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From Topological Strings to Integrable Hierarchies & Back

Lecture #3:

* GW Theory of $[\mathbb{C}^3/\mathbb{Z}_3]$
* Givental Formalism
* C.R.C. á la Coates-Corti-Iritani-Tseng

§1. The orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$

$\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$ where $\mathbb{C}^3 \circlearrowleft \mathbb{Z}_3 : \omega \mapsto \begin{bmatrix} \omega & & \\ & \omega & \\ & & \omega \end{bmatrix}$

Interesting because:

- GW inv'ts are hard to compute: they are NOT reconstructed via WDVV from a finite # of base cases
- NOT Hard Lefschetz $\Rightarrow$ C.R.C. á la Bryan-Graber won't work
- has a well studied crepant resolution $= K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$
- Mirror Symmetry: $\mathcal{X}$ is the orbifold point for $K_{\mathbb{P}^2}$ $\Rightarrow$ $\Rightarrow$ [A.B.K.M.] (~~computed~~) predicted invariants in all genera

In mathematics $g=0$ invariants are computed:

[C.C.I.T.] compute non-equiv. invariants using Givental's formalism

[—, Cadman] compute equiv. invariants via auxiliary localization on gerbes

[Bayer, Cadman] compute invariants by sequence of transformations on moduli spaces of weighted stable maps.

$$I[\mathbb{C}^3/\mathbb{Z}_3] = \underbrace{[\mathbb{C}^3/\mathbb{Z}_3]_1}_{\text{age } 0} \amalg \underbrace{B\mathbb{Z}_{3\,\omega}}_{\text{age } 1} \amalg \underbrace{B\mathbb{Z}_{3\,\bar\omega}}_{\text{age } 2}$$

$$H^*_{CR}(\mathcal{X}) = H^0 \oplus H^2 \oplus H^4 = \langle 1 \rangle_{\mathbb{C}} \oplus \langle \omega \rangle_{\mathbb{C}} \oplus \langle \bar\omega \rangle_{\mathbb{C}}$$

- ω 'behaves like' a divisor
- $\bar\omega$ 'behaves like' a codimension 2 cycle.

$\mathcal{X}$ is a CY 3-fold $\Rightarrow$ $\operatorname{virdim} \overline{\mathcal{M}}_0(\mathcal{X},0) = 0$

$\Rightarrow$ only (non equiv.) invariants are codimension 1.

However, there is NO divisor equation for orbifold divisors $\Rightarrow$

$$\langle \omega^{3n} \rangle_0^{\mathcal{X},0} \quad \text{are all interesting!}$$

Non-equiv potential:

$$\boxed{F_{\mathcal{X}} = \frac{1}{3} x_1 x_\omega x_{\bar\omega} + \sum_{n>0} \langle \omega^{3n} \rangle_0^{\mathcal{X},0} \frac{x^{3n}}{3n!}}$$

Can make the theory richer by introducing the action of a $(\mathbb{C}^*)^3$ on $\mathcal{X}$, and considering insertions in equivariant cohomology:

$$H^*_{\mathbb{C}^{*3}}(pt) = \mathbb{C}[s_1, s_2, s_3]$$

$$\boxed{F_{\mathcal{X}}^{eq} = \frac{1}{3 s_1 s_2 s_3} x_1^3 + \frac{1}{3} x_1 x_\omega x_{\bar\omega} + \sum_{m,n>0} \langle \omega^m \bar\omega^n \rangle_0^{\mathcal{X},0} \frac{x_\omega^m}{m!} \frac{x_{\bar\omega}^n}{n!}}$$

where:

$\to$ $m + 2n \equiv 0 \pmod 3$

$\to$ $\langle \omega^m, \bar\omega^n \rangle$ is a polynomial in s_1, s_2, s_3, homogeneous of degree n.

Via standard techniques, $[\mathbb{C}^3/\mathbb{Z}_3]$ invariants can be expressed as $\mathbb{Z}_3$-Hodge integrals

$$\langle \omega^m \bar{\omega}^n \rangle = \int_{M_{0,n+m}(B\mathbb{Z}_3,0)} \prod_{i}^{n} ev_i^*(\omega) \prod_{j}^{m} ev_j^*(\bar{\omega})\, e^{eq}(\mathbb{E}^{\omega}(s_1) \oplus \mathbb{E}^{\omega}(s_2) \oplus \mathbb{E}^{\omega}(s_3))$$

Thm (Cadman - -)

- WDVV and 6 auxiliary localization comp. on spaces of maps to gerbes reconstruct all equivariant invariants.
- Relations translate to system of P.D.E.'s on a generating function for $\mathbb{Z}_3$-Hodge Integrals.

§2. Givental's Formalism

Idea: encode GWI's in a geometric object. Exploit the geometry to get information on invariants.

Symplectic Vector Space:

- $\mathcal{H} := H^*_{CR}(\mathfrak{X}) \otimes \mathbb{C}((z^{-1})) \ni f = \sum_{i=N}^{\infty} \frac{\alpha_i}{z^i} \qquad \alpha_i \in H^*_{CR}(\mathfrak{X})$

Rmk: $\mathfrak{X}$ has no curve classes $\Rightarrow$ no quantum parameters. In general $\mathcal{H} = H^*[[q's]] \otimes \mathbb{C}((z^{-1}))$

- $\Omega(f,g) := \mathrm{Res}_{z=0}\, (f(-z), g(z))$

↑ orbifold Poincaré pairing

Decomposition: $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

$$\left.\begin{aligned} \mathcal{H}^+ &= H^*_{CR}(\mathfrak{X}) \otimes \mathbb{C}[z] \\ \mathcal{H}^- &= H^*_{CR}(\mathfrak{X}) \otimes \mathbb{C}[[\tfrac{1}{z}]] \end{aligned}\right\} \text{Lagrangian!}$$

Choosing a pair of dual bases for $H^*_{CR}(\mathcal{X})$, we can give $\mathcal{H}$ Darboux coordinates.

Eg: $\phi_1 = \underline{1}$ $\quad \phi_\omega = \underline{\omega}$ $\quad \phi_{\bar\omega} = \underline{\bar\omega}$

$\phi^1 = 3\cdot\underline{1}$ $\quad \phi^\omega = 3\cdot\underline{\bar\omega}$ $\quad \phi^{\bar\omega} = 3\cdot\underline{\omega}$

Then $f \in \mathcal{H}$ has the form:

$$\boxed{f = \sum q_{k,\alpha}\, z^k \phi_\alpha + \sum p_{\ell,\beta} \frac{1}{z^{\ell+1}} \phi^\beta}$$

with $\quad \Omega\left(z^k \phi_\alpha, \frac{1}{z^{\ell+1}} \phi^\beta\right) = (-1)^k \delta_{k,\ell}\, \delta_{\alpha,\beta}$

Via Ω, $\mathcal{H}$ is identified with $T^*\mathcal{H}^+_*$.

Consider now the full descendant G.W. potential $(g=0)$

$$\mathcal{F}_{\mathcal{X}} = \sum \langle \alpha_1 \psi_1^{k_1}, \dots, \alpha_n \psi_n^{k_n} \rangle \frac{1}{n!} =$$

$$= \sum \langle \phi_{\alpha_1} \psi_1^{k_1}, \dots, \phi_{\alpha_n} \psi_n^{k_n} \rangle \frac{t_{\alpha_1,k_1} \cdot \ldots \cdot t_{\alpha_n,k_n}}{n!}$$

Now if we substitute

$$\begin{cases} t_{1,0} = q_{1,0} + 1 & \text{(dilaton shift)} \\ t_{\alpha,k} = q_{\alpha,k} \end{cases}$$

we can view $\quad \mathcal{F}_{\mathcal{X}} : \mathcal{H}^+ \longrightarrow \mathbb{C}$

the G.W. potential as a function on $\mathcal{H}^+$.

Hence

$$dF \in H^0(T^*\mathcal{H}^+)$$

$$\text{Graph } dF := \mathcal{L}^{\mathcal{X}} \subseteq T^*\mathcal{H}^+ = \mathcal{H}$$

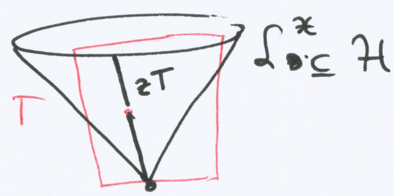
$$\mathcal{L}^{\mathcal{X}} = \left\{ q_{k,\alpha} z^k \phi_\alpha + \frac{\partial F}{\partial q_{k,\alpha}} \frac{1}{z^{k+1}} \phi^\alpha \right\}$$

a formal Lagrangian submanifold of $(\mathcal{H}, \Omega)$

Thm (Coates, Givental):

$\mathcal{L}_{\mathcal{X}}$ is the germ of a Lagrangian cone with vertex at $\underline{0}$

Each tangent space T is tangent precisely along zT



§3. Givental J-function

Givental's J-function is a generating function for invariants with exactly one descendant insertion.

$$J_{\mathcal{X}}(z, t_1, t_\omega, t_{\bar\omega}) = (z+t_1)\phi_1 + t_\omega \phi_\omega + t_{\bar\omega}\phi_{\bar\omega} +$$

$$+ \sum \langle \phi_1^m \phi_\omega^n \phi_{\bar\omega}^p, \phi_\alpha \psi^k \rangle \frac{t_1^m}{m!} \frac{t_\omega^n}{n!} \frac{t_{\bar\omega}^p}{p!} \frac{1}{z^{k+1}} \phi^\alpha$$

$$= z + \tau + \sum \left\langle \tau^n, \frac{\phi_\alpha}{z-\psi} \right\rangle \frac{\phi^\alpha}{n!}$$

Key point $\boxed{\Gamma(J(-z, t_1, t_\omega, t_{\bar\omega})) \in \mathcal{L}^{\mathcal{X}} \;!!!}$

§4. Tseng's Quantum R.R.

[Tseng]: can recover $J_{\mathcal{X}}$ from $J_{B\mathbb{Z}_3}$ via a natural transformation.

Since $J_{B\mathbb{Z}_3}$ is known from work of Jarvis + Kimura, can explicitly recover $J_{\mathcal{X}}$.

$$H^*_{CR}(B\mathbb{Z}_3) = H^0_{CR} = \langle \phi_1, \phi_\omega, \phi_{\bar\omega} \rangle_{\mathbb{C}} \qquad \tau = \sum \tau^i \phi_i$$

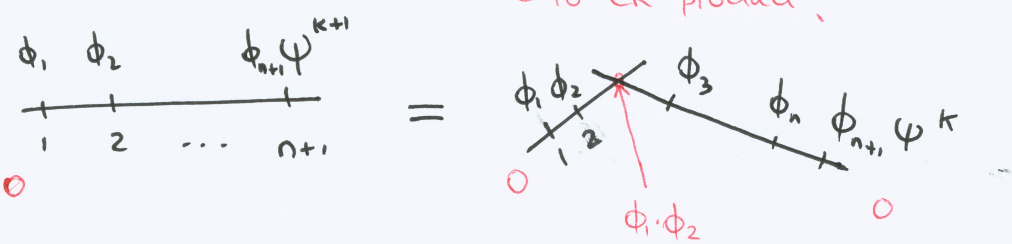
- TOPOLOGICAL RECURSION RELATIONS :

$J_{B\mathbb{Z}_3}$ satisfies following D.E.'s

$$\boxed{z \frac{\partial}{\partial \tau^i} \frac{\partial}{\partial \tau^j} J = \sum c_{ij}^{k} \frac{\partial}{\partial \tau^k} J} \qquad (\bullet)$$

coeff. of quantum product - in this case = to CR product.

$(\bullet)$:



→ [Jarvis-Kimura]: $H^*_{CR}(B\mathbb{Z}_3) \cong \mathbb{C}[\mathbb{Z}_3]$ is a semisimple algebra, with semisimple basis

$$f_0 = \frac{1}{3}\phi_1 + \frac{1}{3}\phi_\omega + \frac{1}{3}\phi_{\bar\omega}$$

$$f_1 = \frac{1}{3}\phi_1 + \frac{\bar\omega}{3}\phi_\omega + \frac{\omega}{3}\phi_{\bar\omega} \qquad (\Rightarrow f_i f_j = \delta_{ij})$$

$$f_2 = \frac{1}{3}\phi_1 + \frac{\omega}{3}\phi_\omega + \frac{\bar\omega}{3}\phi_{\bar\omega}$$

In this coord system $(\bullet)$ is diagonalized and solved by:

$$J_{B\mathbb{Z}_3} = z\, e^{\overbrace{(u_0 f_0 + u_1 f_1 + u_2 f_2)}^{\tau}/z} =$$

$$= z \sum \frac{\tau_1^{k_1}}{k_1!} \frac{\tau_\omega^{k_\omega}}{k_\omega!} \frac{\tau_{\bar\omega}^{k_{\bar\omega}}}{k_{\bar\omega}!} \frac{1}{z^{k_1+k_\omega+k_{\bar\omega}}} \left(\phi^{\omega}\right)^{\frac{\ell+2m}{3}}$$

And $\Gamma(J_{B\mathbb{Z}_3}) \in \mathcal{L}^{B\mathbb{Z}_3}$.

[Tseng]: there is an (explicitly described) linear map

$$\Delta : \mathcal{H}_{B\mathbb{Z}_3} \to \mathcal{H}_{\mathcal{X}} \quad \text{s.t.}$$

$$\Delta(\mathcal{L}^{B\mathbb{Z}_3}) = \mathcal{L}^{\mathcal{X}}$$

Applying Δ, get

NEW PART.

$$\Delta(J_{B\mathbb{Z}_3}) = +z \sum \begin{pmatrix}\text{same as}\\ \text{above}\end{pmatrix} \cdot \prod_{\substack{0 \le b < \frac{\ell+2m}{3} \\ \langle b \rangle = \langle \frac{\ell+2m}{3} \rangle}} (s_1 + \tfrac{1}{3} + bz)(s_2 + \tfrac{1}{3} + bz)(s_3 + \tfrac{1}{3} + bz)$$

$\Big\}$ (*)

which specializes at $\tau_{\bar\omega} = 0$ to the I-function of $\mathcal{X}$

$$\boxed{I = z e^{\tau_1/z} \sum \frac{(\tau')^{\ell}}{z^{\ell}\,\ell!} \prod_{\substack{i=1 \\ (*)}}^{3} (s_i + \tfrac{1}{3} + bz)\, \phi_{\omega\ell}}$$

But now you can just compare these 2 slices of $\mathcal{L}^{\mathcal{X}}$!

$$J = -z + \tau'\phi_1 + \tau^{\omega}\phi_\omega + \mathcal{O}(z^{-1})$$

$$I = -z + t'\phi_1 + \underbrace{\begin{pmatrix}\text{expression}\\ \text{involving}\\ \Gamma \text{ functions}\end{pmatrix}}_{(**)} \phi_\omega + \mathcal{O}(z^{-1})$$

$$\begin{cases} \tau' = t' \\ \tau^{w} = ((**)) \end{cases}$$ is naturally the mirror map.

Inverting the mirror map allows to read off GWI's for $\mathcal{X}$.

Remarks:

- Inverting typically can't be done in closed form, but can be coded to whatever power.
- Bayer-Cadman give a combinatorial formula for this inverse.

§ C.R.C. á la CCIT

There is a degree preserving $\mathbb{C}((z^{-1}))$ linear symplect. iso

$$\mathbb{U} : \mathcal{H}_{\mathcal{X}} \longrightarrow \mathcal{H}_{Y}$$

and a choice of analytic continuations of $\mathcal{L}^{\mathcal{X}}, \mathcal{L}^{Y}$ such that $\mathbb{U}(\mathcal{L}^{\mathcal{X}}) = \mathcal{L}^{Y}$

Further:

(1) $\mathbb{U}(1_{\mathcal{X}}) = 1_{Y} + O(z^{-1})$

(2) $\mathbb{U} \circ (p \cdot -) = \pi^{*}p \cdot \mathbb{U}(-)$ $\quad p$ an untwisted divisor

(3) $\mathbb{U}(\mathcal{H}_{\mathcal{X}}^{+}) \oplus \mathcal{H}_{Y}^{-} = \mathcal{H}_{Y}$

(4) entries of $\mathbb{U}$ don't depend on the quantum parameters.

Tom Coates checked C.R.C. in this form for

$$\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$$

$$Y = K_{\mathbb{P}^2}.$$