Lecture 1

Renzo Cavalieri

Enumerative Geometry

Enumerative geometry is an ancient branch of mathematics that is concerned with counting geometric objects that satisfy a certain number of geometric conditions. Here are a few examples of typical enumerative geometry questions:

Q_1**:** How many lines pass through **2** points in the plane?

Q_2**:** How many conics pass through **5** points in the plane?

Q_3**:** How many rational[1] cubics (i.e. having one node) pass through **8** points in the plane?

Q_d**:** How many rational curves of degree d pass through $\mathbf{3d-1}$ points in the plane?

OK, well, these are all part of one big family...here is one of a slightly different flavor:

Q_l**:** How many lines pass through **4** lines in three dimensional space?

Some Observations:

1. I've deliberately left somewhat vague what the ambient space of our geometric objects: for one, I don't want to worry too much about it; second, if you like, for example, to work over funky number fields, then by all means these can still be interesting questions. In order to get nice answers we will be working over the complex numbers (where we have the fundamental theorem of algebra working for us). Also, when most algebraic geometers say things like "plane", what they really mean is an appropriate compactification of it...there's a lot of reasons to prefer working on compact spaces...but this is a slightly different story...

2. You might complain that the questions may have different answers, because I said nothing about how the points are distributed on the plane. Even in Q_1, if you take the two points to coincide, then you actually have

[1] Rational means that "it can be parameterized". I.e. there exists a function from a line to your curve that is generically one-to-one. Alternatively, your curve is the image in the plane of $(f(t), g(t))$, where f and g are polynomials of degree 3.

infinitely many lines going through them... that's why, in enumerative questions like the Q_d's, it is somewhat implicit that the points are taken to be **in general position**. What does this mean exactly? Well, there is a technical definition which I do not want to get into at this point, but think of it this way. If you were to be blinfolded and spun around before tossing each point onto the plane, then with probability one they will land in general position.

In other words, what I am saying is that the disposition of the points is not "too special" (e.g. if the two points don't coincide for Q_1) then there should be one nice finite answer for all the Q_d's.

Problem 1. *What does "too special" mean for Q_2 and Q_3? And if you like this game...try and see if you can say it for general d...*

3. When your points wander around and get "out" of general position often times your number of solutions to an enumerative question jumps to ∞. However, it **should not** (and in fact it doesn't!) jump from a finite number to another finite number. This somewhat heuristic idea is called the **principle of conservation of numbers**, and was used in the old days to solve enumerative questions: if you are able to place your points in any position (even if it is special) in such a way that you are able to find an answer and it is finite, then that is the right answer for your question in general!

   Problem 2 (Challenge)**.** *Use the principle of conservation of numbers to answer Q_l.*

4. Notice that for an enumerative question to have any hope to have a good answer, you have to impose the right number of conditions to your objetcs...in all of the Q_d's, if you ask for incidence to more than $3d-1$ points, then you find no curve at all; if you ask for fewer than $3d-1$ points, then you get infinitely many curves...

   Problem 3. *Using the footnote, figure out why $3d-1$ is the right number of points for the Q_d's.*

Solving Q_2

Any conic in the plane is the zero set of a degree 2 polynomial in x, y:

$$C = \{a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0\}$$

Therefore I can think that the sextuple of numbers $(a_1, \dots, a_6)$ identifies uniquely a conic in the plane. I.e. there is a function:

$$\left\{ \begin{array}{c} \text{sextuples} \\ (a_1, \dots, a_6) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{conics in} \\ \text{the plane} \end{array} \right\}$$

There is a lot of redundancy, however, since any two proportional sextuples will identify the same conic. But if we mod out by the equivalence relation

$$(a_1, \ldots, a_6) \sim (\lambda a_1, \ldots, \lambda a_6),$$

for any $\lambda \neq 0$, then we obtain a bijection:

$$\left\{ \begin{array}{c} \text{sextuples} \\ (a_1, \ldots, a_6) \end{array} \right\} / \sim \quad \hookrightarrow \quad \left\{ \begin{array}{c} \text{conics in} \\ \text{the plane} \end{array} \right\}$$

OK, so we have gotten ourselves an "algebraic" way of thinking of our set of conics. Can we translate in this language what it means for a conic to pass through a given point, say, for example $P = (8, 23)$?

It means that:

$$8^2 a_1 + (8)(23) a_2 + 23^2 a_3 + 8 a_4 + 23 a_5 + a_6 = 0$$

In other words, passing through a point corresponds to satisfying a linear equation in the coefficients $(a_1, \ldots, a_6)$.

Our question Q_2 therefore translates to solving a **homogeneous linear system** of five equations in six variables. Linear algebra now tells us that if the rank of the corresponding matrix is 5 (which is precisely our general position condition!) then there is exactly a one parameter homogenous family of solutions - which is precisely a proportionality class of sextuples - which is precisely **one conic**!

Problem 4. *Now that you know what the number is, devise an efficient method to find the solution given* 5 *specific points. I mean, nobody really wants to solve a linear system of rank* 5 *if he has the option not to, right?*

Geometric Interpretation

What we did in the previous paragraph has a geometric interpretation: the moduli space of all conics in the plane is a 5-dimensional projective space (you don't know what that is? Hang on, it is coming up in the next section! For the moment think of $\mathbb{C}^5$). Passing through one point defines a hyperplane - and five general hyperplanes in a five dimensional space intersect in exactly one point.

This suggests a general (geometric) strategy to approach an enumerative question:

1. Understand the moduli space of (all) the geometric objects you are looking for.

2. Translate the geometric conditions you want to satisfy to subvarieties of your moduli spaces.

3. Intersect the above subvarieties.

Thus an enumerative question is really, in disguise, a question about **intersection theory on moduli spaces**.

This new point of view is interesting, fascinating and powerful...however it doesn't mean that we have an easy way to the solution of enumerative questions...for example, for the Q_d's, you can solve Q_3 by generalizing the method of Q_2:

- the space of all cubics is $\mathbb{P}^9$.
- passing through a point is still a hyperplane.
- having a node corresponds to a degree 12 hypersurface in $\mathbb{P}^9$.

And therefore there are 12 rational cubics through 8 points in the plane.

The number of quartics (that incidentally is 620) was obtained in a similar way after a huge amount of hard work, which showed that this was not the right avenue to use in pursuing a solution to the general Q_d.

The key to solve this question, was to change the point of view and consider the "right" moduli spaces for the problem. The problem was solved in the nineties by none the less than M. Kontsevich...but I am getting ahead of myself...we'll come back to this.

Our First Moduli Spaces: $\mathbb{P}^n$

We want projective space $\mathbb{P}^n$ to be the moduli space of lines through the origin in $\mathbb{C}^{n+1}$. We also will see that it is a compactification[2] of affine space $\mathbb{C}^n$. Of course I am going to cheat a bit and present the dimension two case, and over the reals too - and will leave it to you as a useful exercise to generalize to arbitrary dimensions and to complex numbers.

$\mathbb{P}^2$: Take One!

One way to identify a line through the origin in $\mathbb{R}^3$ is to simply give a point on it, provided that the point is different from the origin. This means giving a triple of complex numbers $(z_1, z_2, z_3) \neq (0,0,0)$. Again, there is redundancy in this description, since two triples identify the same line if they are proportional to each other.

We therefore need to mod out by the equivalence relation

$$(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3),$$

$\lambda \neq 0$, to obtain:

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \smallsetminus \{(0,0,0)\}}{\sim} \tag{1}$$

[2]This means that $\mathbb{P}^n$ is a compact space and it contains $\mathbb{C}^n$ as a dense open set.

This is a nice very symmetric description to describe $\mathbb{P}^2$ as a set, and it also provides a set of **homogenous coordinates**. However it doesn't tell us much about the structure of $\mathbb{P}^2$.

Problem 5. *Does a polynomial F(X, Y, Z) give a function on* $\mathbb{P}^2$*?*

$\mathbb{P}^2$: Take Two!

One way to get rid of (most of) the redundancy is, instead of picking any point in $\mathbb{R}^3$ to represent a line, to allow only points that live on a sphere.

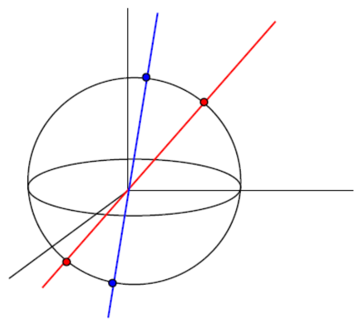

Figure 1: Lines through the origin in $\mathbb{R}^3$ intersect the sphere in two antipodal points.

If we do so, each line corresponds to precisely two antipodal points on the sphere, and therefore

$$\mathbb{P}^2 = \frac{\text{Sphere}}{P \sim -P} \tag{2}$$

This allows us to give a topology to $\mathbb{P}^2$, namely the quotient topology induced from the map from the sphere. Also, since we know the sphere is compact and the image of a compact space via a continuous function is compact, we immediately deduce that $\mathbb{P}^2$ is **compact**.

Problem 6. *Try to prove that* $\mathbb{P}^2$ *is* ***non-orientable***[3]*. One way to show this is to show that it contains a Mobius strip.*

$\mathbb{P}^2$: Take Three!

Yet another way to parameterize lines through the origin in $\mathbb{R}^3$ is the following: consider the plane $\{z = 1\}$. Most every line hits this plane in precisely one point. Unfortunately we are missing some lines...namely all those that live in

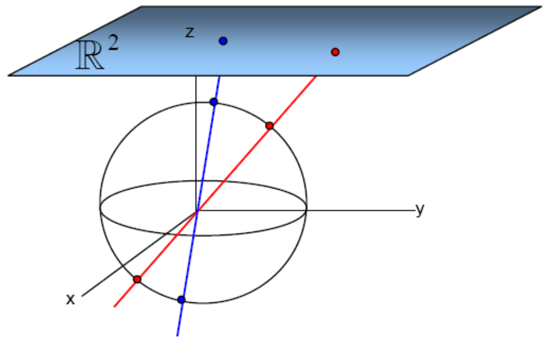

Figure 2: Mapping from lines in $\mathbb{R}^3$ to the plane $z = 1$.

the plane $\{z = 0\}$. Notice that such set of all lines through the origin in a plane is precisely a projective space of dimension 1 (a projective line).

As a set,

$$\mathbb{P}^2 = \mathbb{R}^2 \sqcup \mathbb{P}^1. \tag{3}$$

Problem 7. *Show that with the topology given above, $\mathbb{R}^2$ is an open dense set in $\mathbb{P}^2$. This shows that $\mathbb{P}^2$* ***is a compactification*** *of the plane.*

$\mathbb{P}^2$: Take Three and a Half!

The previous section should leave us slightly unhappy, because the asymmetry of it: the projective plane knows nothing about any particular $\mathbb{R}^2$ being special...therefore our idea is now to consider all possible planes to play the role of screens. This defines an **atlas** for $\mathbb{P}^2$ that allows us to show that $\mathbb{P}^2$ is in fact a smooth differentiable manifold.

Just to be lazy, instead of considering all charts, we choose a minimal atlas consisting of three charts.

$$\begin{array}{cccc} \varphi_z : & U_z = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, y) & \mapsto & (x : y : 1) \end{array}$$

$$\begin{array}{cccc} \varphi_y : & U_y = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, z) & \mapsto & (x : 1 : z) \end{array}$$

$$\begin{array}{cccc} \varphi_x : & U_x = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (y, z) & \mapsto & (1 : y : z) \end{array}$$

[3]Here it is essential that we work over $\mathbb{R}$.

Problem 8. *Describe the transition functions and check that they are differentiable on the overlaps.*

Problem 9. *Define a topology on $\mathbb{P}^2$ as follows: a set $U \subset \mathbb{P}^2$ is open if all of its preimages $\varphi_x^{-1}(U)$, $\varphi_y^{-1}(U)$, $\varphi_z^{-1}(U)$ are open sets of the plane with the euclidean topology. Show that this is indeed a topology, that it makes the three φ maps continuous, and that the images $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$, $\varphi_z(\mathbb{R}^2)$ become open dense sets of $\mathbb{P}^2$. Show that this coincides with the quotient topology defined before!*

Some more food for thoughts...

Problem 10.

1. *Prove that the following are equivalent definitions for the concept of a **line** in $\mathbb{P}^2$:*
   (a) *the set of solutions of a **homogeneous** degree 1 polynomial in X, Y, Z. I.e. the set of points in $\mathbb{P}^2$ that satisfy an equation of the form:*
   $$aX + bY + cZ = 0.$$
   (b) *a line in one of the charts, plus a uniquely determined point in the complement of the chart.*
   (c) *a plane through the origin in $\mathbb{R}^3$.*
2. *Prove that **any** two lines in $\mathbb{P}^2$ intersect in exactly one point.*
3. *In general it makes no sense to ask "where does a polynomial in X, Y, Z vanish in $\mathbb{P}^2$... for example, take the polynomial*
   $$f(X, Y, Z) = X + Y + Z^2$$
   *$f(-2, 1, 1) = 0$, and $f(-4, 2, 2) \neq 0$... but $(-2 : 1 : 1)$ and $(-4, 2, 2)$ are **the same point** in $\mathbb{P}^2$.*
   However, in exercise 1, we have defined lines as the solutions of certain polynomial equations...what saved the day in that case? In general, under which conditions are the zeroes of a polynomial a well-defined notion in $\mathbb{P}^2$? Make your guess for what should be a projective algebraic curve of degree d.
4. *Decide whether the following plane conic (ordinary plane, not $\mathbb{P}^2$!) is a parabola, an ellipse or a hyperbola.*
   $$x^2 + 4xy + 4y^2 + 342x + 57y - 22 = 0$$
   ***Hint:** think of the plane as one chart for $\mathbb{P}^2$. Using what you discovered in exercise 3, think of how to view this conic in $\mathbb{P}^2$, then ask yourself how do an ellipse, a parabola, a hyperbola intersect the line at infinity (i.e. the complement of the chart).*

5. *Show that the complex projective line is the one point compactification of $\mathbb{C}$, and it is therefore homeomorphic to a sphere.*
6. *Try to generalize all of this to $\mathbb{P}^n$.*

What Do We Want From A Moduli Space?

Let us extrapolate from the previous discussion what are the "qualities" we appreciate in a moduli space $\mathcal{M}$:

m1: Points in the space $\mathcal{M}$ are in bijection with the objects we wish to parameterize.

m2: The moduli space has a natural topology (differentiable structure, algebraic structure ... in general a structure similar to the objects you wish to parameterize). Such topology agrees with the intuitive notion of "small perturbation of the objects".

m3: Families of objects, i.e. a morphism of spaces

$$\begin{array}{c} Y \\ \downarrow \\ X, \end{array}$$

where the preimage of any $x \in X$ is one of our objects, should correspond to functions

$$f : X \to \mathcal{M}.$$

m4: If the moduli space is a compactification of some other natural object, then what you have to add to compactify is some combination of "smaller moduli spaces of the same type". I know, this is kind of vague...but think of how $\mathbb{P}^n$ compactifies $\mathbb{C}^n$ by adding a $\mathbb{P}^{n-1}$. Hopefully we will see more examples of this idea.

$G(k, n)$: Projective Space's Big Brothers

Let us consider an n-dimensional vector space V, and choose once and for all a basis $e_1, \ldots, e_n$. For a fixed $k \leq n$ we call **Grassmannian** the moduli space of linear subspaces of V of dimension k. We denote this space by $G(k, n)$.

Problem 11. *Convince yourself that $G(1, n+1) = \mathbb{P}^n$. Also, $G(n, n+1) \cong \mathbb{P}^n$*

We will try to get an intuition about the following
Fact: *$G(k, n)$ is a smooth compact (in fact projective) manifold of dimension $k(n-k)$.*

But all of this will have to wait until next time...in the mean time, you can start and think about the specific case of $G(2, 3)$ (where you know that the answer should be $\mathbb{P}^2$ AND you can draw pictures!), by trying to unravel the following mystery picture.

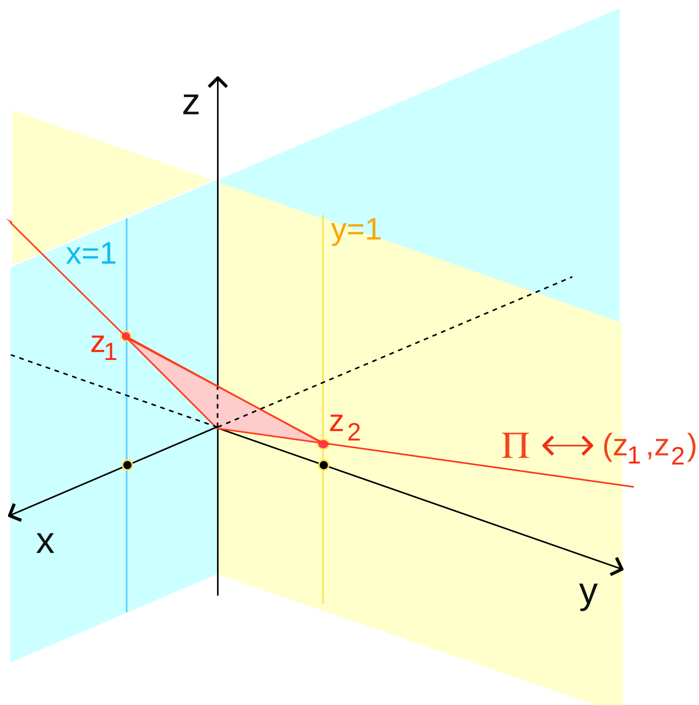

Figure 3: A natural chart for $G(2,3)$.