# Wall Crossings for Double Hurwitz Numbers

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# Coauthors

This is joint work with Paul Johnson (University of Michigan) and Hannah Markwig (Goettingen).

# Double Hurwitz numbers (geometry)

The *Double Hurwitz number* $H^g(\alpha, -\beta)$ :

"number" of degree $d$ covers $E \xrightarrow{\pi} \mathbb{P}^1$ such that:

- $E$ is a (connected) curve of genus $g$.
- $\pi$ is unramified over $\mathbb{P}^1 \setminus \{0, p_1, \ldots, p_r, \infty\}$;
- $\pi$ ramifies with profile $\alpha$ over 0.
- $\pi$ ramifies with profile $\beta$ over $\infty$.
- $\pi$ has simple ramification over the other $p_i$'s.
- The preimages of 0 and $\infty$ are marked.

The above number is weighted by the number of automorphisms of such covers.

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# Double Hurwitz Numbers (combinatorics)

The *Double Hurwitz number* $H^g(\alpha, -\beta)$ :

"number" of $\sigma_0, \tau_1, \dots, \tau_r, \sigma_\infty \in S_d$ such that:

- $\sigma_0$ has cycle type $\alpha$;
- $\tau_i$'s are simple transpositions;
- $\sigma_\infty$ has cycle type $\beta$;
- $\sigma_0 \tau_1 \dots \tau_r \sigma_\infty = 1$
- the subgroup generated by such elements acts transitively on the set $\{1, \dots, d\}$

The above number is multiplied by the automorphisms of the permutations $\sigma_0, \sigma_\infty$ and divided by $d!$.

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# Examples

A couple silly examples:

$$H^0((1,1),-(1,1)) = 2$$

$$H^0((2,1),-(2,1)) = 4$$

# Paul's insight

One can count Hurwitz numbers by counting movies of the monodromy representation, i.e. organizing the count by the cycle types of the successive products:

$$\sigma_0, \sigma_0\tau_1, \sigma_0\tau_1\tau_2 \cdots$$

Via the cut and join equations, such movies can be represented by trivalent graphs. Hence:

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# The formula

**Theorem (C, Johnson, Markwig, 2008)**

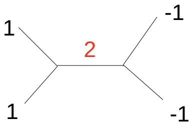
$$H^g(\alpha, -\beta) = Aut(\alpha)Aut(\beta) \sum_{\Gamma} \frac{1}{Aut(\Gamma)} \prod_{IE} e_i \qquad (1)$$

- $\Gamma$ are trivalent, oriented, genus $g$ graphs.
- the edges have positive integer weights satisfying the 0-tension condition at each vertex.
- the ends are labelled by the parts of $\alpha$ and $\beta$.
- the vertices are totally ordered (compatibly with the edges).
- *IE* stands for “internal edges".

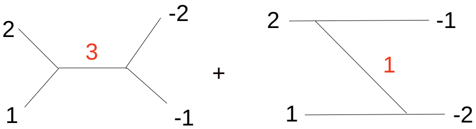
# Examples

Back to the silly examples:

$$H^0((1,1),-(1,1)) = 2$$




$$H^0((2,1),-(2,1)) = 4$$



# Structure of double Hurwitz Numbers

Think of double Hurwitz numbers as functions in the entries of the partitions:

$$\begin{array}{rccc} H^g(-): & \mathcal{H} \subseteq \mathbb{R}^{\ell(\alpha)+\ell(\beta)} & \rightarrow & \mathbb{R} \\ & (\alpha_1, \ldots, -\beta_{\ell(\beta)}) & \mapsto & H^g(\alpha, -\beta) \end{array}$$

where

$$\mathcal{H} = \left\{ \sum_{i=1}^{\ell(\alpha)+\ell(\beta)} x_i = 0 \right\}$$

# Goulden, Jackson, Vakil:

1. $\mathcal{H}$ is subdivided into a finite number of chambers, inside each of which the Hurwitz numbers are polynomials in the entries of the partitions.
2. polynomials have nonzero coefficients between top degree $4g - 3 + \ell(\alpha) + \ell(\beta)$ and bottom degree $2g - 3 + \ell(\alpha) + \ell(\beta)$.
3. polynomials are either even or odd (according to parity of leading degree).
4. polynomials should be interpreted as intersection numbers on some moduli space.

# The genus 0 story

In genus 0, Shapiro, Shadrin, Vainshtein settle the story:

1. They describe the location of the walls.
2. They give a closed formula for the Hurwitz number in one chamber.
3. They describe wall crossing formulas at any wall. It has the flavor of a degeneration formula.

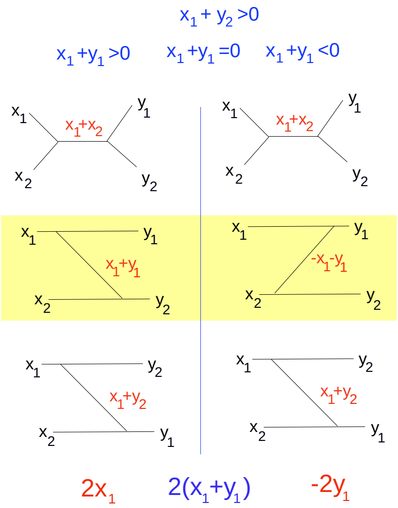
# Specifically:

Wall: $\sum_I x_i = 0$.

Set $\Delta = \sum_I x_i$.

$$\begin{aligned} WC(x) &= P_{\sum_I x_i > 0}(x) - P_{\sum_I x_i < 0}(x) \\ &= \binom{r}{r_1, r_2} \Delta H^0(x_I, -\Delta) H^0(x_{I^c}, \Delta) \end{aligned}$$

# Example



$x_1 + y_2 > 0$
$x_1 + y_1 > 0$
$x_1 + y_1 = 0$
$x_1 + y_1 < 0$
$x_1$
$x_2$
$y_1$
$y_2$
$x_1 + x_2$
$x_1 + y_1$
$-x_1 - y_1$
$x_1 + y_2$
$2x_1$
$2(x_1 + y_1)$
$-2y_1$

# Main Result

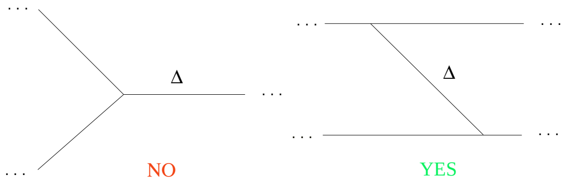
Keeping fingers crossed:

**Theorem** (C, Johnson, Markwig, 2009)

Location of walls and degeneration-type wall crossing formulas are described in arbitrary genus $g$.

# Sketch of proof in $g = 0$

- The graphs that can contribute to the wall crossing must contain an edge labelled $\Delta$.
- Further, this edge must be allowed to "flip".




- When cutting along $\Delta$, note that the graph of each connected component is a graph used to compute the Hurwitz numbers corresponding to the splitting of inputs and outputs prescribed by the wall.

- Conversely, start from two graphs used to compute the Hurwitz numbers for the connected components. Each such graph comes with a total order of the vertices, and there are $\binom{r}{r_1, r_2}$ ways to merge the orders on each components to a total order of ALL vertices. Gluing the graphs along $\Delta$, you obtain a graph contributing to the wall crossing!
- The contributions:

WC:

$$\prod_{IE \in \Gamma} e_k = \Delta \prod_{IE \in \Gamma_1} e_i \prod_{IE \in \Gamma_2} e_j$$

H:

$$\prod_{IE \in \Gamma_1} e_i \prod_{IE \in \Gamma_2} e_j$$

# Remark

What makes our proof nice and easy is that we have a geometric bijection between the graphs contributing to both sides of the formula. The polynomial contributions of each graph then just follow along for a ride!

$$\Gamma \overset{\varphi}{\leftrightarrow} (\Gamma_1, \Gamma_2, m(\Gamma_1, \Gamma_2))$$

# The woes of higher genus:

1. There is a $g$ dimensional polytope parameterizing admissible flows on a graph with assigned ends. The polynomial contribution to the Hurwitz numbers is obtained by summing the product of the edges over all integral points of this polytope.
2. There is NOT a natural bijection between the graphs appearing on two sides of the formula. A very subtle inclusion/exclusion is required.

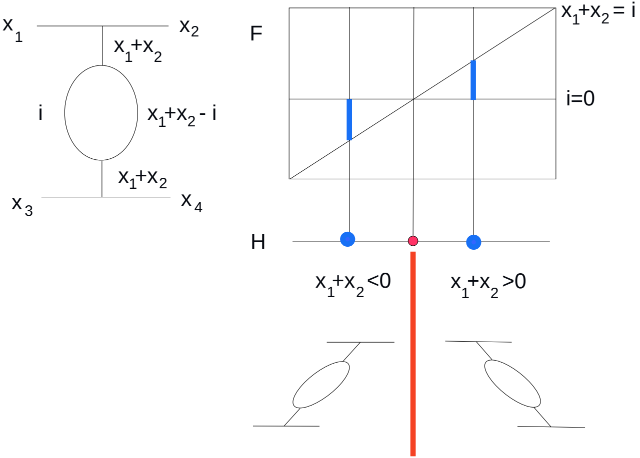
# The precise statement of the theorem

**Theorem** (C, Johnson, Markwig, 2009)

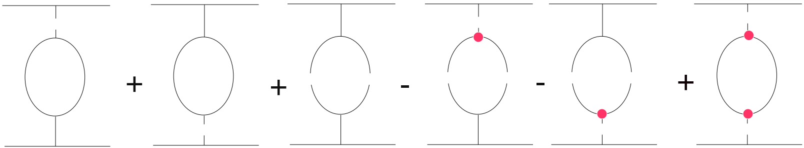
For the wall $\Delta = \sum_I x_i = 0$

$$WC(x) = \sum_{\substack{r_1+r_2+r_3=r \\ |\lambda|=|\eta|=\Delta}} (-1)^{r_2} \binom{r}{r_1, r_2, r_3} \frac{\prod \lambda_i}{\ell(\lambda)!} \frac{\prod \eta_j}{\ell(\eta)!} H^{r_1}(x_I, -\lambda) H^{r_2 \bullet}(\lambda, -\eta) H^{r_3}(\eta, x_{I^C})$$

# An example

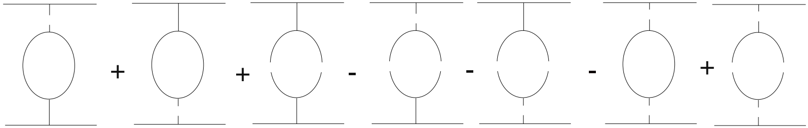



$$\frac{WC_\Gamma(x)}{(x_1+x_2)^2} = \sum_0^{x_1+x_2} i(x_1+x_2-i) - \sum_{x_1+x_2}^{0} i(x_1+x_2-i) = {\color{red}2} \sum_0^{x_1+x_2} i(x_1+x_2-i)$$




$$\binom{4}{1} + \binom{4}{3} + \binom{4}{2} - \binom{4}{1,1,2} - \binom{4}{2,1,1} + \binom{4}{1,2,1} = {\color{red}2}$$

# A better way to get that 2




$$\binom{4}{1}+\binom{4}{3}+\binom{4}{2}-\binom{4}{1,1,2}-\binom{4}{2,1,1}-\binom{4}{1,2,1}+\binom{4}{1,1,1,1}=2$$

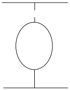
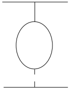
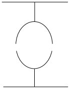
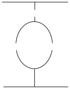
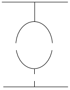
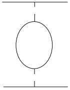
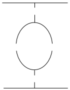
# The Hurwitz cuts inclusion exclusion

In general:

1. Given a graph $\Gamma$ there is a set of “cuttable edges".
2. Must cut cuttable edges in all possible legal ways.
3. Only cut edges are allowed to flop when reglued.
4. Direction of reglued edges is determined by merging of orders of vertices of components.
5. Inclusion/exclusion on number of connected components.

With this procedure, graphs contributing to the wall crossing are matched with graphs used to compute the Hurwitz numbers corresponding to the cuts of the graph, just like in genus 0.

# That 2 yet again

| |  |  |  |  |  |  |  | |
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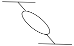
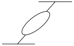
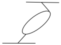
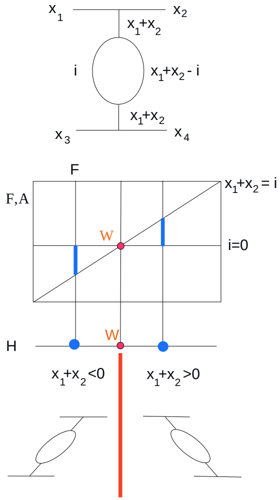
|  | + | + | + | - | - | - | + | 1 |
|  | | | | | | | + | 1 |
|  | | | | | - | | + | 0 |

# Recap: the general strategy of proof

For a topological type of graph Γ:

$$\begin{array}{ccccc} F_p^{(g)}, A_p & \hookrightarrow & \mathcal{F}, \mathcal{A}, \mathcal{W} & \longrightarrow & \mathbb{R}^E, \prod e_i \\ \downarrow & & \downarrow & & \downarrow \\ p & \hookrightarrow & H, W & \hookrightarrow & \mathbb{R}^{EE} \end{array}$$

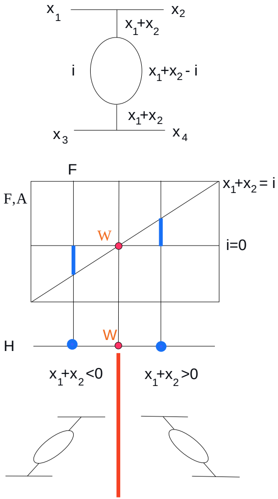


Checking the wall crossing formula for $\Gamma$:

$$\begin{array}{ccc} H_g(F_{p_1}, A_{p_1}) & \xrightarrow{\quad T \quad} & H_g(F_{p_2}, A_{p_2}) \\ \downarrow & & \downarrow \\ p_1 \longrightarrow & H, W & \longleftarrow p_2 \end{array}$$

amounts to checking, if
$V = \sum C_i \in H_g(F_{p_1}, A_{p_1})$:

$$V - T(V) = I/E(\Gamma)$$



...to get to the end of the story, there is still one or two ideas and a fair amount of technicalities, but maybe let's stop here for now.

Thank You!