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From topological Strings to Integrable Hyerarchies and back.

(Trieste, Sept 2008)

Lectures on:

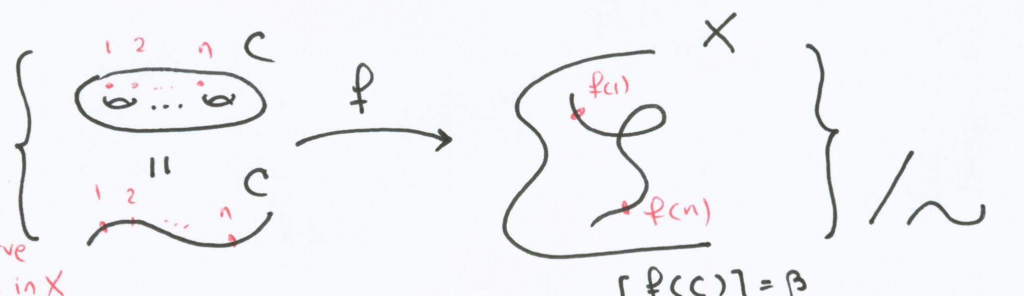
GROMOV-WITTEN THEORY of ORBIFOLDS

Lecture # 1:

* Moduli Spaces of Stable Maps
* G.W. Invariants
* G.W. theory for Orbifolds
* WDVV
* Localization

§1. Moduli Spaces of Stable maps

$$M_{g,n}(X,\beta) = \left\{ \quad \right\} /\sim$$



g: genus of source curve

n: marks on C

X: fixed target space

β: a curve class in X, $\beta \in H_2(X)$

Compactify by:

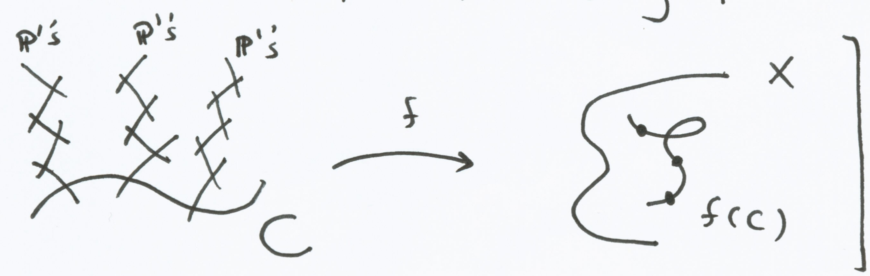
(1) $f_*[C] = \beta$

(2) allow C to be a nodal curve, BUT

(3) require f to have a finite # of automorphisms

(prevents out of control bubbling of chains of contracted $\mathbb{P}$'s)

No:



Good News:

(•) $\bar{\mathcal{M}}_{g,n}(X,\beta)$ is a compact moduli space

(••) The boundary of $\bar{\mathcal{M}}_{g,n}(X,\beta)$ is <u>MODULAR</u>.

(•••) The family of $\bar{\mathcal{M}}_{g,n}(X,\beta)$ comes with natural maps:

* forgetful maps

$$\begin{cases} \text{points: } \pi : \bar{\mathcal{M}}_{g,n+1}(X,\beta) \to \bar{\mathcal{M}}_{g,n}(X,\beta) \\ \qquad (C, f, p_1, \dots, p_{n+1}) \mapsto (C, f, p_1, \dots, p_n) \\ \text{maps: } f : \bar{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \bar{\mathcal{M}}_{g,n} \\ \qquad (C, f) \longmapsto C \end{cases}$$

* gluing maps

$$\begin{cases} gl_{12} : \bar{\mathcal{M}}_{g_1,\circ}(X,\beta_1) \underset{f_\circ = f_\star}{\times} \bar{\mathcal{M}}_{g_2,\star}(X,\beta_2) \to \bar{\mathcal{M}}_{g_1+g_2}(X,\beta_1+\beta_2) \\ gl_{irr} : \bar{\mathcal{M}}_{g-1,\{\circ,\star\}}(X,\beta)_{\{f_\circ = f_\star\}} \longrightarrow \bar{\mathcal{M}}_g(X,\beta) \end{cases}$$

* evaluation maps

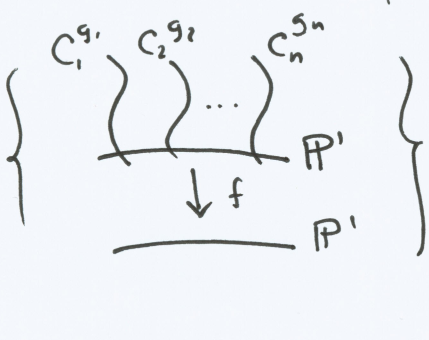
$$ev_i : \bar{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X$$
$$(C, f, p_1, \dots, p_n) \longrightarrow f(p_i)$$

Bad News:

(•) $\bar{\mathcal{M}}_{g,n}(X,\beta)$ are Deligne-Mumford stacks.

(••) singular, not equidimensional.

E.g. $\bar{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \to$ a compactification of $\emptyset$ by:



has components of dimensions from $2g$ to $3g-1$.

Fixes:

(•) Accept Stacks as a fact of life.

(••) VIRTUAL FUNDAMENTAL CLASS (Li-Ruan, Behrend-Fantechi...):

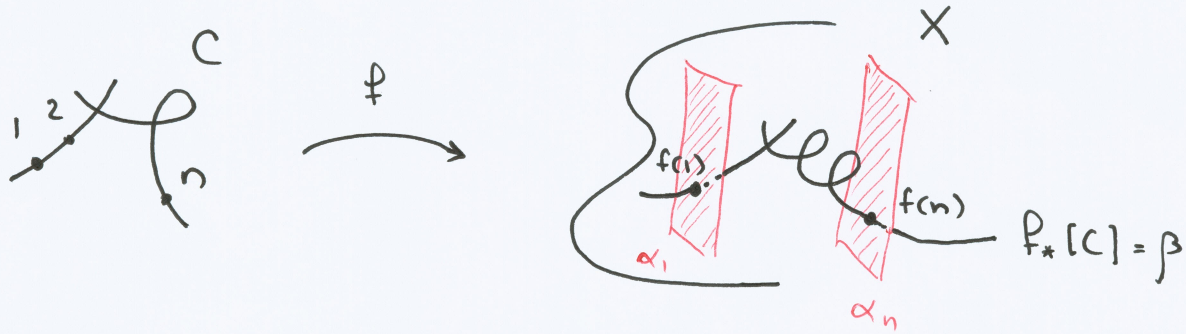
there exists a cycle $1^{vir} \in A_{exp.dim.}(\mathcal{M})$ that has all the good properties of a fundamental class (namely, it plays well with all the natural maps above)

§2. Gromov-Witten Invariants

Consider $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and choose n cycles $\alpha_1, \dots, \alpha_n \in \begin{cases} H^*(X) \\ A^*(X) \end{cases}$

$$\langle \alpha_1, \dots, \alpha_n \rangle_g^{X,\beta} := \left(ev_1^*(\alpha_1) \cup \dots \cup ev_n^*(\alpha_n) \right) \cap 1^{vir} \quad \left(= \int_{[\mathcal{M}]^{vir}} \prod ev_i^*(\alpha_i) \right)$$

"Morally" is the cycle of maps $f: C \to X$ such that:



"Morally" because this holds true when the moduli space is smooth and of the ~~right~~ expected dimension. ($\Rightarrow 1^{vir} = 1$)

I.e.:

- $g = 0$
- X convex ($\mathbb{P}^n$, Grassmannians, homog. varieties etc)

One can view G.W.I. as:

(1) numbers;

(2) multi linear functions:

$$H^*(X)^{\otimes n} \longrightarrow H^*(\bar{M}_{g,n}(X,\beta))$$

(3) a Coh.F.T.:

(a collection of multi-linear functions "compatible" with the gluing maps)

(4) A formal power series (G.W. potential) $\in \mathbb{C}[[x, q]]$:

- $e_1, \dots, e_N$ a basis for $H^*(X)$
  $x = \sum x_i e_i$
- $C_1, \dots, C_r$ a basis for $H_2(X)$
  $\beta = \sum \beta_i C_i$

$$\mathcal{F}_X := \sum_{g,n,\beta} \langle x^n \rangle_g^{X,\beta} \frac{1}{n!} q^\beta = \sum \langle e_1^{n_1} \dots e_N^{n_N} \rangle_g^{X,\beta} \frac{x_1^{n_1}}{n_1!} \dots \frac{x_N^{n_N}}{n_N!} q_1^{\beta_1} \dots q_r^{\beta_r}$$

(5) $(g=0)$ A family of Frobenius algebras on $H^*(X)[[q]]$, parameterized by $H^*(X)$:

$$\gamma_1 \underset{x}{*} \gamma_2 := \sum_{n \geq 0} \left(ev_{3*} \left(ev_1^*(\gamma_1) \cup ev_2^*(\gamma_2) \cup \prod_4^{n+3} ev_i^*(x) \right) q^\beta \right)_{M_{0,n+3}(X,\beta)}$$

$$= \sum \langle \gamma_1, \gamma_2, x^n, \gamma^3 \rangle_0^{X,\beta} \gamma_3 q^\beta$$

Unravelling this definition in coordinates, one sees that the structure constants of the quantum product are 3^{rd}-dvt. of the G.W. potential

$$e_\alpha \underset{x}{*} e_\beta = \sum \frac{\partial^3 \mathcal{F}}{\partial x_\alpha \partial x_\beta \partial x_\gamma} e^\gamma$$

This leads to:

(6) A Frobenius manifold structure on $H^*(X)[[q]]$

manifold M s.t. -> $T_m M$ is a Frob. Algebra.

-> metric, 1 vector field are flat.

-> integrability condition.

(7) A Lagrangian cone in an ∞-dimensional symplectic vector space. This is Givental's point of view and will be explored in the 3^{rd} lecture.

§3. Orbifold G.W. theory

$\mathcal{X}$ orbifold $\leftrightarrow$ space locally modelled on quotient M/G $\longleftrightarrow$ D.M. stack

(M: mfold; G: finite group)

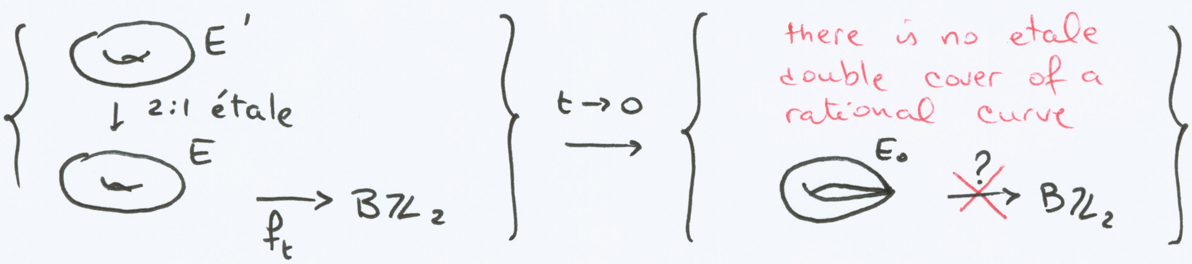
<u>Intuition</u>: $x \in \mathcal{X}$ comes with a finite group G_x $\left(\begin{array}{l}\text{isotropy group}\\ \text{stabylizer}\end{array}\right)$

whose elements should be thought of as "automorphisms of x".

<u>Example</u>: $BG = [pt/G]$ a moduli space for principal G-bundles.

$I\mathcal{X}$ = inertia orbifold $= \mathcal{X} \times_\Delta \mathcal{X} = \{(x,g) : x \in \mathcal{X}, g \in G_x\}$

If one tries to reproduce GW theory with an orbifold target, things go bad. Simple example that illustrates non-properness of such moduli spaces:



(Chen-Ruan; Abramovich-Graber-Vistoli); introduce following modifications:

(1) $C \longrightarrow$ orbicurve (AGV: twisted stable curve)

(2) $f \longrightarrow$ representable (map on stabylizers is injective / fibers are schemes)

(3) ev maps take value in $H^*_{CR}(\mathcal{X}) := H^*(I\mathcal{X})$

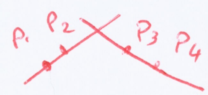
§4 WDVV

Consider the forgetful morphism:

$$\phi_{1234} : \bar{M}_{0,n}(X,\beta) \longrightarrow \bar{M}_{0,4} \cong \mathbb{P}^1$$

$$P_1 = \left[{}_{12} \times {}^{34} \right] \sim P_2 = \left[{}^{14} \times {}^{23} \right] \Rightarrow D_1 = \underbrace{[\phi^{-1}(P_1)]}_{} \sim D_2 = [\phi^{-1}(P_2)]$$

divisor of maps from nodal curves with



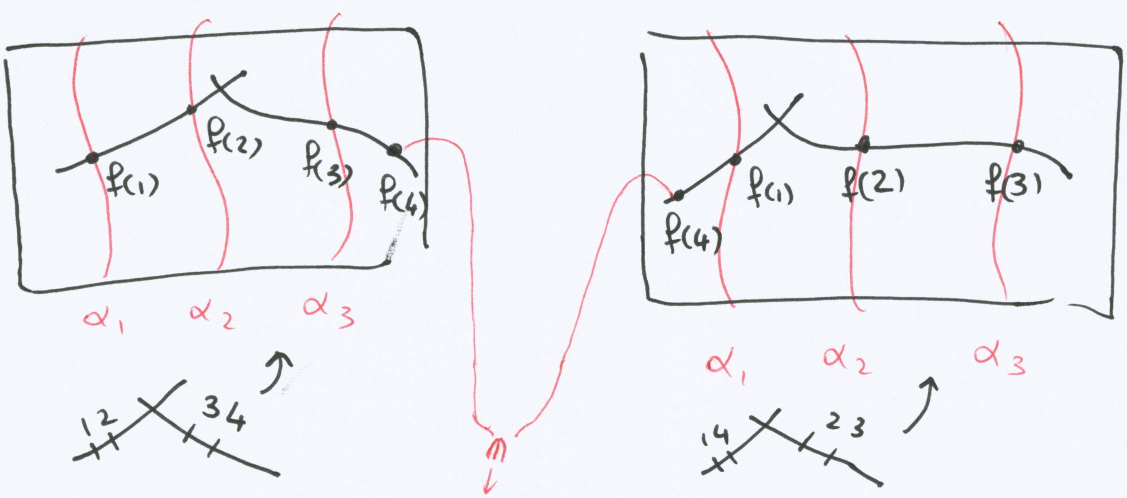
and other data distributed in all possible ways.

Therefore

$$\prod_1^n ev_i^*(\alpha_i) \cap D_1 = \prod_1^n ev_i^*(\alpha_i) \cap D_2$$ a nontrivial relation

between GWI-like objects.

This relation $\Rightarrow$ associativity of the quantum product!



$$(\alpha_1 * \alpha_2) * \alpha_3 = \alpha_1 * (\alpha_2 * \alpha_3)$$

On the GW potential:

$$\boxed{F_{x_i x_j x_k}\, g^{kk'}\, F_{x_{k'} x_l x_m} = F_{x_i x_m x_k}\, g^{kk'}\, F_{x_{k'} x_j x_l}}$$

§5 Atyiah-Bott Localization

$X^{\circlearrowleft T} \Rightarrow$ can induce naturally a torus action on $\bar{M}_{g,n}(X,\beta)$

<u>A-B Localization thm</u>: F_i the fixed loci for T-action.

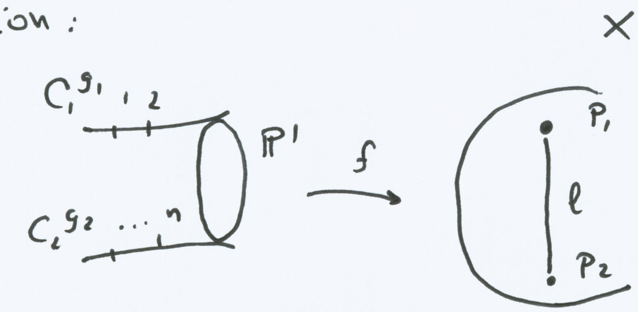
$H_T^*(pt) = \mathbb{C}[t]$

$$H_T^*(\bar{M}_{g,n}(X,\beta)) \otimes \mathbb{C}(t) \longrightarrow \bigoplus_{F_i} H_T^*(F_i) \otimes \mathbb{C}(t)$$

$$\alpha \longmapsto \sum_{F_i} \frac{\alpha|_{F_i}}{e^{eq}(N_{F_i/M})}$$

is an <u>ISOMORPHISM</u>.

Prototypical situation:



P_1, P_2 fixed points
ℓ a T-inv't line.

A T-fixed map: $f : \mathbb{P}^1 \to \ell$ a multiple cover ramified only over P_1, P_2

$C^{g_1} \to P_1$

$C^{g_2} \to P_2$

Locus of such maps $\cong B\mathbb{Z}_n \times \bar{M}_{g_1,n_1} \times \bar{M}_{g_2,n_2}$

$\Rightarrow$ computing GWI's on X is reduced to intersecting tautological classes on moduli spaces of curves. (Mumford; Faber-Pandharipande) handle this systematically via G.R.R.