ON HURWITZ THEORY AND APPLICATIONS

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2.1. **Curves/Riemann Surfaces** 101. In this section we recall some basic facts in the theory of algebraic curves and Riemann Surfaces. There are several excellent references that can be looked at, for example [**?**], [**?**], or [HM98].

The object of our study can be viewed equivalently as algebraic of complex analytic objects. It is very useful to keep in mind this equivalence.

Definition 2.1 (for algebraic geometers)**.** A (projective) **curve** is equivalently:

- a projective algebraic variety (over the complex numbers) of dimension 1.
- a field extension of $\mathbb{C}$ of transcendence degree 1.

Note: For a passionately pure algebraic geometer there is no need to have $\mathbb{C}$ as the ground field. Most features of the theory will hold over k an algebraically closed field of characteristic 0. Many surprises make the day of arithmetic geometers electing to work over finite fields of fields of positive characteristics. Here we do not dare to venture into this mysterious yet fascinating territory.

Definition 2.2 (for complex analysts)**.** A (compact) **Riemann Surface** is a compact complex analytic manifold of dimension 1.

We abuse of notation and allow Riemann Surface to have nodal singularities. It is a remarkable feature of the theory that we do not need to consider any worse type of degenerations of the smooth objects to have compact moduli spaces.

Exercise 1. A Riemann Surface is orientable. Check that the Cauchy-Riemann equations imply that any holomorphic atlas is a positive atlas.

Topologically a smooth Riemann surface is just a connected sum of g tori. The number g, the genus, is an important discrete invariant. Simple things become extremely confusing when one starts to deal with nodal or disconnected curves, so we spell out once and for all the relevant definitions.

Definition 2.3.

(1) If X is a smooth curve, the **genus** of X is equivalently:
   - the number of holes of the corresponding topological surface.
   - $h^0(X, K_X)$: the dimension of the space of global sections of the canonical line bundle.
   - $h^1(X, \mathcal{O}_X)$.

(2) If X is a nodal, connected curve, the **geometric genus** of X is the genus of the normalization of X (i.e. the genus of the smooth curve obtained by pulling the nodes apart).

(3) If X is a nodal, connected curve, the **arithmetic genus** of X is $h^1(X, \mathcal{O}_X) = h^0(X, \omega_X)$ (i.e. the genus of the curve obtained by smoothing the node).

(4) If X is a disconnected curve, the **geometric genus** is the sum of the genera of the connected components.

(5) If X is a disconnected curve, the **arithmetic genus**

$$g := 1 - \chi(\mathcal{O}_X) = 1 - h^0(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X).$$

In other words, you subtract one for every additional connected components.

The arithmetic genus is constant in families, and therefore we like it best. Unless otherwise specified genus will always mean arithmetic genus. See figure 2.1 for an illustration.

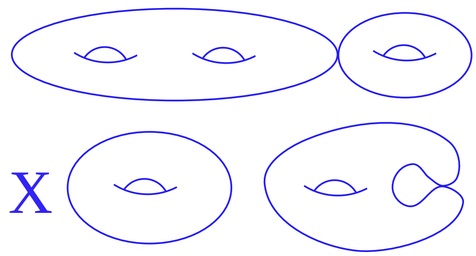


FIGURE 1. A (disconnected) curve of arithmetic genus 4 and geometric genus 5.

Exercise 2. Check that all definitions are consistent when X is a smooth connected curve.

Fact (Riemann-Roch theorem for curves)**.** If L“ = ”$\mathcal{O}(D)$ is a line bundle (or invertible sheaf) on a curve X, then:

$$h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g$$

or equivalently:

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = c_1(L)$$

Exercise 3. For X a smooth connected curve, check that K_X has degree $2g - 2$.

2.2. **Maps of Curves.** Of the several excellent references for this section, my favorite is [**?**]. It is a simple fact from complex analysis that any map of Riemann Surfaces can be given local expression $z \mapsto z^k$, with $k > 1$ only at a finite number of points.

Definition 2.4. A map $f : X \to Y$ of Riemann Surfaces is a **ramified cover**(see Figure 2):

- $B \subset Y$ is a finite set called **branch locus**;
- $f_{|X \setminus f^{-1}(B)} : f^{-1}(B) \to Y \setminus B$ is a degree d topological covering map;
- for $x \in f^{-1}(B)$ the local expression of f at x is $F(z) = z^k$; the number $k := r_f(x)$ is the **ramification order** of f at x.
- $R \subseteq f^{-1}(B) := \{x \in X \ s.t. \ r_f(x) > 1\}$ is called the **ramification locus**.

Viceversa, every branched cover identifies a unique map of Riemann Surfaces:

Fact (Riemann Existence Theorem)**.** If Y is a compact Riemann Surface and $f^\circ : X^\circ \to Y \setminus B$ a topological cover, then there exist a

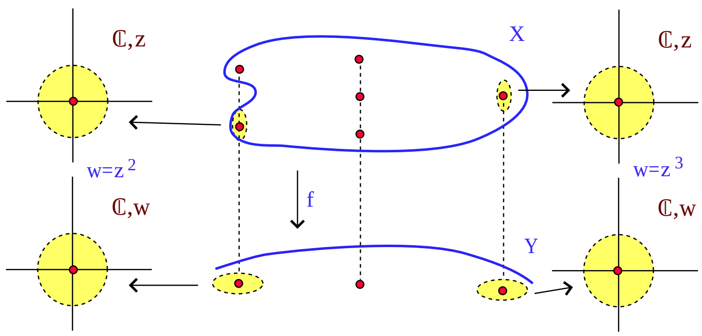


FIGURE 2. A ramified cover of degree 3.

unique smooth compact Riemann Surface X, obtained from the topological surface X° by adding a finite number of points and a unique map f of Riemann Surfaces extending f°.

Finally, a theorem which will be fundamental for us and relates the various discrete invariants of curves and maps introduced so far.

Theorem 2.5 (Riemann-Hurwitz)**.** *For a map of smooth Riemann Surfaces $f: X \to Y$:*

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} (r_f(x) - 1). \tag{1}$$

Exercise 4. Prove the Riemann Hurwitz theorem in two ways:

(1) Topologically: compute the euler characteristic of X by lifting a triangulation on Y where the branch locus is contained in the set of vertices of the triangulation.
(2) Analytically/Algebro Geometrically: compute the degree of the divisor of the pullback via f of a meromorphic one-form on Y. Note that this gives the degree of K_X.

Definition 2.6. Let $f: X \to Y$ be a map of Riemann Surfaces, $y \in Y$, $\{x_1, \ldots, x_n\} = f^{-1}(x)$, then the (unordered) collection of integers $\{r_f(x_1), \ldots, r_f(x_n)\}$ is called the **ramification profile** of f at y. We think of this set as a partition of d and denote it $\eta(v)$ (or simply η). If $\eta(y) = (2, 1, \ldots, 1)$, then f has **simple ramification** over y.

We are ready for our first definition of Hurwitz numbers.

Definition 2.7 (Geometry)**.** Let $(Y, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus h. Let $\underline{\eta} = (\eta_1, \ldots, \eta_s)$ be a

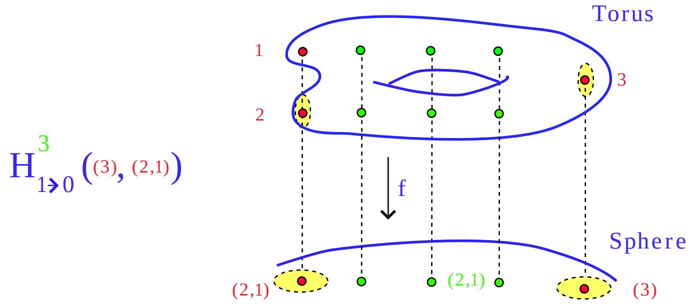


FIGURE 3. The covers contributing to a given Hurwitz Number.

vector of partitions of the integer d. We define the *Hurwitz number*:

$$H^r_{g\to h,d}(\underline{\eta}) \quad := \quad \text{weighted number of} \left\{ \begin{array}{c} \textit{degree } d \textit{ covers} \\ X \xrightarrow{f} Y \textit{ such that :} \\ \bullet \text{ X } \textit{is connected of genus } \text{g}; \\ \bullet\ f \textit{ is unramified over} \\ X \setminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\}; \\ \bullet\ f \textit{ ramifies with profile } \eta_i \textit{ over } q_i; \\ \bullet\ f \textit{ has simple ramification over } p_i; \\ \circ \textit{ preimages of each } q_i \textit{ with same} \\ \textit{ramification are distinguished by} \\ \textit{appropriate markings.} \end{array} \right\}$$

Each cover is weighted by the number of automorphisms.

Figure 3 might help visualize the features of this definition.

Remarks:

(1) For a Hurwitz number to be nonzero, r,g, h and $\underline{\eta}$ must satisfy the Riemann Hurwitz formula (1). The above notation is always redundant, and it is common practice to omit appropriate unnecessary invariants.

(2) The last condition $\circ$ was recently introduced in [GJV03], and it is well tuned to the applications we have in mind. These Hurwitz numbers differ by a factor of $\prod \mathrm{Aut}(\eta_i)$ from the classically defined ones where such condition is omitted.

(3) One might want to drop the condition of X being connected, and count covers with disconnected domain. Such Hurwitz numbers are denoted by $H^\bullet$ (To my knowledge Okounkov and

Pandharipande started the now common convention of using $\bullet$ to denote a disconnected theory).

Example 2.8.

-
$$H^0_{0\to 0,d}((d),(d)) = \frac{1}{d}$$
-
$$H^4_{1\to 0,2} = \frac{1}{2}$$
-
$$H^3_{1\to 0,2}((2),(1,1)) = 1$$

This is a very beautiful geometric definition, but it is extremely impractical. A reasonably simple Hurwitz number such as $H^4_{1\to 0,3}((3))$ is already out of our reach.

2.3. **Representation Theory.** The problem of computing Hurwitz numbers is in fact a discrete problem and it can be approached using the representation theory of the symmetric group. A standard reference here is [FH91].

Given a branched cover $f : X \to Y$, pick a point y_0 not in the branch locus, and label the preimages $1, \dots, d$. Then one can naturally define a group homomorphism:

$$\begin{array}{rccc} \varphi_f : & \pi_1(Y \setminus B, y_0) & \to & S_d \\ & \gamma & \mapsto & \sigma_\gamma : \{i \mapsto \tilde{\gamma}_i(1)\}, \end{array}$$

where $\tilde{\gamma}_i$ is the lift of γ starting at i ($\tilde{\gamma}_i(0) = i$). This homomorphism is called the **monodromy representation**, and its construction is illustrated in Figure 4.

Remarks:

(1) A different choice of labelling of the preimages of y_0 corresponds to composing φ_f with an inner automorphism of S_d.
(2) If $\rho \in \pi_1(Y \setminus B, y_0)$ is a little loop winding once around a branch point with profile η, then σ_ρ is a permutation of cycle type η.

Viceversa, the monodromy representation contains enough information to recover the topological cover of $Y \setminus B$, and therefore, by the Riemann existence theorem, the map of Riemann surfaces. To count covers we can count instead (equivalence classes of) monodromy representations. This leads us to the second definition of Hurwitz numbers.

Definition 2.9 (Representation Theory)**.** Let $(Y, p_1, \dots, p_r, q_1, \dots, q_s)$ be an $(r + s)$-marked smooth Riemann Surface of genus g, and $\underline{\eta} =$

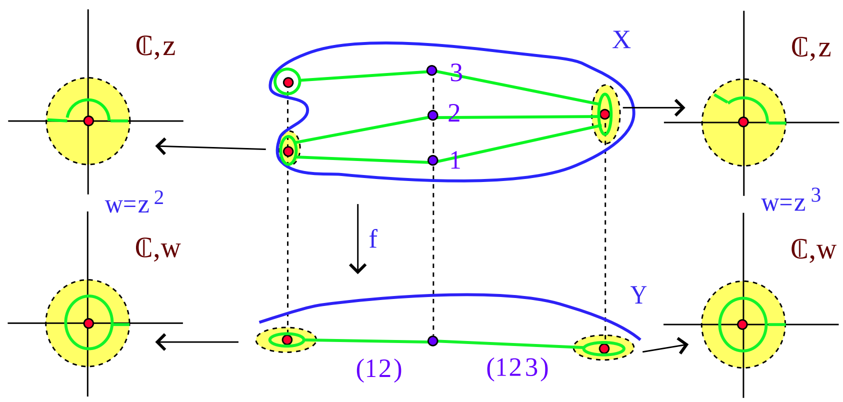


FIGURE 4. Sketch of the construction of the monodromy representation for the cover f.

$(\eta_1, \dots, \eta_s)$ a vector of partitions of the integer d:

$$H^r_{g\to h,d}(\underline{\eta}) := \frac{|\{\underline{\eta}\text{-monodromy representations } \varphi^{\underline{\eta}}\}|}{|S_d|} \prod \mathrm{Aut}\eta_i, \tag{2}$$

where an $\underline{\eta}$-monodromy representation is a group homomorphism

$$\varphi^{\underline{\eta}} : \pi_1(Y \setminus B, y_0) \to S_d$$

such that:

• for ρ_{q_i} a little loop winding around q_i once, $\varphi^{\underline{\eta}}(\rho_{q_i})$ has cycle type η_i.
• for ρ_{p_i} a little loop winding around p_i once, $\varphi^{\underline{\eta}}(\rho_{p_i})$ is a transposition.
$\star$ $\mathrm{Im}(\varphi^{\underline{\eta}}(\rho_{q_i}))$ acts transitively on the set $\{1, \dots, d\}$.

Remarks:

(1) To count disconnected Hurwitz numbers just remove the last condition $\star$.
(2) Dividing by $d!$ accounts simoultaneously for automorphisms of the covers and the possible relabellings of the preimages of y_0.
(3) $\prod \mathrm{Aut}\eta_i$ is non-classical and it corresponds to condition $\circ$ in Definition 2.7.

Exercise 5. Check with this definition the Hurwitz numbers in Example 2.8. Compute $H^4_{1\to 0,3}((3)) = 9$ and $H^4_{0\to 0,3} = 4$.

2.4. $h = 0$**, Disconnected, Unlabelled.** We restrict our attention to the target genus 0, disconnected theory, where the connection with representation theory can be carried even further. Also, it is more convenient to work with the classical definition of Hurwitz numbers, so in this section we drop condition $\circ$ of Definition 2.7. In this case Definition 2.9 can be reformulated:

$$H^\bullet_g(\underline{\eta}) = \frac{1}{d!}|\{(\sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_r) \ s.t. \ \sigma_1 \ldots \sigma_s \tau_1 \ldots \tau_r = Id\}|, \tag{3}$$

where:

- σ_i has cycle type η_i;
- τ_i is a transposition.

Equation (3) recasts the computation of Hurwitz numbers as a multiplication problem in the class algebra of the symmetric group. Recall that $\mathcal{Z}(\mathbb{C}[S_d])$ is a vector space of dimension equal the number of partitions of d, with a natural basis indexed by conjugacy classes of permutations.

$$\mathcal{Z}(\mathbb{C}[S_d]) = \bigoplus_{\eta \vdash d} \mathbb{C} \ C_\eta,$$

where

$$C_\eta = \sum_{\sigma \in S_d \text{ of cycle type } \eta} \sigma.$$

We use $|C_\eta|$ to denote the number of permutations of cycle type η. We also use the notation $C_{Id} = Id$ and $C_\tau = C_{(2,1^{d-2})}$. Then the Hurwitz number is the coefficient of the identity in the appropriate product of elements of the class algebra:

$$H^\bullet_g(\underline{\eta}) = \frac{1}{d!}[C_{Id}]C_{\eta_1} \cdot \ldots \cdot C_{\eta_s} \cdot C_\tau^r. \tag{4}$$

It is a classical fact that $\mathcal{Z}(\mathbb{C}[S_d])$ is a semisimple algebra with semisimple basis indexed by irreducible representations of S_d, and change of bases essentially given by the character table:

$$\mathrm{e}_\lambda = \frac{\dim\lambda}{d!} \sum_{\eta \vdash d} \mathcal{X}_\lambda(\eta) C_\eta \tag{5}$$

and

$$C_\eta = |C_\eta| \sum_{\lambda \text{ irrep. of } S_d} \frac{\mathcal{X}_\lambda(\eta)}{\dim\lambda} \mathrm{e}_\lambda. \tag{6}$$

$\mathcal{X}_\lambda(C_\eta)$	C_{Id}	C_τ	$C_{(3)}$
1	1	1	1
-1	1	-1	1
P	2	0	1

$$e_1 = \frac{1}{6}(C_{Id} + C_\tau + C_{(3)}) \qquad C_{Id} = e_1 + e_{-1} + e_P$$

$$e_{-1} = \frac{1}{6}(C_{Id} - C_\tau + C_{(3)}) \qquad C_\tau = 3e_1 - 3e_{-1}$$

$$e_P = \frac{1}{3}(2C_{Id} \quad - C_{(3)}) \qquad C_{(3)} = 2e_1 + 2e_{-1} - e_P$$

Table 1. All you have always wanted to know about S_3 (and never dared to ask).

Assuming without loss of generality that $r = 0$, we can finally rewrite equation (4):

$$\begin{aligned} H_g^\bullet(\underline{\eta}) &= \frac{1}{d!}[C_{Id}]\sum_\lambda \prod_{i=1}^{s} \left(|C_{\eta_i}| \frac{\mathcal{X}_\lambda(\eta_i)}{\dim\lambda} \right) e_\lambda \\ &= \frac{1}{d!}\sum_\lambda \prod_{i=1}^{s} \left(|C_{\eta_i}| \frac{\mathcal{X}_\lambda(\eta_i)}{\dim\lambda} \right) \frac{\dim\lambda}{d!}\mathcal{X}_\lambda(Id) \\ &= \left(\frac{1}{d!}\right)^2 \sum_\lambda (\dim\lambda)^{2-n} \prod_{i=1}^{s} |C_{\eta_i}|\mathcal{X}_\lambda(\eta_i). \end{aligned} \tag{7}$$

Example 2.10. Let us revisit the computation of $H_1((3))$. In this case the condition of a point with full ramification forces all covers to be connected, so $H = H^\bullet$. The symmetric group S_3 has three irreducible representations, the trivial and alternating one dimensional representations, and a two dimensional representation obtained by quotienting the permutation representation by the invariant small diagonal line. In Table 2.4 we recall the character table of S_3 and the transformations from the conjugacy class basis to the representation basis.

We have:

$$\begin{aligned} H_1((3)) &= \frac{1}{6}[C_{Id}]C_{(3)}C_\tau^4 \\ &= \frac{1}{6}[C_{Id}](2\cdot 3^4 e_1 + 2\cdot(-3)^4 e_{-1}) \\ &= \frac{1}{6}\left(\frac{2\cdot 3^4}{6} + \frac{2\cdot 3^4}{6}\right) = 9 \end{aligned}$$

2.5. **Disconnected to Connected: the Hurwitz Potential.** The character formula is an efficient way to describe disconnected Hurwitz numbers (provided one has a good handle on the characters of the appropriate symmetric group, which is in itself a complicated matter). We now investigate how to relate the disconnected theory to the connected theory. Let us begin by observing a simple example:

Example 2.11. We have seen in Exercise 5 that $H_{0,3} = 4$. From the character formula:

$$H^{\bullet}_{0,3} = \frac{1}{36}(2 \cdot 3^4) = \frac{9}{2} = 4 + \frac{1}{2}$$

The last $\frac{1}{2}$ is the contribution of disconnected covers, consisting of an ellyptic curve mapping to the line as a double cover and of a line mapping isomorphically. The relationship between connected and disconnected Hurwitz numbers is systematized in the language of generating functions.

Definition 2.12. The **Hurwitz Potential** is a generating function for Hurwitz numbers. As usual we present it with as many variables as possible, keeping in mind that in almost all applications one makes a choice of the appropriate variables to mantain:

$$\mathcal{H}(p_{i,j}, u, z, q) := \sum H^r_{g\to 0,d}(\underline{\eta})\ p_{1,\eta_1} \cdot \ldots \cdot p_{s,\eta_s} \frac{u^r}{r!} z^{1-g} q^d,$$

where:

- $p_{i,j}$, for i and j varying among non-negative integers, index ramification profiles. The first index i keeps track of the branch point, the second of the profile. For a partition η the notation $p_{i,\eta}$ means $\prod_j p_{i,\eta_j}$.
- u is a variable for unmarked simple ramification. Division by $r!$ reflects the fact that these points are not marked.
- z indexes the genus of the cover (more precisely it indexes the euler characteristic, which is additive under disjoint unions).
- q keeps track of degree.

Similarly one can define a disconnected Hurwitz potential $\mathcal{H}^{\bullet}$ encoding all disconnected Hurwitz numbers.

Silly but Important Convention: we choose to set $p_{i,1} = 1$ for all i. This means that an unramified poing sitting above a branch point is not "recorded". With this convention, the monomial in p_i's (for a fixed i) has weighted degree at most (but not necessarily equal) the exponent of the variable q.

Fact. The connected and disconnected potentials are related by exponentiation:

$$1 + \mathcal{H}^{\bullet} = \mathrm{e}^{\mathcal{H}} \tag{8}$$

Exercise 6. Convince yourself of equation (8). To me, this is one of those things that are absolutely mysterious until you stare at it long enough that, all of a sudden, it becomes absolutely obvious...

Example 2.11 revisited: the information we observed before is encoded in the coefficient of u^4zq^3 in equation (8):

$$H^\bullet_{0,3}\frac{u^4}{4!}zq^3 = H_{0,3}\frac{u^4}{4!}zq^3 + \frac{1}{2!}2\left(H_{1,2}\frac{u^4}{4!}q^2\right)(H_{0,1}zq).$$

Exercise 7. Check equation (8) in the cases of $H^\bullet_{-1,4}$, $H^\bullet_{-1}((2,1,1),(2,1,1))$ and $H^\bullet_{-1}((2,1,1),(2,1,1),(2,1,1),(2,1,1)$. All these Hurwitz numbers equal $\frac{3}{4}$.

Remark 2.13. Unfortunately I don't know of any particular efficient reference for this section. The book [**?**] contains more information that one might want to start with on generating functions; early papers of various subsets of Goulden, Jackson and Vakil contain the definitions and basic properties of the Hurwitz potential.

2.6. **Higher Genus Target.** Hurwitz numbers for higher genus targets are determined by genus 0 Hurwitz numbers. In fact something much stronger holds true, i.e. target genus 0, 3-pointed Hurwitz numbers suffice to determine the whole theory. The key observation here are the degeneration formulas.

Theorem 2.14. *Let $\mathfrak{z}(\nu)$ denote the order of the centralizer of a permutation of cycle type ν. Then:*

(1)

$$H^{0,\bullet}_{g\to 0}(\eta_1,\ldots,\eta_s,\mu_1,\ldots,\mu_t) = \sum_{\nu\vdash d}\mathfrak{z}(\nu)H^{0,\bullet}_{g_1\to 0}(\eta_1,\ldots,\eta_s,\nu)H^{0,\bullet}_{g_2\to 0}(\nu,\mu_1,\ldots,\mu_t)$$

with $g_1+g_2+\ell(\nu)-1=g$.

(2)

$$H^{0,\bullet}_{g\to 1}(\eta_1,\ldots,\eta_s) = \sum_{\nu\vdash d}\mathfrak{z}(\nu)H^{0,\bullet}_{g-\ell(\nu)\to 0}(\eta_1,\ldots,\eta_s,\nu,\nu).$$

These formulas are called degeneration formulas because geometrically they correspond to simoltaneously degenerating the source and the target curve, as illustrated in Figure 5. Proving the degeneration formulas geometrically however gives rise to subtle issues of infinitesimal automorphisms (that explain the factor of $\mathfrak{z}\nu$). However a combinatorial proof is straightforward.

Proof of (1): recall that

$$d!H^{0,\bullet}_{g\to 0}(\eta_1,\ldots,\eta_s,\mu_1,\ldots,\mu_t) = |\{\sigma_1,\ldots,\sigma_s,\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t\}|,$$

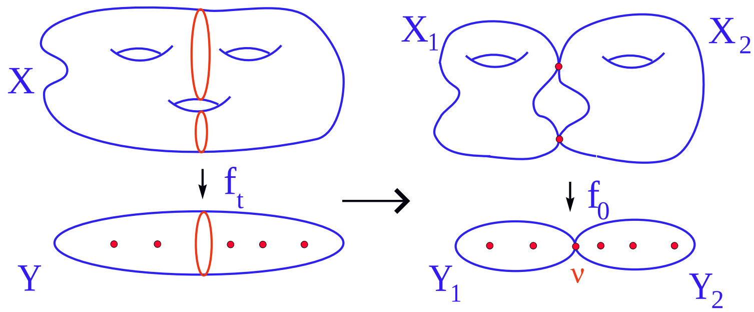


FIGURE 5. Degeneration of a cover to a nodal cover. Note that source and target degenerate simoultaneously and the ramification orders on both sides of the node match.

where the permutations have the appropriate cycle type, and the product of all permutations is the identity. Define $\pi = \sigma_1 \dots \sigma_s$, then

$$|\{\sigma_1, \dots, \sigma_s, \pi^{-1}, \pi, \tilde{\sigma}_1, \dots, \tilde{\sigma}_t\}| = \sum_{\nu \vdash d} \frac{1}{|C_\nu|} |\{\sigma_1, \dots, \sigma_s, \pi_1\}||\{\pi_2, \tilde{\sigma}_1, \dots, \tilde{\sigma}_t\}|$$

where in the RHS π_1 and π_2 have cycle type ν and we require the products of the permutations in the two sets to equal the identity. We must divide by $|C_\nu|$ because in the LHS we want the two newly introduced permutations to be inverses of each other, and not just in the same conjugacy class. But now we recognize that the term on the RHS is:

$$\sum_{\nu \vdash d} \frac{1}{|C_\nu|} d! H^{0,\bullet}_{g_1 \to 0}(\eta_1, \dots, \eta_s, \nu) d! H^{0,\bullet}_{g_2 \to 0}(\nu, \mu_1, \dots, \mu_t)$$

The proof is finally concluded by observing the identity $|C_\nu| \mathfrak{z}(\nu) = d!$.

Exercise 8. Prove part (2) of Theorem 5.

It is now immediate to observe that applying iteratively the two recursions above one can describe a formula for all Hurwitz numbers. Combining this with formula (7) one obtains the general character formula for Hurwitz numbers, sometimes referred in the literature as Burnside's formula:

$$H^{0,\bullet}_{g \to h}(\underline{\eta}) = \sum_{\lambda} \left(\frac{d!}{\dim \lambda} \right)^{2h-2} \prod_{i=1}^{s} \frac{|C_{\eta_i}| \mathcal{X}_\lambda(\eta_i)}{\dim \lambda}$$

3. Moduli Spaces

3.1. **Quick and Dirty Introduction.** The concept of moduli space is central in algebraic geometry. In a sense, the point of view of modern algebraic geometry is that every space should be thought as a moduli space. While it is impossible to do justice to such a rich subject in a few pages, I wish to give some intuitive ideas that might help read the more rigorous literature on the field. Another very friendly introduction is given by the first chapter of [HM98].

Informally, a moduli space for (equivalence classes of) geometric objects of a given tipe consists of:

(1) a set $\mathcal{M}$ whose points are in bijective correspondence with the objects we wish to parameterize;
(2) the notion of functions to $\mathcal{M}$, described in terms of families of objects:

$$Hom(B, \mathcal{M}) \quad \leftrightarrow \quad \begin{array}{ccc} X_b & \rightarrow & X \\ \downarrow & & \downarrow \\ b & \rightarrow & B \end{array}$$

Remarks:

(1) In modern language we are describing the (scheme) structure of $\mathcal{M}$ by describing its **functor of points**.
(2) a family of objects naturally gives rise to a function to $\mathcal{M}$, but the other implication is much trickier. When this is the case we say that $\mathcal{M}$ is a **fine** moduli space. This in particular implies that there is a family $\mathcal{U} \rightarrow \mathcal{M}$ (called **universal family**) such that the fiber over each $m \in \mathcal{M}$ is the object parameterized by m. Also, every family is obtained by pullback from the universal family.
(3) it is often the case that it is not possible to obtain a fine moduli space (this typically happens when the objects one wishes to parameterize have automorphisms). In this case one must make a choice:
    (a) be satisfied with a scheme $\mathcal{M}$ whose points are in bijective correspondence with the objects to parameterize, plus some universality condition (for any other space $\mathcal{N}$ whose points have such a property, there exists a unique map $\mathcal{M} \rightarrow \mathcal{N}$). In this case we say $\mathcal{M}$ is a **coarse** moduli space.
    (b) forget the idea of $\mathcal{M}$ being a scheme, and allow it to be some categorical monstruosity (in modern language called

a **stack**), that has the property of recovering the equivalence between families and functions to $\mathcal{M}$. The mantra here is that to do (a good amount of) geometry on a stack one very seldom has to meddle with the categorical definitions but uses the above equivalence to translate geometric questions from the stack to families of objects.

Exercise 9. Familiarize yourself with these concepts by looking at the following more or less silly examples/exercises.

(1) Any scheme X is a fine moduli space...for itself, i.e. for the functor describing families of points of X.
(2) Note that the points of the cuspidal cubic $X = \{y^2 = x^3\}$ are in bijection with the points of $\mathbb{P}^1$, but X is not a coarse moduli space for "families of points of $\mathbb{P}^1$".
(3) Consider the moduli space for equivalence classes of unit length segments in the real plane up to rigid motions. What is the coarse moduli space? Show that this is not a fine moduli space by constructing two non-isomorphic families of segments. Understand that the lack of fine-ness comes from the fact that you can flip the segment.
(4) Let us NOT-define but introduce our favorite stack: $\mathcal{B}G = [pt/G]$ is the stack quotient of a point by a group G. We do not define what this is, but define that functions $B \to \mathcal{B}G$ correspond to principal G-bundles $P \to B$. Convince yourself that the moduli space considered in the previous paragraph is $\mathcal{B}\mathbb{Z}_2$.

3.2. **Various Moduli Spaces Related to Curves and their Maps.** We are concerned with the interactions among different moduli spaces that have to do with curves. Here we introduce the characters. In general $\mathcal{M}$ denotes a moduli space parameterizing smooth objects and $\overline{\mathcal{M}}$ denotes some suitable compactification obtained by allowing mild degenerations of the objects. Curves can acquire only nodal singularities, and **stable** always means "with finitely many automorphisms". All these moduli spaces are really (Deligne-Mumford) stacks.

$\overline{\mathcal{M}}_{g,n}$: the moduli space of **stable curves** of genus g with n marked points. Here stability means that every rational component must have at least three special points (nodes or marks), and that a smooth ellptic curve needs to have at least one mark. This is a smooth stack of dimension $3g - 3 + n$, connected, irreducible. See [HM98] for more.

$\overline{\mathcal{M}}_{g,n}(X,\beta)$**:** the space of **stable maps** to X of degree $\beta \in H_2(X)$. A map is stable if every contracted rational component has three special points. If $g = 0$ and X is convex then these are smooth schemes, but in general these are nasty creatures even as stacks. They are singular and typically non-equidimensional. Luckily deformation theory experts can construct a Chow class of degree in the expected dimension, and many of the formal properties of the fundamental class, called a **virtual fundamental class**. Intersection theory on these spaces is then rescued by capping all classes with the virtual fundamental class. Good references for people interested in these spaces are [**?**] and [**?**].

$Hurw_{g\to h,d}(\underline{\eta}) \subset Adm_{g\to h,d}(\underline{\eta})$**:** the **Hurwitz spaces** parameterize degree d covers of smooth curves of genus h by smooth curves of genus g. A vector of partitions of d specifies the ramification profiles over marked points on the base. All other ramification is required to be simple. Hurwitz spaces are typically smooth schemes (unless the ramification profiles are chosen in very particular ways so as to allow automorphisms), but they are obviously non compact. The **admissible cover** compactification, consisting of degenerating simoultaneosly target and cover curves, was introduced in [HM82]. In [ACV01], the normalization of such space is interpreted as a (component of a) space of stable maps to the stack $\mathcal{B}S_d$. Without going into the subtleties of stable maps to a stack, we understand that by admissible cover we always denote the corresponding smooth stack.

Remark 3.1. The geometry of these moduli spaces is richly interlaced by a number of natural maps, such as evaluation maps at marked points, gluing maps, maps that forget points, functions, target, source etc... we will introduce these maps as we will need to use them.

Exercise 10. A nice feature of all these spaces is that the boundary is modular, i.e. it is built up of similar types of moduli spaces, but with smaller type of invariants. Understand this statement for the spaces we introduced.

Exercise 11. Familiarize yourself with the statement that the universal family for $\overline{\mathcal{M}}_{g,n}$ is isomorphic to $\overline{\mathcal{M}}_{g,n+1}$. Describe the $(n+1)$-pointed curves parameterized by the image of a section $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \mathcal{U}$. We call this (boundary!) divisor D_i.

Exercise 12. Observe that there is a natural branch map from $Hurw_{g\to h,d}(\underline{\eta})$ to an appropriate fiber product of a moduli space of pointed curves and a quotient of (another) moduli space of pointed curves by the action of a symmetric group. Make this statement precise. What is the degree of this branch map?

Exercise 13. Describe the moduli space $M_g(\mathbb{P}^1, 1)$ and the stable maps compactification $\overline{M}_g(\mathbb{P}^1, 1)$.

Exercise 14. The **hyperelliptic locus** is the subspace of $\overline{M}_g$ parameterizing curves that admit a double cover to $\mathbb{P}^1$. Understand the hyperelliptic locus as the moduli space $Adm_{g\to 0,2}((2), \dots, (2))$ and subsequently as a stack quotient of $\overline{\mathcal{M}}_{0,2g+2}$ by the trivial action of $\mathbb{Z}_2$.

3.3. **Tautological Bundles on Moduli Spaces.** We define bundles on our moduli spaces by describing them in terms of the geometry of families of objects. In other words, for any family $X \to B$, we give a bundle on B constructed in some canonical way from the family X. This insures that this assignment is compatible with pullbacks (morally means that we are thinking of B as a chart and that the bundle patches along various charts). This is the premium example of the philosophy of doing geometry with stacks. We focus on two particular bundles that will be important for our applications.

3.3.1. *The Cotangent Line Bundle and ψ classes.* An excellent reference for this section, albeit unfinished and unpublished, is [Koc01].

Definition 3.2. The ***i*-th cotangent line bundle** $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ is globally defined as the restriction to the i-th section of the relative dualizing sheaf from the universal family:

$$\mathbb{L}_i := \sigma_i^*(\omega_\pi).$$

The first Chern class of the cotangent line bundle is called ψ **class**:

$$\psi_i := c_1(\mathbb{L}_i).$$

This definition is slick but unenlightening, so let us chew on it a bit. Given a family of marked curves $f : X \to B (= \varphi_f : B \to \overline{\mathcal{M}}_{g,n})$, the cotangent spaces of the fibers X_b at the i-th mark naturally fit together to define a line bundle on the image of the i-th section, which is then isomorphic to the base B. This line bundle is the pullback $\varphi_f^*(\mathbb{L}_i)$. Therefore informally one says that the cotangent line bundle is the line bundle whose fiber over a moduli point is the cotangent line of the parameterized curve at the i-th mark.

The cotangent line bundle arises naturally when studying the geometry of the moduli spaces, as we quickly explore in the following exercises.

Exercise 15. Convince yourself that the normal bundle to the image of the i-th section in the universal family is naturally isomorphic to $\mathbb{L}_i^\vee$ (This is sometimes called the i-th tangent line bundle and denoted $\mathbb{T}_i$).

Exercise 16. Consider an irreducible boundary divisor $D \cong \overline{M}_{g_1,n_1+\bullet} \times \overline{M}_{g_2,n_2+\star}$. Then the normal bundle of D in the moduli space is naturally isomorphic to the tensor product of the tangent line bundles of the components at the shadows of the node:

$$N_{D/\overline{M}_{g,n}} \cong \mathbb{L}_\bullet^\vee \boxtimes \mathbb{L}_\star^\vee$$

Is this statement consistent with the previous exercise? Why?

3.3.2. *The Hodge Bundle.*

Definition 3.3. The **Hodge bundle** $\mathbb{E}(= \mathbb{E}_{g,n})$ is a rank g bundle on $\overline{\mathcal{M}}_{g,n}$, defined as the pushforward of the relative dualizing sheaf from the universal family. Over a curve X, the fiber is canonically $H^0(X, \omega_X)$ (i.e. the vector space of holomorphic 1-forms if X is smooth). The Chern classes of $\mathbb{E}$ are called λ classes:

$$\lambda_i := c_i(\mathbb{E}).$$

We recall the following properties([Mum83]):

Vanishing: ch_i can be written as a homogeneous quadratic polynomial in λ classes. Thus:

$$\mathrm{ch}_i = 0 \quad \text{for} \quad i > 2g. \tag{9}$$

Mumford Relation: the total Chern class of the sum of the Hodge bundle with its dual is trivial:

$$c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1. \tag{10}$$

Hence $\mathrm{ch}_{2i} = 0$ if $i > 0$.

Separating nodes:

$$\iota^*_{g_1,g_2,S}(\mathbb{E}) \cong \mathbb{E}_{g_1,n_1} \oplus \mathbb{E}_{g_2,n_2}, \tag{11}$$

where with abuse of notation we omit pulling back via the projection maps from $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$ onto the factors.

Non-separating nodes:

$$\iota^*_{irr}(\mathbb{E}) \cong \mathbb{E}_{g-1,n} \oplus \mathcal{O}. \tag{12}$$

Remark 3.4. We define the Hodge bundle and λ classes on moduli spaces of stable maps and Hurwitz spaces by pulling back via the appropriate forgetful morphisms.

Exercise 17. Use the above properties to show vanishing properties of λ-classes:

(1) $\lambda_g^2 = 0$ if $g > 0$.
(2) $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$. If now we allow marked points, then the vanishing holds on "almost all" the boundary, but one needs to be more careful. Describe the vanishing locus of $\lambda_g\lambda_{g-1}$ in this case.
(3) λ_g vanishes on the locus of curves not of compact type (i.e. where the geometric and arithmetic genera are different).

4. Atyiah-Bott Localization

The localization theorem of [AB84] is a powerful for the intersection theory of moduli spaces that can be endowed with a torus action. In this section we present the basics of this techniques following [HKK$^+$03] and focus on one particular application, the evaluation of the hyperelliptic locus in the tautological ring of $\mathcal{M}_g$.

4.1. **Equivariant Cohomology.** Let G be group acting on a space X. According to your point of view G might be a compact Lie group or a reductive algebraic group. Then G-equivariant cohomology is a cohomology theory developed to generalize the notion of the cohomology of a quotient when the action of the group is not free. The idea is simple: since cohomology is homotopy invariant, replace X by a homotopy equivalent space $\tilde{X}$ on which G acts freely, and then take the cohomology of $\tilde{X}/G$. Rather than delving into the definitions that can be found in [HKK$^+$03], Chapter 4, we recall here some fundamental properties that we will use:

(1) If G acts freely on X, then
$$H^*_G(X) = H^*(X/G).$$
(2) If X is a point, then let EG be any contractible space on which G acts freely, $BG := EG/G$, and define:
$$H^*_G(pt.) = H^*(BG).$$
(3) If G acts trivially on X, then
$$H^*_G(X) = H^*(X) \otimes H^*(BG).$$

Example 4.1. If $G = \mathbb{C}^*$, then $EG = S^\infty$, $BG := \mathbb{P}^\infty$ and
$$H^*_{\mathbb{C}^*}(pt.) = \mathbb{C}[\hbar],$$
with $\hbar = c_1(\mathcal{O}(1))$.

Remark 4.2. Dealing with infinite dimensional spaces in algebraic geometry is iffy. In [Ful98], Fulton finds an elegant way out by showing that for any particular degree of cohomology one is intereted in, one can work with a finite dimensional approximation of BG. Another route is to instead work with the stack $\mathcal{B}G = [pt./G]$. Of course the price to pay is having to formalize cohomology on stacks...here let us just say that $\mathcal{O}(1) \to \mathcal{B}\mathbb{C}^*$, pulled back to the class of a point, is a copy of the identity representation $Id : \mathbb{C}^* \to \mathbb{C}^*$.

Let $\mathbb{C}^*$ act on X and let F_i be the irreducible components of the fixed locus. If we-push forward and then pull-back the fundamental class of F_i we obtain

$$i^* i_*(F_i) = e(N_{F_i/X}).$$

Since $N_{F_i/X}$ is the moving part of the tangent bundle to F_i, this euler class is a polynomial in $\hbar$ where the $\hbar^{codim(F_i)}$ term has non-zero coefficient. This means that if we allow ourselved to invert $\hbar$, this euler class becomes invertible. This observation is pretty much the key to the following theorem:

Theorem 4.3. *The maps:*

$$\bigoplus_i H^*(F_i)(\hbar) \overset{\sum \frac{i_*}{e(N_i)}}{\longrightarrow} H^*_{\mathbb{C}^*}(X) \otimes \mathbb{C}(\hbar) \overset{i^*}{\longrightarrow} \bigoplus_i H^*(F_i)(\hbar)$$

are inverses (as $\mathbb{C}(\hbar)$-algebra homomorphisms) of each other. In particular, since the constant map to a point factors (equivariantly!) through the fixed loci, for any equivariant cohomology class α:

$$\int_X \alpha = \sum_i \int_{F_i} \frac{i^*(\alpha)}{e(N_{F_i/X})}$$

In practice, one can reduce the problem of integrating classes on a space X , which might be geometrically complicated, to integrating over the fixed loci (which are hopefully simpler).

Example 4.4 (The case of $\mathbb{P}^1$)**.** Let $\mathbb{C}^*$ act on a two dimensional vector space V by:

$$t \cdot (v_0, v_1) := (v_0, tv_1)$$

This action defines an action on the projectivization $\mathbb{P}(V) = \mathbb{P}^1$. The fixed points for the torus action are $0 = (1 : 0)$ and $\infty = (0 : 1)$. The canonical action on $T_{\mathbb{P}}$ has weights $+1$ at 0 and -1 at ∞. Identifying $V \setminus 0$ with the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ minus the zero section, we get a canonical lift of the torus action to $\mathcal{O}_{\mathbb{P}^1}(-1)$, with weights $0, 1$. Also, since $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(-1)^\vee$, we get a natural linearization for $\mathcal{O}_{\mathbb{P}^1}(1)$ as well (with weights $0, -1$). Finally, by thinking of $\mathbb{P}^1$ as the projectivization of an equivariant bundle over a point, we obtain:

$$H^*_{\mathbb{C}^*}(\mathbb{P}^1) = \frac{\mathbb{C}[H,\hbar]}{H(H-\hbar)}.$$

The Atyiah-Bott isomorphism now reads:

$$\begin{array}{ccc} \mathbb{C}(\hbar)_0 \oplus \mathbb{C}(\hbar)_\infty & \leftrightarrow & H^*_{\mathbb{C}^*}(\mathbb{P}) \otimes \mathbb{C}(\hbar) \\ (1,0) & \to & \frac{H}{\hbar} \\ (0,1) & \to & \frac{H-\hbar}{-\hbar} \\ (1,1) & \leftarrow & 1 \\ (\hbar,0) & \leftarrow & H \end{array}$$

4.2. **Applying the Localization Theorem to Spaces of Maps.** Kontsevich and Manin first applied the localization theorem to smooth moduli spaces of maps in []. Graber and Pandharipande ([]) generalized this technique to the general case of singular moduli spaces, showing that localization "plays well" with the virtual fundamental class. Several subsequent applications by Okounkov-Pandharipande, Graber-Vakil, Bertram and many other have crowned it as an extremely powerful technique for intersection theory on the moduli space of stable maps. In [Cav06b], [Cav05], the author began applying localization to moduli spaces of admissible covers, technique that was subsequently framed into the larger context of orbifold Gromov Witten theory via the foundational work of [AGV06].

Let X be a space with a $\mathbb{C}^*$ action, admitting a finite number of fixed points P_i, and of fixed lines l_i (NOT pointwise fixed). Typical examples are given by projective spaces, flag varieties, toric varieties... Then:

(1) A $\mathbb{C}^*$ action is naturally induced on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ by postcomposition.
(2) The fixed loci in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parameterize maps from nodal curves to the target such that (see Figure **??**):
    - components of arbitrary genus are contracted to the fixed points P_i.
    - rational components are mapped to the fixed lines as d-fold covers fully ramified over the fixed points.

    In particular

$$F_i \cong \prod \overline{\mathcal{M}}_{g_j,n_j} \times \prod \mathcal{B}\mathbb{Z}_{d_k}.$$

(3) The **"virtual"** normal directions to the fixed loci correspond essentially to either smoothing the nodes of the source curve (which by exercise 16 produces sums of ψ classes and equivariant weights), or to deforming the map out of the fixed points

and lines. This can be computed using the deformation exact sequence ([HKK+03], (24.2)), and produces a combination of equivariant weights and λ classes.

MAKE PICTURE!!

The punchline is, one has reduced the tautological intersection theory of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to combinatorics, and Hodge integrals (i.e. intersection theory of λ and ψ classes). From a combinatorial point of view this can be an extremely complicated and often unmanageable problem, but in principle application of the Grothendieck-Riemann-Roch Theorem and of Witten Conjecture/Kontsevich's Theorem completely determine all Hodge integrals. Carel Faber in [**?**] explained this strategy and wrote a Maple code that can handle efficiently integrals up to a certain genus and number of marks.

5. Evaluation of The Hyperelliptic Locus

We apply localization to moduli spaces of admissible covers to give a proof of a theorem of Faber and Pandharipande that bypasses the use of Grothendieck-Riemann-Roch and only relies on the combinatorics of simple Hurwitz numbers. This proof is independent of the original proof and it is the $d = 2$ case of Theorem 0.2 in [BCT06].

Theorem 5.1 ([FP00], Corollary of Proposition 3)**.** *Denote by H_g a $(2g+2)!$ cover of the hyperlliptic locus in $\mathcal{M}_g$ obtained by marking the Weierstrass points. Then:*

$$\sum_{g=1}^{\infty}\left(\int_{H_g}\lambda_g\lambda_{g-1}\right)\frac{x^{2g-1}}{(2g-1)!}=\frac{1}{2}\tan\left(\frac{x}{2}\right). \tag{13}$$

Observations.

(1) Since the class $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$, the above integral can be performed on the closure of the hyperelliptic locus. We choose the open statement because of the original application that led to study the problem: the class of the hyperelliptic locus is a generator for the socle in the tautological ring of $\mathcal{M}_g$.

(2) Choosing the appropriate generating function packaging is key to solving these questions. While (13) is probably the most appealing form of the result, we prove the equivalent integrated version:

$$\mathcal{D}_1(x):=\sum_{g=1}^{\infty}\left(\int_{H_g}\lambda_g\lambda_{g-1}\right)\frac{x^{2g}}{(2g)!}=-\ln\cos\left(\frac{x}{2}\right).$$

(3) We essentially use the identification described in Exercise 14:

$$\overline{H_g} \cong Adm_{g\to 0,2}((2),\dots,(2)) \cong \text{``}\frac{1}{2}\text{''}\overline{\mathcal{M}}_{0,2g+2}.$$

to translate the geometric problem into a combinatorial one.

5.1. **Outline of proof.** We first introduce generating functions for other Hodge integrals on the Hyperelliptic locus:

$$\mathcal{D}_i(x) := \sum_{g\geq i}\left(\int_{\overline{H}_g}\lambda_g\lambda_{g-i}\psi^{i-1}\right)\frac{x^{2g}}{2g!}. \tag{14}$$

For reasons that will become evident in a few lines, we also define

$$\mathcal{D}_0(x) := \frac{1}{2}. \tag{15}$$

Now the proof of (13) follows from combining the following two ingredients. First a way to describe all $\mathcal{D}_i$'s in terms of $\mathcal{D}_1$.

Lemma 5.2 ([Cav06a],Theorem 1)**.**

$$\mathcal{D}_i(x) = \frac{2^{i-1}}{i!}\mathcal{D}_1^i(x).$$

or equivalently

$$\sum_{i=0}^{\infty}\mathcal{D}_i(x) = \frac{1}{2}\ e^{2\mathcal{D}_1(x)}$$

Second, an interesting way to write the identity $0 = 0$:

Lemma 5.3. *The integral*

$$\int_{Adm_{g\to\mathbb{P}^1}} e(R^1\pi_*f^*(\mathcal{O}\oplus\mathcal{O}(-1))) = 0 \tag{16}$$

implies the relation:

$$\frac{1}{2}(\cos(x)-1) = \frac{1}{2}\sin(x)\left(\sum_{i=0}^{\infty}\int\mathcal{D}_i(x)\right) \tag{17}$$

Exercise 18. Given the two lemmas, conclude the proof. This is in fact a Calc II exercise!

Remarks.

(1) Note that the auxiliary integral (16) is on a moduli space of admissible covers of a parameterized $\mathbb{P}^1$. This allows the moduli space to have a $\mathbb{C}^*$ action. The fixed loci however are boundary strata consisting of products of admissible cover spaces of an unparameterized rational curve, illustrated in Figure 6.

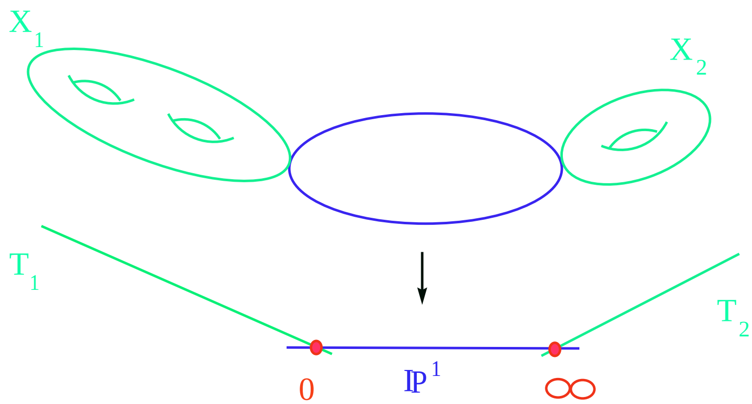


FIGURE 6. The fixed loci for integrals of admissible covers to a parameterized $\mathbb{P}^1$ consist of covers where all ramification happens over 0 and ∞ - or rather over rational tails sprouting from these points.

(2) Lemma 5.2 can also be proven by computing via localization the following auxiliary integrals:

$$A_k := \int_{Adm_{g\to\mathbb{P}^1}} \lambda_g\lambda_{g-k} ev_1^*(0) ev_2^*(0) ev_3^*(\infty) = 0$$

For $k > 1$, the integral is 0 for trivial dimension reasons. After localization one obtains a relation among $\mathcal{D}_i(x)$'s with $i \le k$, which allows to compute $\mathcal{D}_k(x)$ inductively.

(3) Lemma 5.2 generalizes nicely to the case of multiple ψ insertions. For I a multi-index of size $i-1$

$$\sum_{g\ge i}\left(\int_{\overline{H}_g} \lambda_g\lambda_{g-i}\psi^I\right)\frac{x^{2g}}{2g!} = \binom{i-1}{I}\frac{2^{i-1}}{i!}\mathcal{D}_1^i(x). \tag{18}$$

Formula (18) was experimentally discovered independently by the author and Danny Gillam in 2007. A formal proof of (18) was recently given by the author and his PhD student Dusty Ross.

(4) Deriving relation (17) from the auxiliary integral (16) requires an appropriate choice of lifting of the torus action to the bundles $\mathcal{O}$ and $\mathcal{O}(-1)$. A lifting that has opposite weights for the two bundles over 0, gives, by Mumford relation (10) a Hurwitz number contribution by the moduli spaces of the covers sprouting over 0. Since hyperelliptic Hurwitz numbers are trivial, the

generating functions for them are the $\frac{1}{2}\sin(x)$ and $\frac{1}{2}(\cos(x)-1)$ that we see in relation (17).

6. Simple Hurwitz Numbers and the ELSV Formula

The name **simple Hurwitz number** (denoted $H_g(\eta)$) is reserved for Hurwitz numbers to a base curve of genus 0 and with only one special point where arbitrary ramification is assigned. In this case the number of simple ramification, determined by the Riemann-Hurwitz formula, is

$$r = 2g + d - 2 + \ell(\eta). \tag{19}$$

In this case, our combinatorial definition (3) of Hurwitz number simplifies further to count (up to an appropriate multiplicative factor) the number of ways to factor a (fixed) permutation $\sigma \in C_\eta$ into r transpositions that generate S_d:

$$H_g(\eta) = \frac{1}{\prod \eta_i} |\{(\tau_1, \dots, \tau_r \ s.t.\ \tau_1 \cdot \dots \cdot \tau_r = \sigma \in C_\eta, \langle \tau_1, \dots, \tau_r \rangle = S_d)\}| \tag{20}$$

Exercise 19. Prove that (20) is indeed equivalent to (3).

The first formula for simple Hurwitz number was given and "sort of" proven by Hurwitz in 1891 ([**?**]):

$$H_0(\eta) = r! d^{n-3} \prod \frac{\eta_i^{\eta^i}}{\eta_i!}.$$

Particular cases of this formula were proven throughout the last century, and finally the formula became a theorem in 1997 ([]). In studying the problem for higher genus, Goulden and Jackson made the following conjecture.

Conjecture. *For any fixed values of $g, n := \ell(\eta)$:*

$$H_g(\eta) = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} P_{g,n}(\eta_1 \dots, \eta_n), \tag{21}$$

where $P_{g,n}$ is a symmetric polynomial in the η_i's with:

- *deg $P_{g,n} = 3g - 3 + n$;*
- *$P_{g,n}$ doesn't have any term of degree less than $2g - 3 + n$;*
- *the sign of the coefficient of a monomial of degree d is $(-1)^{d-(3g+n-3)}$.*

In [ELSV01] Ekehdal, Lando, Shapiro and Vainshtein prove this formula by establishing a remarkable connection between simple Hurwitz numbers and tautological intersections on the moduli space of curves.

Theorem 6.1 (ELSV formula)**.** *For all values of $g, n = \ell(\eta)$ for which the moduli space $\overline{\mathcal{M}}_{g,n}$ exists:*

$$H_g(\eta) = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \ldots + (-1)^g \lambda_g}{\prod (1 - \eta_i \psi_i)}, \tag{22}$$

Remark 6.2. Goulden and Jackson's polynomiality conjecture is proven by showing the coefficients of $P_{g,n}$ as tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Using our standard multi-index notation:

$$P_{g,n} = \sum_{k=0}^{g} \sum_{|I_k| = 3g-3+n-k} (-1)^k \left(\int \lambda_k \psi^{I_k} \right) \eta^{I_k}$$

Remark 6.3. The polynomial $P_{g,n}$ is a generating function for all Hodge integrals on $\overline{\mathcal{M}}_{g,n}$, and hence a good understanding of this polynomial can yield results about the intersection theory on the moduli space of curves. In fact the *ELSV* formula has given rise to several remarkable applications:

[**?**]**:** Okounkov and Pandharipande use the ELSV formula to give a proof of Witten's conjecture, that an appropriate generating function for the ψ intersections satisfies the KdV hierarchy. The ψ intersections are the coefficients of the leading terms of $P_{g,n}$, and hence can be reached by studying the asymptotics of Hurwitz numbers:

$$\lim_{N \to \infty} \frac{P_{g,n}(N\eta)}{N^{3g-3+n}}$$

[GJV06]**:** Goulden, Jackson and Vakil manage to get a handle on the lowest order terms of $P_{g,n}$ to give a new proof of the λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi^I = \binom{2g-3+n}{I} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}$$

We sketch a proof of the *ELSV* formula following [GV03]. The strategy is to evaluate an integral via localization, fine tuning the geomety in order to obtain the desired result.

We introduce a new moduli space, which is in some sense a hybrid between stable maps and admissible covers. Moduli spaces of relative stable maps were first introduced in the symplectic category by A. Li and Ruan ([]), and then portered to the algebraic category by Jun Li in [Li02](CHECK IF IT'S THE RIGHT REFERENCE).

We denote

$$\mathcal{M} := \overline{\mathcal{M}}_g(\mathbb{P}^1, \eta\infty)$$

parameterizing maps of degree d, $f : X \to \mathbb{P}^1$, such that, when X is a smooth curve, the profile over ∞ is η. The degenerations we include to compactify are twofold:

- away from the preimages of ∞ we have degenerations of "stable maps" type: we can have nodes and contracting components for the source curve, and nothing happens to the target $\mathbb{P}^1$;
- when things collide at ∞, then the degeneration is of "admissible cover" type: a new rational component sprouts from $\infty \in \mathbb{P}^1$, the special point carrying the profile requiremente transfers to this component. Over the node we require nodes for the source curve, with maps satisfying the kissing condition.

A slightly more formal description of these spaces can be found in [**?**], Section 4.10. The space $\mathcal{M}$ has virtual dimension $r = 2g + d + \ell(\eta)$ and admits a globally defined branch morphism ([**?**]):

$$br : \mathcal{M} \to Sym^r(\mathbb{P}^1) \cong \mathbb{P}^r.$$

The simple Hurwitz number:

$$H_g(\eta) = \deg(br) = br^* \cap [\mathcal{M}]^{vir}$$

can now interpreted as an intersection number on a moduli space with a torus action and evaluated via localization. The map br can be made $\mathbb{C}^*$ equivariant by inducing the appropriate action on $\mathbb{P}^r$. The key point is now to choose the appropriate equivariant lift of the class of a point in $\mathbb{P}^r$. Recalling that choosing a point in $\mathbb{P}^r$ is equivalent to fixing a branch divisor, we choose the $\mathbb{C}^*$ fixed point corresponding to stacking all ramification over 0. Then there is a unique fixed locus contributing to the localization formula, depicted in Figurefig:ELSV, which is essentially isomorphic to $\overline{\mathcal{M}}_{g,n}$ (up to some automorphism factors coming from the automorphisms of the bubbles over $\mathbb{P}^1$).

The *ELSV* formula falls immediately out of the localization formula.

7. Double Hurwitz Numbers

Double Hurwitz numbers count covers of $\mathbb{P}^1$ with special ramification profiles over two points, that for simplicity we assume to be 0 and ∞. Double Hurwitz numbers are classically denoted $H^r_g(\mu, \nu)$; in [**?**] we start denoting double Hurwitz numbers $H^r_g(\mathbf{x})$, for $\mathbf{x} \in H \subset \mathbb{R}^n$ an integer lattice point on the hyperplane $\sum x_i = 0$. The subset of positive coordinates corresponds to the profile over 0 and the negative coordinates to the profile over ∞. We define $\mathbf{x_0} := \{x_i > 0\}$ and $\mathbf{x}_\infty := \{x_i < 0\}$.

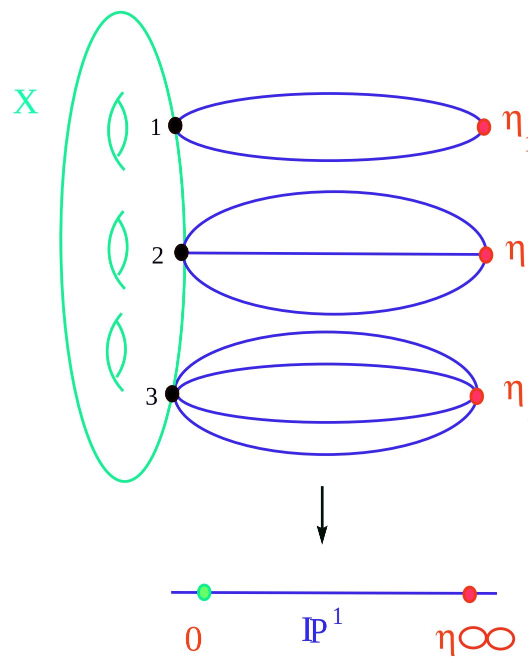


FIGURE 7. the unique contributing fixed locus in the localization computation proving the *ELSV* formula.

The number r of simple ramification is given by the Riemann-Hurwitz formula,

$$r = 2g - 2 + n$$

and it is independent of the degree d. In [GJV03], Goulden, Jackson and Vakil start a systematic study of double Hurwitz numbers and in particular invite us to consider them as a function:

$$H_g^r(-) : \mathbb{Z}^n \cap H \to \mathbb{Q}. \tag{23}$$

They prove some remarkable combinatorial property of this function:

Theorem 7.1 (GJV)**.** *The function $H_g(-)$ is a piecewise polynomial function of degree $4g - 3 + n$.*

And conjecture some more:

Conjecture (GJV)**.** *The polynomials describing $H_g^r(-)$ have degree $4g - 3 + n$, lower degree bounded by $2g - 3 + n$ and are even or odd polynomials (depending on the parity of the leading coefficient).*

Later, Shapiro, Shadrin and Vainshtein explore the situation in genus 0. They describe the location of all walls, and give a geometrically suggestive formula for how the polynomials change when going across a wall.

Theorem 7.2 (SSV). *The chambers of polynomiality of $H^r_g(-)$ are bounded by* ***walls*** *corresponding to the* ***resonance*** *hyperplanes W_I, given by the equation*

$$W_I = \left\{ \sum_{i \in I} x_i = 0 \right\},$$

for any $I \subset \{1, \ldots, n\}$.

Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two chambers adjacent along the wall W_I, with $\mathfrak{c}_1$ being the chamber with $x_I < 0$. The Hurwitz number $H^r_g(\mathbf{x})$ is given by polynomials, say $P_1(\mathbf{x})$ and $P_2(\mathbf{x})$, on these two regions. A wall crossing formula is a formula for the polynomial

$$WC^r_I(\mathbf{x}) = P_2(\mathbf{x}) - P_1(\mathbf{x}).$$

Genus 0 wall crossing formulas have the following inductive description:

$$WC^r_I(\mathbf{x}) = \delta \binom{r}{r_1, r_2} H^{r_1}(\mathbf{x_I}, \delta) H^{r_2}(\mathbf{x_{I^c}}, -\delta), \tag{24}$$

where $\delta = \sum_{i \in I} x_i$ is the distance from the wall at the point we evaluate the wall crossing.

Remarks.

(1) This formula appears not to depend on the particular choice of chambers $\mathfrak{c}_1$ and $\mathfrak{c}_2$ that border on the wall, but only upon the wall W_I; however the polynomials for the simpler Hurwitz numbers in the formula depend on chambers themselves.
(2) The walls W_I correspond to values of $\mathbf{x}$ where the cover could potentially be disconnected, or where $\mathbf{x}_i = 0$. In the first case the formula reminds of a boundary divisor degeneration formula, and somehow begs for a geometric understanding.
(3) Crossing this second type of wall corresponds to moving a ramification between 0 and ∞. In the traditional view of double Hurwitz numbers, these were viewed as separate problems: the lenght of the profiles over 0 and ∞ were fixed separately, rather than just the total length. However, here we see that it is natural to treat them as part of the same problem: in genus 0 the wall crossing formula for $\mathbf{x}_i = 0$ is trivial - and as such identical to all other wall crossing formulas. This motivates our $\mathbf{x}$ replacing μ, ν in our notation. Note that I am note just being cute here. Let me preview that in higher genus this second type

of wall crossing are not trivial any more, while still obeying the same wall crossing formulas as wall crossing of the first type.

The way Goulden, Jackson and Vakil prove their result is similar to [**?**]: they compute double Hurwitz numbers by counting decorated ribbon graphs on the source curve. A ribbon graph is obtained by pulling back a set of segments from the base curve (connecting 0 to the simple ramification points) and then stabylizing. Each ribbon graph comes with combinatorial decorations that are parameterized by integral points in a polytope with linear boundaries in the x_i's. Standard algebraic combinatorial techniques then show that such counting yields polynomials so long as the topology of the various polytopes does not change. The downside of this approach is that these are pretty "large" polytopes and it is hard to control their topology.

Shapiro, Shadrin and Vainshtein go at the problem with a geometric angle, and are able to prove the wall crossing formulas using some specific properties of intersections of ψ classes in genus 0. Since descendants become quickly more mysterious in higher genus, their approach didn't generalize.

In what follows I'd like to present the approach of [] to this problem. Motivated by tropical geometry, we are able to compute double Hurwitz numbers in terms of some trivalent polynomially weighted graphs (that can be thought as tropical covers, even though this point of view is not necessary other than to give the initial motivation) that are, in a sense, "movies of the monodromy representation". These graphs give a straightforward and clean proof of the genus 0 situation. In [] we show that in higher genus each graph Γ comes together with a polytope P_Γ (with homogenous linear boundaries in the x_i) and we have to sum the polynomial weight of the graph over the integer lattice points of P_Γ. It is again standard (think of it as a discretization of integrating a polynomial over a polytope) to show that this contribution is polynomial when the topology of the polytope does not change. The advantage here is that we have shoved most of the complexity of the situation in the polynomial weights of the graph: our polytopes are only g dimensional and it is possible to control their topology. Thus in [], we are able to give a complete description of the situation for arbitrary genus.

7.1. **The Cut and Join Recursions and Tropical Hurwitz Numbers.** The *Cut and Join equations* are a collection of recursions among Hurwitz numbers. In the most elegant and powerful formulation they are expressed as one differential operator acting on the Hurwitz potential. Our use of cut and join here is unsophisticated, so we limit ourselves to a basic discussion, and refer the reader to [**?**] for a more in-depth presentation.

Let $\sigma \in S_d$ be a fixed element of cycle type $\eta = (n_1, \ldots, n_l)$, written as a composition of disjoint cycles as $\sigma = c_l \ldots c_1$. Let $\tau = (ij) \in S_d$ vary among all transpositions. The cycle types of the composite elements $\tau\sigma$ are described below.

cut: if i, j belong to the same cycle (say c_l), then this cycle gets "cut in two": $\tau\sigma$ has cycle type $\eta' = (n_1, \ldots, n_{l-1}, m', m'')$, with $m' + m'' = n_l$. If $m' \neq m''$, there are n_l transpositions giving rise to an element of cycle type η'. If $m' = m'' = n_l/2$, then there are $n_l/2$.

join: if i, j belong to different cycles (say c_{l-1} and c_l), then these cycles are "joined": $\tau\sigma$ has cycle type $\eta' = (n_1, \ldots, n_{l-1} + n_l)$. There are $n_{l-1}n_l$ transpositions giving rise to cycle type η'.

Example 7.3. Let $d = 4$. There are 6 transpositions in S_4. If $\sigma = (12)(34)$ is of cycle type $(2, 2)$, then there are 2 transpositions ((12) and (34)) that "cut" σ to give rise to a transposition and $2 \cdot 2$ transpositions $((13), (14), (23), (24))$ that "join" σ into a four-cycle.

For readers allergic to notation, Figure **??** illustrates the above discussion. MAKE FIGURE in PDF

Let us now specialize our definition of Hurwitz number by counting monodromy representations to the case of double Hurwitz numbers.

$$H_g^r(\mathbf{x}) := \frac{|\mathrm{Aut}(\mathbf{x_0})||\mathrm{Aut}(\mathbf{x}_\infty)|}{d!} |\{\sigma_0, \tau_1, \ldots, \tau_r, \sigma_\infty \in S_d\}|$$

such that:

- σ_0 has cycle type $\mathbf{x_0}$;
- τ_i's are simple transpositions;
- σ_∞ has cycle type $\mathbf{x}_\infty$;
- $\sigma_0\tau_1 \ldots \tau_r\sigma_\infty = 1$
- the subgroup generated by such elements acts transitively on the set $\{1, \ldots, d\}$.

The key insight of [**?**] is that we can organize this count in terms of the cycle types of the composite elements

$$C_{\mathbf{x_0}} \ni \sigma_0, \sigma_0\tau_1, \sigma_0\tau_1\tau_2, \ldots, \sigma_0\tau_1\tau_2 \ldots \tau_{r-1}, \sigma_0\tau_1\tau_2 \ldots \tau_{r-1}\tau_r \in C_{\mathbf{x}_\infty}$$

At each step the cycle type can change as prescribed by the cut and join recursions, and hence for each possibility we can construct a graph with edges weighted by the multiplicites of the cut and join equation. In [] we call such graphs monodromy graphs.

Definition 7.4. For fixed g and $\mathbf{x} = (x_1, \ldots, x_n)$, a graph Γ is a **monodromy graph** if:

(1) Γ is a connected, genus g, directed graph.
(2) Γ has n 1-valent vertices called *leaves*; the edges leading to them are *ends*. All ends are directed inward, and are labeled by the weights $x_1, \ldots, x_n$. If $x_i > 0$, we say it is an *in-end*, otherwise it is an *out-end*.
(3) All other vertices of Γ are 3-valent, and are called *internal vertices*. Edges that are not ends are called *internal edges*.
(4) After reversing the orientation of the out-ends, Γ does not have directed loops, sinks or sources.
(5) The internal vertices are ordered compatibly with the partial ordering induced by the directions of the edges.
(6) Every internal edge e of the graph is equipped with a *weight* $w(e) \in \mathbb{N}$. The weights satisfy the *balancing condition* at each internal vertex: the sum of all weights of incoming edges equals the sum of the weights of all outgoing edges.

So we can compute Hurwitz number as a weighted sum over monodromy graphs. Keeping in account the various combinatorial factors one obtains:

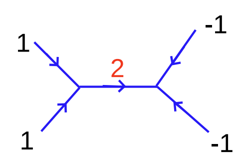
Lemma 7.5 ([], Lemma 4.1)**.** *The Hurwitz number is computed as:*

$$H_g(\mathbf{x}) = \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|} \varphi_\Gamma, \tag{25}$$

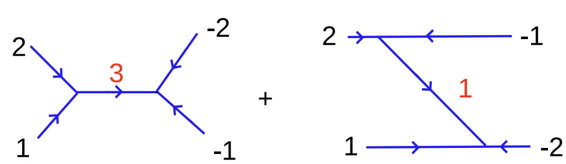
where the sum is over all monodromy graphs Γ for g and $\mathbf{x}$, and φ_Γ denotes the product of weights of all internal edges.

Example 7.6. Here are a couple silly examples computed using formula 25.

$H^0((1,1), -(1,1)) = 2$



$H^0((2,1), -(2,1)) = 4$



Now instead of individual numbers, we want to compute the Hurwitz functions. In figure **??** we illustrate the situation. We change a little the notation, using y's for the coordinates at ∞. We observe that different graphs contribute according to the sign of $x_1 + y_1$. This gives us

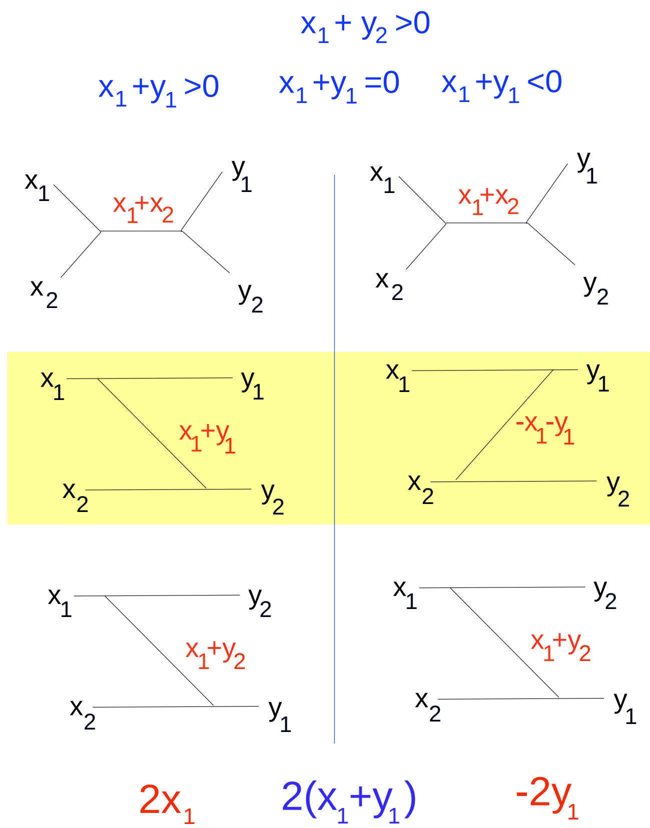


FIGURE 8. Computing double Hurwitz numbers using Lemma 7.5 and observing the wall crossing.

two different polynomials, the difference of which is given by formula 24 (in a trivial way since three pointed genus zero double Hurwitz numbers are trivially seen to be 1).

7.2. **Genus** 0**.** In this section we show how this point of view leads to fairly elementary proofs of Theorems 7.1 and 7.2.

(1) Each edge weight is linear homogeneous in the x_i's and we have $3g - 3 + n$ internal edges. Therefore each contributing graph gives a polynomial of the appropriate degree. A graph contributes if and only if all edge weights are positive. It's clear then that the regions of polynomiality are precisely where the signs of all edge weights persist.

(2) For a given wall W_I, graphs that appear on both sides of the wall give no contribution to the wall crossing. A graph appears on both sides of the wall if and only if no edge changes sign

across the wall, i.e. if and only if there is no edge with weight $\delta = \pm \sum_{i\in I} x_i$.

(3) So in order to compute the wall crossing formula, we only need to focus on graphs that contain an edge labelled $\pm\delta$. In particular if the edge is labelled δ, the graph appears on one side of the wall, if it is labelled $-\delta$ the graph appears on the other side of the wall. Keeping in mind that the wall crossing formula is the subtraction of the Hurwitz polnomyals on either side of the wall, the polynomial contribution from any graph (no matter on which side of the wall it is on) Γ is δ times the product of the weights of all other internal edges.

(4) Now look at the RHS of (24). Each of the Hurwitz numbers appearing can be computed as a weighted sum over the appropriate monodromy graphs, and therefore the product of Hurwitz numbers can be computed as a weighted sum over pairs of graphs. For a fixed pair of graphs the polynomial contribution is δ times the product of internal edges of both graphs.

(5) We now prove formula (24) by exhibiting a bijection that preserves polynomial contributions between:
   (a) the set L of graphs contributing to the wall crossing.
   (b) a set R of cardinality $\binom{r}{r_1,r_2}$ times the cardinality of the set of pairs of graphs contributing to the product of Hurwitz numbers on the right.

   First we describe R as the set of triples $(\Gamma_1, \Gamma_2, \mathfrak{o})$, where Γ_1 and Γ_2 are graphs contributing to the respective Hurwitz numbers, and $\mathfrak{o}$ is a total ordering ther vertices of $\Gamma_1 \cup \Gamma_2$ compatible with the individual vertex orderings of both graphs.

Exercise 20. Convince yourself that R has indeed the desired cardinality.

Now consider the functions

$$Cut : L \to R$$

that cuts each graph along the edge labelled δ, and

$$Glue : R \to L$$

that glues the two graphs Γ_1, Γ_2 along the ends labelled δ orienting the new edge as prescribed by the total order $\mathfrak{o}$.

Exercise 21. Convince yourself that Cut and $Glue$ are inverses to each other.

8. Higher Genus

Lemma 7.5 is not special to genus 0, and gives us a combinatorial recipe for computing arbitrary double Hurwitz numbers. However two significant complications arise when trying to generalize the previous theorems to higher genus:

(1) In genus 0 the balancing condition and the weights at the ends determine uniquely the weights of all internal edges. It is easy to see that in higher genus this is not true, and that in fact each genus adds a degree of freedom in the choice of weights. In other words, given a directed, vertex ordered graph with labelled ends, there is a g-dimensional polytope parameterizing internal edge weights compatible with the end weights and the balancing condition. The bounds of this polytope are linear in the x_i's.

(2) In genus 0 a graph contributing to the wall crossing had a unique edge labelled δ, and consequently a unique way to be disconnected into two smaller graphs. In higher genus this is not the case any more: there are multiple edges that can be "cut" when crossing the wall, and multiple ways to disconnect the graph. A careful project of inclusion/exclusion is then required to obtain the result.

To recover Theorem **??** one has to deal with (1), and it is really not too bad. Our Hurwitz number is still expressed as a finite sum over graphs, except now for each graph the contribution is a homogenous polynomial of degree $3g-3+n$ in the x_i's plus g new variables, that we need to "integrate" over the integer lattice points of a g dimensional polytope. You can think of this as a generalization of continuous integration, or as a repeated application of the power sum formulas; in either case it is not hard to see that the result is locally a polynomial in the x_i's of degree $4g-3+n$.

Where does piecewise polynomiality kick in? Well, as you move the x_i's around the various graph polytopes change their topology (some face could get hidden or uncovered by other faces translating around). That is precisely where the walls are. Again a little bit of analysis shows that this happen precisely when multiple hyperplanes defining the polytope intersect non-transversally, and this happens when the graph can be disconnected.

Here we wish to focus on the much more subtle (and interesting) case of generalizing the wall crossing formula. Rather than a complete discussion of the proof (a hopefully reasonable outline of which can be found in section 4 of [**?**]), here we would like to illustrate the salient ideas through two examples, each one specifically tuned to illustrate dealing with points (1) and (2) above. But before we embark into this endeavour, let us start with stating the result.

Theorem 8.1.
(26)

$$WC_I^r(\mathbf{x}) = \sum_{\substack{s+t+u=r \\ |\mathbf{y}|=|\mathbf{z}|=|\mathbf{x}_I|}} (-1)^t \binom{r}{stu} \frac{\prod \mathbf{y}_i}{\ell(\mathbf{y})!} \frac{\prod \mathbf{z}_j}{\ell(\mathbf{z})!} H^s(\mathbf{x}_I, \mathbf{y}) H^{t\bullet}(-\mathbf{y}, \mathbf{z}) H^u(\mathbf{x}_{I^c}, -\mathbf{z})$$

Here $\mathbf{y}$ *is an ordered tuple of* $\ell(\mathbf{y})$ *positive integers with sum* $|\mathbf{y}|$*, and similarly with* $\mathbf{z}$*.*

8.1. **The simple cut: Hyperplane Arrangements and The "Cut-Glue" Geometric Bijection.** In this section we focus on an example of how a geometric bijection can be defined between graphs contributing to the wall crossing, and graphs contributing to the degeneration formula on the RHS of (26). The idea distills to its simplest in the case of graphs that can be cut in only one way.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $r = 6$ (i.e. $g = 2$). Let $I = \{1, 3\}$, and let $\mathfrak{c}_1$ be an H-chamber next to the wall W_I satisfying $x_1 + x_3 \leq 0$. $\mathfrak{c}_2$ is the opposite H-chamber. In Figure 9 we consider a graph Γ contributing to the wall crossing for this particular wall. Really, we want to think of Γ as an un-directed graph. The direction of the edges we show in this picture are to be thought of as a choice of reference orientation, and we understand that putting a positive weight on an edge preserves the reference orientation, while a negative weight reverses it. Now we observe:

- For any choice of labeling of the ends, there is a 2-dimensional plane of possible weights for the edges satisfying the balancing condition.
- This plane is subdivided by hyperplanes (lines) whose equations are given precisely by the internal edge weights.
- Chambers for this hyperplane arrangement correspond to orientations of the edges of the graph. Unbounded chambers correspond to orientations with directed loops, bounded chambers correspond to orientation with NO directed loop.
- We give each chamber a multiplicity consisting of the number of vertex orderings of the directed graph compatible with the directions of the edges. Note that unbounded chambers have multiplicity 0.

Figure 9 illustrate this situation over two points living on either side of wall W_I.

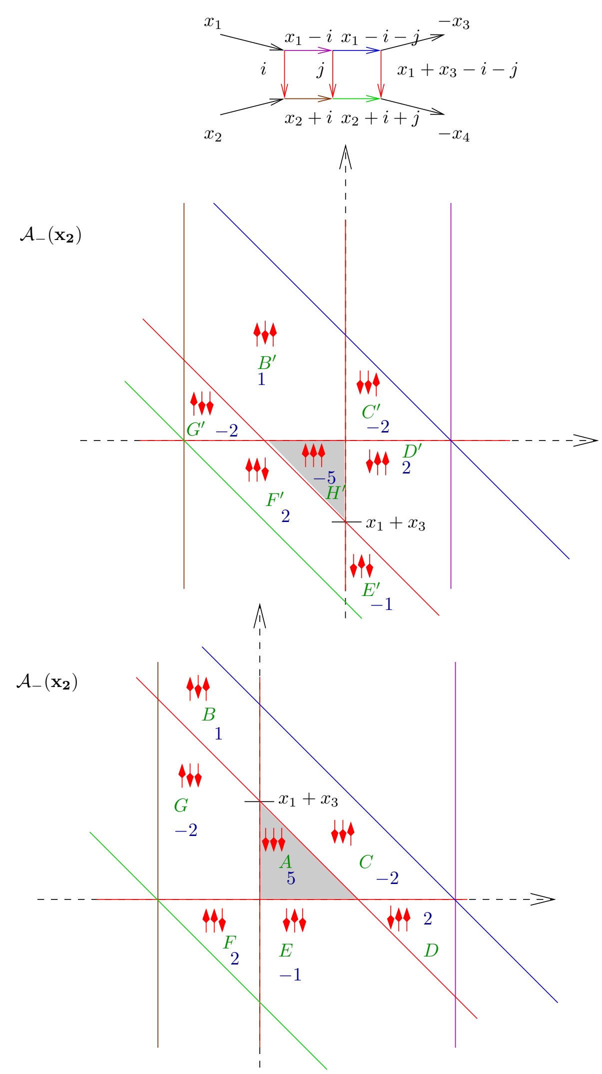


FIGURE 9. Hyperplane arrangements corresponding to orientations of edges of a graph.

At the wall the the three red hyperplanes corresponding to the three edges that can be cut to disconnect the graph, meet in codimension 2.

On one side of the wall, these three hyperplanes form a simplex which vanishes when we hit the wall (*vanishing F-chamber*). A new simplex reappears on the other side of the wall (an *appearing F-chamber*).

The directed graph corresponding to the appearing chamber has a flow from top to bottom, that can only be realized when $x_1+x_3 \geq 0$, i.e. on side "2" of the wall. Thus we can see from the graphs whether an F-chamber is vanishing/appearing or not. The 6 neighboring chambers appear on both sides of the wall.

When computing the Hurwitz number, for each directed graph we need to sum the product of internal edges over the integer points of the corresponding chamber. Note that for all directed graphs with the same underlying graph, the product of internal edges ($= \varphi_\Gamma$) differs at most by a sign, and that this signs alternates in adjacent chambers. To understand the contribution to P_2, we sum the polynomial $\varphi_{\mathcal{A}}$ (weighted with sign and multiplicity) over the lattice points in each of the chambers A, B, …, G. For the polynomial P_1, we have to play the same game with the chambers B', …, H' on top, however, we evaluate this polynomial now at the point $\mathbf{x}_2$ which is not in $\mathfrak{c}_1$ but in $\mathfrak{c}_2$. Thus, we have to "carry" the chambers $B', \ldots, H'$ over the wall, i.e. we need to interpret the region bounded e.g. by the defining hyperplanes of B' on the other side of the wall in terms of the chambers $A, \ldots, G$. **Note:** there are several things to be careful about signs in these discussions, but I would like to make the expositional choice of completely ignoring any sign issue, and ask for your trust that everything pans out as desired at the end of the day. Or rather, if you are really hardcore...

Exercise 22. Check my signs! They should be defined in a natural and consisten way till the end of the paper...hopefully, that is.

We now express each of the chambers $B', \ldots, H'$ as a formal signed sum of the chambers $A, \ldots, G$.

$$H' = A \qquad B' = B - A \qquad C' = C + A \qquad D' = D - A$$
$$E' = E + A \qquad F' = F - A \qquad G' = G + A.$$

The only chamber on side 1 which contains B in its support when interpreted on side 2 is B', and the B coefficient for B' is $+1$. In the difference $P_2(\mathbf{x}_2) - P_1(\mathbf{x}_2)$ the two summands $\sum_B 1 \cdot \varphi_{\Gamma_B} - \sum_B 1 \cdot \varphi_{\Gamma_B}$ cancel. This is a general fact.

Fact. Only appearing chambers contribute to the wall crossing. We must be careful though. Chambers on side 1 of the wall (which are certainly not appearing!), when transported to side 2 DO contribute to an appearing chamber.

Since the polynomial we are integrating over A is always (up to sign) the same, what we really must be concerned with is the multiplicities of the contributions to A by A itself and by the chambers on side 1 that map to A.

$$\begin{aligned} &\sum_A \big(5-(-5)+1-(-2)+2-(-1)+2-(-2)\big)\varphi_{\mathcal{A}} \\ &= \sum_A 20 \cdot \varphi_{\mathcal{A}} = \sum_A \binom{6}{3} \cdot \varphi_{\mathcal{A}}. \end{aligned} \tag{27}$$

If we cut the graph Γ at the three edges, then the upper part Γ_u contributes to the Hurwitz number $H^3(x_1, x_3, -i, -j, -x_1-x_3+i+j)$ and the lower part Γ_l contributes to the Hurwitz number $H^3(x_2, x_4, i, j, -x_2-x_4-i-j)$. In fact the pair (Γ_u, Γ_l) appears 6 times in the product of Hurwitz numbers, corresponding to all ways of labelling the three cut edges. Then note that to compute the pair of Hurwitz numbers we must sum over all $i \geq 0$, $j \geq 0$ and $x_1 + x_3 - i - j \geq 0$ (the simplex A) the product of internal edges of the two connected components times the connecting edges, hence just the polynomial $\varphi_{\mathcal{A}}$. Then the contribution to the right hand side of (26) by pair of graphs that glue to Γ is $6 \sum_A \binom{6}{3} \cdot \frac{\varphi_{\mathcal{A}}}{6}$, i.e. (27).

We want to take this a little further, and interpret this equality geometrically. The factor $\binom{6}{3}$ counts the ways to merge two orderings of the vertices of Γ_1 and Γ_2 to a total ordering of all vertices. Then re-gluing the cut graphs with the extra data of this merging gives a bijection with the directed, vertex-ordered graphs, contributing to (27). So in this case we have a direct generalization of the $Cut - Glue$ correspondence of section 7.2

Remark 8.2. Or course to turn this idea into a proof one needs to formalise things. In [], we interpret the bounded chambers above as a basis of the g-th relative homology of the hyperplane arrangement, and the process of "carrying the chambers over the wall" as a Gauss Manin connection on the corresponding homological bundle. Then the core of our idea is a combinatorial formula for this Gauss Manin connection in terms of cutting and regluing of graphs.

8.2. **The egg: Inclusion/Exclusion.** In section 8.1 we focused on a graph with a wealth of possible edge orientations and hence a very interesting associated hyperplane arrangement, but only one way to disconnect along the wall. Here we go to the other extreme and observe

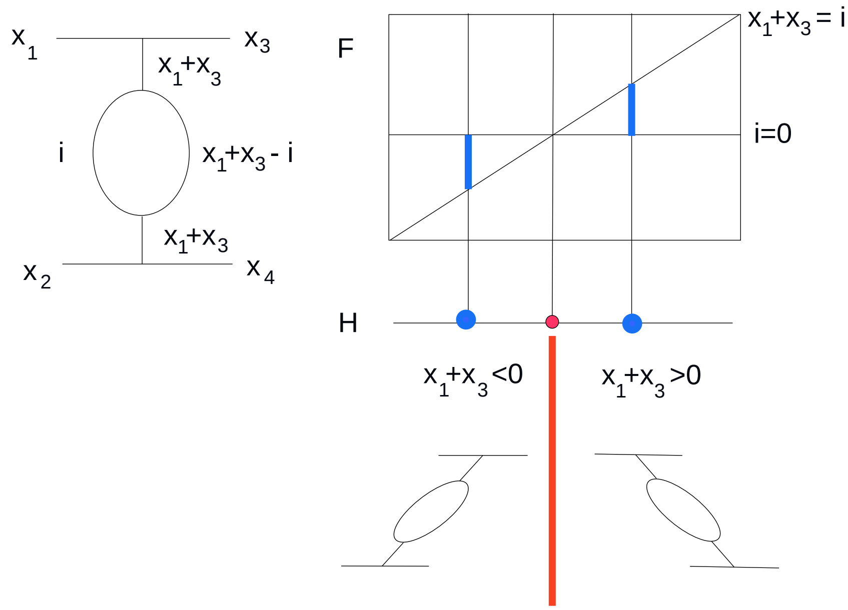


FIGURE 10. The egg graph and its orientations on either side of the wall.

a graph with trivial orientation choices but many possible ways to disconnect. Consider "the egg" in Figure 10, and again the wall W_I given by the equation $x_1 + x_3 = 0$.

There is only one possible orientation for the egg on each side of the wall and one possible vertex ordering for each of these orientations. The chamber on the left hand side is an appearing chamber, and the chamber on the left, when transported to the right, covers it with multeplicity -1.

The coefficient in front of $\sum_C \varphi_G amma$ for the egg contribution to the wall crossing is therefore:

$$1 - (-1) = 2$$

Good news first: let us check that our formula indeed gives us a 2. If we disconnect the egg in all possible ways that give us at most three connected component (see Figure 11) and look at the appopriate coefficients, we get:

$$\binom{4}{1} + \binom{4}{3} + \binom{4}{2} - \binom{4}{1,1,2} - \binom{4}{2,1,1} + \binom{4}{1,2,1} = 2$$

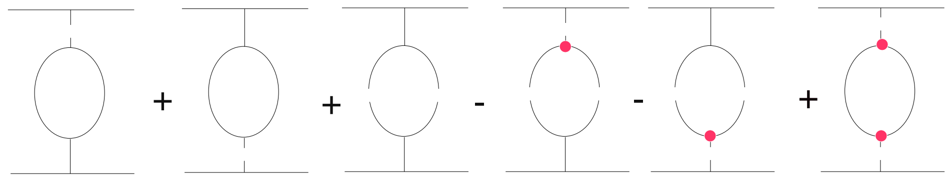


FIGURE 11. The light cuts of the egg graph.

But now for the bad news. We would like to prove this by giving a geometric bijection between the graphs contributing to the wall crossings and the regluings of the cut graphs where we allow to reorient the cut edges. To go from the left side egg to the right side egg we need to reorient all internal edges, and there is no cut in Figure **??** that allows us to do that!

After much crying and gnashing of teeth, this lead us to think that maybe we should allow ourselves to do more general cuts, in fact to cut the graphs in all possible ways, and organize our inclusion/exclusion in terms of the number of connected components of the cut graph. Luckily, in this example we obtain the desired 2, as shown in Figure 12.

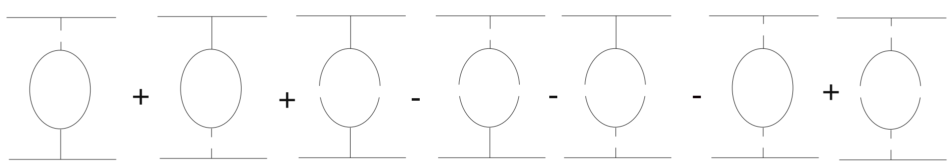
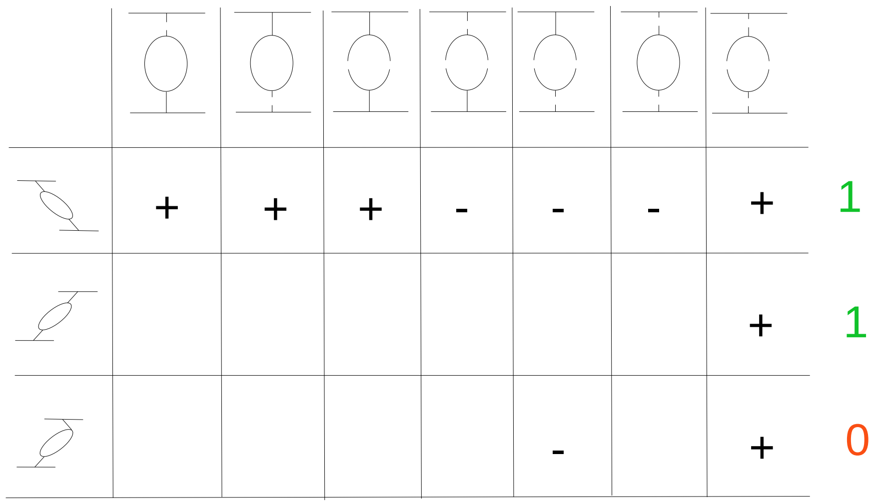


FIGURE 12. The general cuts of the egg graph.

$$\binom{4}{1}+\binom{4}{3}+\binom{4}{2}-\binom{4}{1,1,2}-\binom{4}{2,1,1}-\binom{4}{1,2,1}+\binom{4}{1,1,1,1}=2$$

In this cut/glue inclusion exclusion process one introduces a huge number of other graphs (with sinks/sources etc) whose contribution should clearly vanish. In Figure 8.2, we check that the contribution to the inclusion/exclusion process by each of the two contributing egg graphs is indeed 1 (corresponding to the number of vertex orderings of the "good eggs"), and the contribution for a "bad egg" (i.e. an egg with a sink) indeed vanishes.



Therefore, to prove Theorem 8.1, we first prove that our combinatorial recipe for the Gauss Manin connection proves a "heavy formula"', and then show via yet another inclusion/exclusion argument that the two formulas are equivalent.

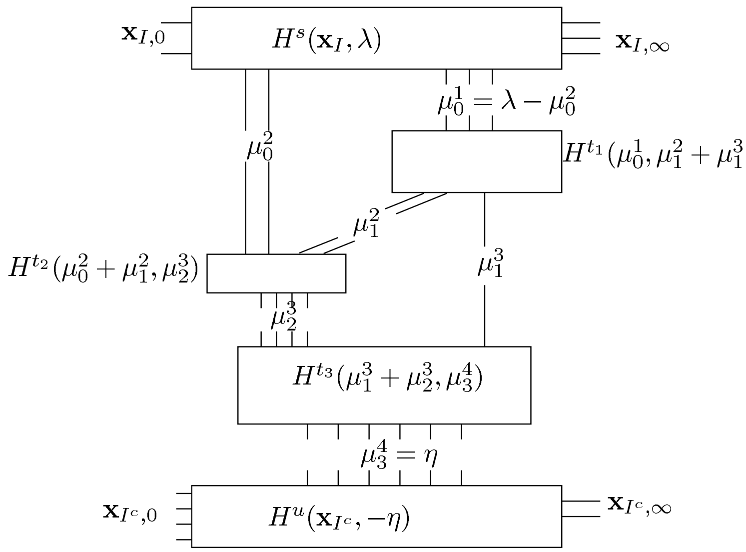
Theorem 8.3 (Heavy Formula)**.**

$$WC_I^r(\mathbf{x}) = \sum_{N=0}^{\infty} \sum_{\substack{s+(\sum_{j=1}^{N} t_j)+u=r \\ |\lambda|=|\eta|=d \\ data\ \ in\ \star}} (-1)^N \binom{r}{s, t_1, \ldots, t_N, u} \frac{\prod \mu_i^j}{\prod \ell(\mu_i^j)!} H^s(\mathbf{x}_I, \lambda) \left(\prod_{j=1}^{N} H^{t_j}(\star) \right) H^u(\mathbf{x}_{I^c}, -\eta) \tag{28}$$

The data denoted by $\star$ is illlustrated in Figure 13: it consists in disconnecting a graph with the right numerical invariants in all possible legal ways, where legal means that the graph obtained by shrinking all connected components to vertices and mantaining the cut edges as edges has no directed loops. The μ_i^j denote the partitions of weights of the edges connecting the i-th to the j-th connected component.

References

[AB84] Michael Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.


$\mathbf{x}_{I,0}$
$H^s(\mathbf{x}_I, \lambda)$
$\mathbf{x}_{I,\infty}$
$\mu_0^1 = \lambda - \mu_0^2$
μ_0^2
$H^{t_1}(\mu_0^1, \mu_1^2 + \mu_1^3)$
μ_1^2
$H^{t_2}(\mu_0^2 + \mu_1^2, \mu_2^3)$
μ_1^3
μ_2^3
$H^{t_3}(\mu_1^3 + \mu_2^3, \mu_3^4)$
$\mu_3^4 = \eta$
$\mathbf{x}_{I^c,0}$
$H^u(\mathbf{x}_{I^c}, -\eta)$
$\mathbf{x}_{I^c,\infty}$

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