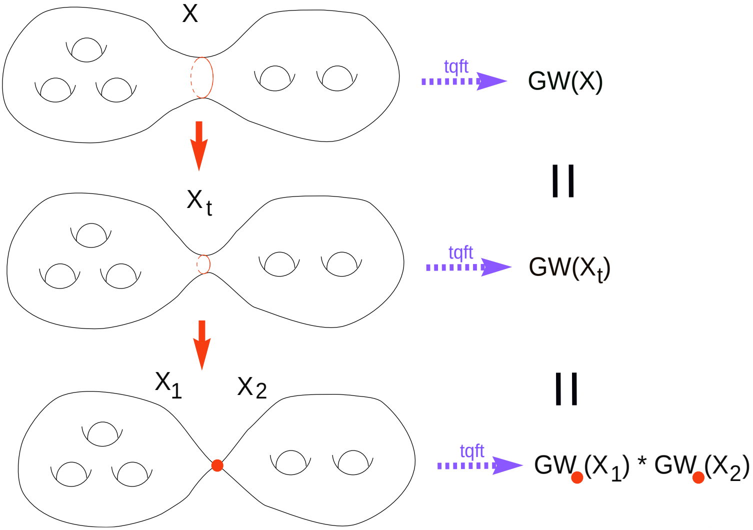

Motivation 3

The evaluation of $\lambda_g\lambda_{g-1}$ on H_g controls the orbifold Gromov-Witten theory of the orbifold quotient

$$\mathfrak{X} = [\mathbb{C}^2/\mathbb{Z}_2]$$

.

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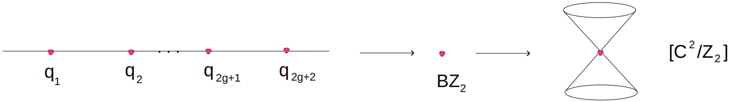

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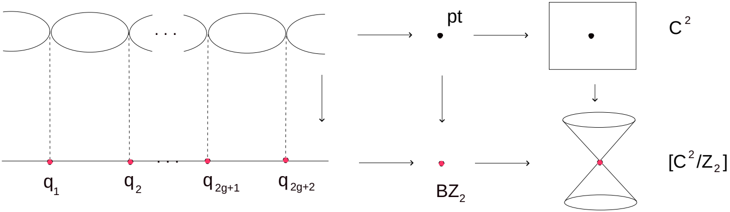

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In the late '90s Faber and Pandharipande computed the evaluation of the (closure of the) hyperelliptic locus in $R^{g-2}(\mathcal{M}_g)$ using a GRR computation by Mumford.

Our philosophy is to understand this geometric problem via purely combinatorial instruments - and give a new proof of such computation.

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The *Hodge bundle*

$$\mathbb{E}_g \to \overline{\mathcal{M}_g}$$

is a rank g vector bundle, whose fiber over a curve C is:

- the holomorphic differential 1-forms on C (if C is smooth).
- the global sections of the relative dualizing sheaf (K_C if C smooth).
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The *Simple Hurwitz number* H^g_η :

"number" of degree d covers $E \xrightarrow{\pi} \mathbb{P}^1$ such that:

- E is a (connected) curve of genus g.
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The above number is weighted by the number of automorphisms of such covers.

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Hurwitz Numbers are Combinatorial

By “identifying" a ramified cover with its monodromy representation, we obtain the following purely combinatorial expressions for simple Hurwitz numbers:

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POP QUIZ: What are all hyperelliptic Hurwitz numbers?

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Generating Functions for Simple Hurwitz Numbers

It's often useful to package Hurwitz numbers for all genera in formal power series form:

$$\mathcal{H}_\eta(u) := \sum H^g_\eta \frac{u^{\varphi(g)}}{\varphi(g)!}.$$

$\varphi(g) = 2g + d + \ell(\eta) - 2 =$ number of simple ramification points not including ∞.

Back to Hyperelliptic numbers

$$\mathcal{H}_{(2)}(u) := \sum H^g_{(2)} \frac{u^{2g+1}}{(2g+1)!} = \frac{1}{2}\sinh(u).$$

$$\mathcal{H}_{(1,1)}(u) := \sum H^g_{(1,1)} \frac{u^{2g+2}}{(2g+2)!} = \frac{1}{2}(\cosh(u)-1).$$

Moduli Spaces of Admissible Covers

Let X be a nodal curve of genus h.
An admissible cover of X of degree d is a finite morphism $\pi : E \longrightarrow X'$ satisfying the following:

- $X' = X \cup T_1 \cup \ldots \cup T_n$ is a nodal curve obtained by attaching rational tails to X.
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   - they are smooth (stacks);
   - the boundary is “combinatorial".
2. They are useful spaces:
   - *Ionel, Graber-Vakil*: applications to the study of the tautological ring of moduli spaces of curves.
   - *Costello, Bryan-Graber-Pandharipande*: orbifold GW theory of Gorenstein stacks.
3. They are handy spaces:
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The Theorems

Faber-Pandharipande New proof (-)

Denote by $\overline{H}_g$ a $(2g+2)!$ cover of the hyperelliptic locus obtained by marking all the Weierstrass points. Then:

$$\mathcal{F}(u) := \sum_{g=1}^{\infty} \left(\int_{\overline{H}_g} \lambda_g \lambda_{g-1} \right) \frac{u^{2g-1}}{(2g-1)!} = \frac{1}{2} \tan\left(\frac{u}{2}\right).$$

This is what we will proof!

A generalization

Looijenga/Bryan-Pandharipande
New proof (Bertram , -, Todorov)

Let $\overline{H}_{dd} \subseteq \overline{\mathcal{M}}_g$ be the closure of the locus of curves that admit a degree d map to $\mathbb{P}^1$ with two fully ramified points (again, all branch locus marked). Then:

$$\sum_{g=1}^{\infty} \left(\int_{\overline{H}_{dd}} \lambda_g \lambda_{g-1} \right) \frac{u^{2g-1}}{(2g-1)!} = \frac{1}{2} \left(\cot\left(\frac{u}{2}\right) - d \cot\left(\frac{du}{2}\right) \right).$$

The strategy

1. Relate the (evaluation of) $\lambda_g \lambda_{g-1}$ to tautological classes with descendants.
2. Find a way to compute the sum of all such classes in terms of $\lambda_g \lambda_{g-1}$.
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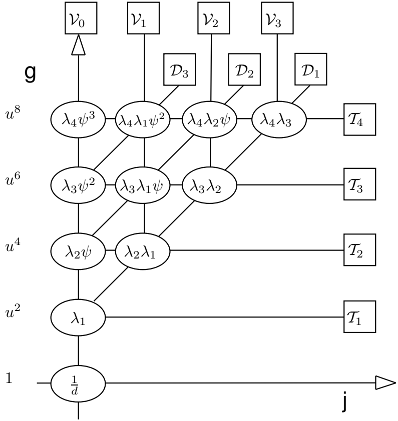
Introducing descendants:



Double Hodge Functions

For any degree d, define

$$\mathcal{L}_i(u) := \sum_{g=i}^{\infty} \left(\int_{\overline{Adm}_{g \xrightarrow{d} 0,(d),(d),(2),\ldots,(2)}} \lambda_g \lambda_{g-i} \psi^{i-1} \right) \frac{u^{2g}}{(2g)!}$$

Then

Theorem (-)

$$\mathcal{L}_i(u) = \frac{d^{i-1}}{i!} \mathcal{L}_1^i(u)$$

Remark: we use the theorem to define $\mathcal{L}_0 = \frac{1}{d}$. This is not a wacky thing to do.

The Calabi-Yau cap

Lemma (-)

$$CY(u) := \frac{1}{2}u + \sum_{g=1}^{\infty}\left(\int_{\overline{Adm}_{g\xrightarrow{2}0,(2),\ldots,(2)}} \lambda_g\lambda_{g-1} + \lambda_g\lambda_{g-2}\psi + \ldots + \lambda_g\psi^{g-1}\right)\frac{u^{2g+1}}{(2g+1)!} =$$

$$= \tan\left(\frac{u}{2}\right).$$

Putting everything together

$$CY(u) \quad \leftrightarrow \quad \sum_{0}^{\infty} \mathcal{L}_i(u) \quad \leftrightarrow \quad \exp(\mathcal{L}_1)(u)$$

$$\mathcal{L}_1(u) \quad \leftrightarrow \quad \mathcal{F}(u)$$

Putting everything together

$$\frac{d}{du}CY(u) \quad = \quad \sum_{0}^{\infty}\mathcal{L}_i(u) \quad = \quad \frac{1}{2}e^{\frac{(\mathcal{L}_1)(u)}{2}}$$

$$\frac{d}{du}\mathcal{L}_1(u) \quad = \quad \mathcal{F}(u)$$