Jacobians of quotients of Artin-Schreier curves

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Abstract. We investigate interesting properties of the Jacobian of a universal family of covers $\psi_C : Z_C \to \mathbb{P}^1_C$ over a scheme C in characteristic p. Since we construct this family as a quotient of certain Artin-Schreier curves, the endomorphism ring of the Jacobian contains many copies of a subfield of $\mathbb{Q}(\zeta_p)$ (which is totally ramified at p). We find the Newton polygon of any geometric fibre of the family using a result of Zhu.

1. Introduction

Arithmetic geometers have been greatly successful at understanding the structure of abelian varieties in characteristic p. For example, the possibilities for the endomorphism ring and the Newton polygon of a principally polarized abelian variety are known. Furthermore, there are many results on the geometry of the loci of these strata in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g.

These results provide a strong motivation to understand how the Torelli locus intersects such strata in $\mathcal{A}_g$. At this moment, there are many open questions on this subject. In particular, it is unknown which endomorphism rings and Newton polygons occur for Jacobians of curves in characteristic p. One indication of the difficulty of these questions is to notice the variety of techniques used in the pursuit of examples and partial results. This paper grew as an attempt to understand these questions and techniques in light of a new example.

In Section 2.1, we present a large dimensional family of curves Z_C whose Jacobians have p-rank 0. We show a large ring of endomorphisms acts upon the Jacobian of Z_C in Section 2.2. The main result is Theorem 2.5 which states that $K^r \hookrightarrow \mathrm{End}(\mathrm{Jac}(Z_C)) \otimes \mathbb{Q}$. Here K and r (defined precisely in Section 2.2) are respectively a subfield of $\mathbb{Q}(\zeta_p)$ and a divisor of $p^a - 1$. This result is interesting since by a pure dimension count the Torelli locus and the locus corresponding to abelian varieties with an action of K should not intersect at all in $\mathcal{A}_g$. It is worth noting that the field K is totally ramified above the prime p. This phenomenon occurs since the curves in the family Z_C are quotients of wildly ramified covers of the projective line. In Section 2.3, we compute the $\mathcal{O}_K$-module structure of

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$H^0(\mathrm{Jac}(Z_C), \Omega^1)$. We also use a result in [**Zhu**] to find the Newton polygon of any geometric fibre of Z_C when p satisfies a certain congruence condition.

As an application, in Section 3 we give some results for the Jacobian J of one fibre of the family Z_C. For this fibre, the action of the Cartier operator C is easy to describe. As a result, we find the $\mathcal{O}_K$-module structure of $H^1(J, \Omega^1)/CH^1(J, \Omega^1)$ which determines the a-number and the reduced a-numbers of J. Furthermore, we directly compute the Newton polygon for this fibre of the family. While the result on the Newton polygon follows directly from the fundamental work [**Kat81**] of Katz (or the recent work of Zhu), we hope that this discussion will illuminate some of the issues behind their more technical proofs.

It was a great opportunity to take part in the 2002 Barrett Lectures conference at the University of Tennessee at Knoxville and to hear the plenary lectures by Y. Ihara and Y. Kawamata. This paper fits into the general context of Ihara's work of determining the action of arithmetic Galois groups on covers of the projective line or on the associated Tate modules. To find the action of $\mathrm{Gal}(\mathbb{Q}_p)$, it is valuable to understand the Newton polygon and the endomorphisms of the Jacobians of the covers in question.

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2. Quotients of Artin-Schreier covers

In this section, we construct a universal family of covers $\psi_C : Z_C \to \mathbb{P}^1_C$ over a scheme C in characteristic p. Then we show that the Jacobians of the curves Z_C in this family have a large endomorphism ring.

2.1. Construction of the relative curve Z_C. Let k be an algebraically closed field of characteristic $p > 2$. Let $q = p^a$ and choose an isomorphism $(\mathbb{F}_q, +) \simeq (\mathbb{Z}/p)^a$. Fix $n, j \in \mathbb{N}$ such that $n|(p^a - 1)$ and $\gcd(j, pn) = 1$. Consider a semi-direct product $I \simeq (\mathbb{Z}/p)^a \rtimes \mu_n$ where the action of μ_n is irreducible.

Consider a cover $\phi : Y \to \mathbb{P}^1_k$ given by the equations $x_1^n = x$ and $y^{p^a} - y = x_1^j f(x^{-1})$ where $f(x^{-1}) \in k[x^{-1}]$ has degree less than j/n. We will call such a cover an *decomposable I-Galois cover with conductor j*. Here ϕ is an I-Galois cover with Galois action given by $\alpha(y, x_1) = (y + \alpha, x_1)$ for $\alpha \in \mathbb{F}_q$ and $h(y, x_1) = (h^j y, hx_1)$ for $h \in \mu_n$. The cover ϕ has branch locus $\{0, \infty\}$; over 0 it has ramification μ_n and over ∞ it is totally ramified. The filtration of higher ramification groups of ϕ satisfies $I_i \simeq (\mathbb{Z}/p)^a$ for $1 \leq i \leq j$ and $I_i = 0$ for $i > j$. By the Riemann-Hurwitz formula, the genus g of Y is $g = (p^a - 1)(j - 1)/2$. We assume throughout that $g \geq 2$.

The cover is called *decomposable* since the $(\mathbb{Z}/p)^a$-Galois subcover $\phi' : Y \to X \simeq \mathbb{P}^1_k$ of ϕ can be viewed as the fibre product of related $\mathbb{Z}/p$-Galois covers. More precisely, let L_1 denote a set of representatives of the $(q-1)/(p-1)$ orbits of $\mathbb{F}_q^*$ under the action of $\mathbb{F}_p^*$. For $\ell \in L_1$, let $\phi_\ell : Y_\ell \to X \simeq \mathbb{P}^1_k$ be the $\mathbb{Z}/p$-Galois Artin-Schreier cover $y^p - y = \ell x_1^j f(x^{-1})$. Then ϕ' dominates the cover ϕ_ℓ for $\ell \in L_1$. Thus it dominates the normalization of the fibre product over X of all ϕ_ℓ. Also the genus of Y and of the fibre product are the same. It follows that the normalization of the fibre product of $\{\phi_\ell |\ \ell \in L_1\}$ is isomorphic to ϕ'.

The following lemma indicates that isomorphism classes of decomposable I-Galois covers with conductor j are essentially determined by the choice of $f(x^{-1}) \in$

$k[x^{-1}]$. Furthermore, it states that there is a universal family of such covers whose dimension is $r(I,j)$ which is approximately j/n. Here $r(I,j)$ is the number of exponents $e \in \mathbb{N}$ satisfying: (i) $1 \leq e \leq j$; (ii) $e \equiv j \bmod n$; (iii) there is no $v \in \mathbb{N}^+$ so that $p^{av}e$ satisfies (i) and (ii). Condition (iii) is important for the following reason: if $g_1(x_1), g_2(x_1) \in k[x_1]$, by [**Pri02**, Lemma 2.1.5] the two covers given by equations $x_1^n = x$ and $y^{p^a} - y = g_i(x_1)$ are isomorphic if and only if $g_1(x_1) = zg_2(x_1) + \delta^{p^a} - \delta$ for some $z \in \mu_{p^a-1}$ and some $\delta \in k[x]$.

Consider the functor $F_{I,j}$ which associates to an irreducible k-scheme S the set of equivalence classes of decomposable I-Galois covers of $\mathbb{P}^1_S$ with conductor j. Here two such covers are *equivalent* if they are isomorphic after a purely inseparable pullback of S. For technical reasons, this functor will only be representable in a category where finite purely inseparable morphisms are invertible. Let $\tau : \phi \to \mathcal{M}_g$ be the forgetful morphism sending ϕ to the isomorphism class of the curve Y.

LEMMA 2.1. *There is a finite quotient $C = C(I,j)$ of $\mathbb{G}_m \times \mathbb{G}_a^{r(I,j)-1}$ by μ_{p^a-1} which is a fine moduli space for the functor $F_{I,j}$ of equivalence classes of decomposable I-Galois covers with conductor j. Thus there is a universal family $\phi_C : Y_C \to \mathbb{P}^1_C$ of equivalence classes of decomposable I-Galois curves with conductor j over $C(I,j)$. The image of ϕ_C under τ in $\mathcal{M}_g$ has dimension $r(I,j)-1$ if $n > 1$ and $r(I,j)-2$ if $n = 1$.*

PROOF. By [**Pri02**, Theorem 2.2.10], in the case $a = 1$, there is a fine moduli space $C = C(I,j)$ for the functor $F_{I,j}$ in a category where finite purely inseparable morphisms are invertible. Here C is constructed as a finite quotient $\mathbb{G}_m \times \mathbb{G}_a^{r(I,j)-1}$ by μ_{p-1}. More specifically: there is a morphism $T : \mathrm{Hom}(\circ, C) \to F_{I,j}(\circ)$ given by using the coordinates of T as the coefficients of the polynomial $f(x^{-1})$; the morphism T induces a bijection between the k-points of C and $F_{I,j}(\mathrm{Spec}(k))$; and if $\phi_S \in F_{I,j}(S)$ then there exists a finite radicial morphism $i : S' \to S$ and a unique morphism $f : S' \to C$ such that $T(f) = i^*\phi_S$ in $F_{I,j}(S')$. The universal family $\phi : Y \to \mathbb{P}^1_C$ is given by $T : \mathrm{Hom}(C,C) \to F_{I,j}(C)$. The proof for $a > 1$ is identical after replacing p with p^a throughout.

By [**Pri02**, Lemma 2.1.2], a subvariety of the family Y_C is isotrivial if and only if it is constant after an automorphism of the projective line which fixes 0 and ∞ if $n > 1$ (resp. fixes ∞ if $n = 1$). Thus the dimension of the image of Y under τ is one (resp. two) less than the dimension of C. □

Note for $n > 1$ that the automorphisms of the projective line which fix 0 and ∞ are only changes in scaling of the variable x, which changes the coefficients of $f(x^{-1})$. So there is a representative of each isomorphism class of the fibres of Y_C for which $f(x^{-1})$ is a monic polynomial. In other words, there is an action of $\mathbb{G}_m$ on C whose orbits correspond to isomorphism classes of the fibres of Y_C.

Suppose $\phi_C : Y_C \to \mathbb{P}^1_C$ is the universal decomposable I-Galois cover with conductor j as above. We will describe ϕ with the equations $x_1^n = x$ and $y^{p^a} - y = x_1^j f(x^{-1})$. Consider the action of a subgroup $H \simeq \mu_n \subset I$ of C-automorphisms on Y_C given by $x_1 \mapsto \zeta_n^i x_1$ and $y \mapsto \zeta_n^{ji} y$ for $0 \leq i \leq n-1$. The condition that $\gcd(j,n) = 1$ implies that H is not normal in I. Let Z_C be the quotient of the family Y_C by H. Then Z_C is a relative curve over C. Let $\psi_C : Z_C \to \mathbb{P}^1_C$ be the quotient cover which is a non-Galois degree p^a cover of $\mathbb{P}^1_C$ (branched over 0 and ∞).

LEMMA 2.2. *The relative curve Z_C has genus $g' = (q-1)(j-1)/2n$ and p-rank 0. The image of Z_C in $M_{g'}$ has dimension $r(I,j)-1$ (resp. $r(I,j)-2$ if $n=1$).*

PROOF. The subgroup H fixes exactly two points of Y_C, namely the point $\phi_C^{-1}(\infty)$ and the point $(x_1, y) = (0,0)$. One can show that $g' = \text{genus}(Z) = (q-1)(j-1)/2n$ by the Riemann-Hurwitz formula applied to the H-Galois cover $Y_C \to Z_C$.

Recall that $\phi'_C : Y_C \to \mathbb{P}^1_C$ is a p-group Galois cover branched at only one point. Thus the p-rank $\sigma(Y_C)$ of Y_C (and thus Z_C) is 0 by the Deuring-Shafarevich formula: $\sigma(Y_C) - 1 = |(\mathbb{Z}/p)^a|(\sigma(X_C)-1) + \Sigma(e_y - 1)$, [**Cre84**, Corollary 1.8].

The image of Y_C in $\mathcal{M}_g$ has dimension $r(I,j)-1$ (resp. $r(I,j)-2$ if $n=1$). For any irreducible k-scheme S, consider an S-point of C and the corresponding H-Galois cover of relative curves $Y_S \to Z_S$. If Y_S is isotrivial then Z_S is isotrivial since there are only finitely many choices for H in $\text{Aut}_S(Y_S)$, [**DM69**, Theorem 1.11]. If Z_S is isotrivial then Y_S is isotrivial since there are only finitely many μ_n-Galois covers of Z_S branched at two fixed points, [**Gro71**, Exposé X, Corollary 2.12]. So the dimension of the image of Z_C in $M_{g'}$ is the same as the dimension of Y_C in $\mathcal{M}_g$. □

2.2. Endomorphism ring of the Jacobian. In this section, we show that the endomorphism ring of the Jacobian of Z_C contains many copies of a subfield of $\mathbb{Q}(\zeta_p)$. In particular, this yields examples of Jacobians which have real multiplication by a totally real field in which the characteristic p ramifies completely. This section is inspired by Ellenberg's paper [**Ell01**] in which he uses I-Galois covers of curves in characteristic 0 to find Jacobians whose endomorphism ring contains a subfield of $\mathbb{Q}(\zeta_p)$.

Recall the definitions of p, q, n, j, I, H, Y_C, and Z_C from the previous section. Let $n_1 = \gcd(n, p-1)$. Let $K = \mathbb{Q}(\zeta_p^{(n_1)})$ be the subfield of $\mathbb{Q}(\zeta_p)$ which has degree $(p-1)/n_1$ over $\mathbb{Q}$. Note that over the prime p, the field K is totally ramified and has uniformizer $(\zeta_p - 1)^{n_1}$.

LEMMA 2.3. *Suppose $r = (q-1)/\text{lcm}(n, p-1)$ and $H\backslash I/H$ is the double coset space of H in I. There is an isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathbb{Q}[H\backslash I/H] \simeq \mathbb{Q} \oplus r\ \mathbb{Q}(\zeta_p^{(n_1)}).$$

PROOF. First, we show that the double cosets $H\backslash I/H$ can be identified with the orbits of $\mathbb{F}_q$ under the action of H. Suppose $g_1 = \alpha_1 h_1,\ g_2 = \alpha_2 h_2 \in I$ with $\alpha_i \in \mathbb{F}_q$ and $h_i \in H \simeq \mu_n$. Then $Hg_1H = Hg_2H$ if and only if $\alpha_1 = h(\alpha_2)$ for some $h \in H$. By the Perlis-Walker Theorem [**PW50**], $\mathbb{Q}[\mathbb{F}_q] = \mathbb{Q} \oplus (q-1)/(p-1)\ \mathbb{Q}(\zeta_p)$. If h is a generator of H, then h^{n/n_1} is the stablizer of $\mathbb{F}_p^* \subset \mathbb{F}_q$. So the double cosets other than H have order $(p-1)/n_1$ in $H\backslash I/H$. Also the number of non-trivial orbits of $\mathbb{F}_q$ under the action of H equals $(q-1)n_1/n(p-1)$, from which the result follows. □

Consider the irreducible representations of I. For each $c \in \mathbb{Z}/n$ there is a representation V_c of I of dimension one which factors through the map $\mathbb{Z}/n \to \mathbb{C}^*$ sending $1 \mapsto e^{2\pi i c/n}$. Let W be the direct sum of the $(q-1)/n$ irreducible representations of I of dimension n. Let W^H be the elements of W fixed by H. So W has dimension $q-1$ and W^H has dimension $(q-1)/n$.

The morphism $i : I \to \text{Aut}(Y_C)$ induces a morphism $i' : \mathbb{Z}[I] \to \text{End}(\text{Jac}(Y_C))$ and defines a morphism $\pi_H = \sum_{h \in H} i'(h) \in \text{End}(\text{Jac}(Y_C))$. Consider the natural action $\mathbb{Q}[H\backslash I/H] \to \operatorname{End}(\pi_H \operatorname{Jac}(Y_C))\otimes\mathbb{Q}$ and the isogeny $\operatorname{Jac}(Z_C) \to \pi_H \operatorname{Jac}(Y_C)$. Let $\ell \neq p$ be a prime. Then there is an action

$$\mathbb{Q}[H\backslash I/H] \to \operatorname{End}(\operatorname{Jac}(Z_C)) \otimes \mathbb{Q} \hookrightarrow \operatorname{End}(H_1(Z_C, \mathbb{Q}_\ell)).$$

It follows that $H_1(Z_C, \mathbb{Q}_\ell)$ is a $\mathbb{Q}[H\backslash I/H]$-module. Computing the image of $\mathbb{Q}[H\backslash I/H]$ in $\operatorname{End}(\operatorname{Jac}(Z_C))\otimes\mathbb{Q}$ is equivalent to computing the dimension of the representation of $\mathbb{Q}[H\backslash I/H]$ on $H_1(Z, \mathbb{Q}_\ell) \simeq \pi_H H^1(Y_C, \mathbb{Q}_\ell)$.

Proposition 2.4. *There is a $\mathbb{Q}[H\backslash I/H]$-module isomorphism $H_1(Z_C, \mathbb{Q}_\ell) \simeq (W^H)^{\oplus(j-1)}$.*

Proof. Consider the representation ρ_Y determined by the action of I on $H^1(Y, \mathbb{Q}_\ell)$. By the Lefschetz fixed point theorem [**Mil80**, V, Corollary 2.8],

$$\chi(\rho_Y) = 2\chi_{\mathrm{triv}} - 2\chi_1 + a_0 + a_\infty.$$

Here a_0 (respectively a_∞) is the character of the Artin representation attached to the branch point 0 (respectively ∞) and χ_1 (resp. $\chi(\rho_Y)$) is the character of the regular representation (resp. of ρ_Y).

Let $\chi(W)$ be the character associated to the representation W. Then $\chi(W)(g) = 0$ if $g \notin (\mathbb{Z}/p)^a$, $\chi(W)(g) = 0$ if $\mathrm{id} \neq g \in (\mathbb{Z}/p)^a$, and $\chi(W)(\mathrm{id}) = q - 1$. Also, $a_\infty(g) = -1$ if $g \notin (\mathbb{Z}/p)^a$, and $a_\infty(g) = -(j-1)$ if $\mathrm{id} \neq g \in (\mathbb{Z}/p)^a$, and $a_\infty(\mathrm{id}) = (p^a - 1)(j-1) + p^a(n-1)$. Also $a_0(g) = 0$ if $g \notin \mathbb{Z}/n$, $a_0(g) = -p^a$ if $\mathrm{id} \neq g \in \mathbb{Z}/n$, and $a_0(\mathrm{id}) = p^a(n-1)$.

Let $m = \langle \chi(W), \chi(\rho_Y)\rangle / \langle \chi(\rho_Y), \chi(\rho_Y)\rangle$. For this m and for some integer m_c for each $c \in \mathbb{Z}/n$, the character $\chi(\rho_Y)$ decomposes as

$$\chi(\rho_Y) = m\chi(W) + \sum_{c\in\mathbb{Z}/n} m_c \chi(V_c).$$

If $c \neq 0$ then only the trivial element of V_c is fixed by H. Thus $H_1(Z, \mathbb{Q}_\ell) \simeq (W^H)^{\oplus m}$.

We compute $m = j - 1$ by finding the inner product of $\chi(W)$ with the other characters:

$$\langle \chi(W), \chi(W)\rangle = p(p-1); \quad \langle \chi(W), \chi_{\mathrm{triv}}\rangle = 0; \quad \langle \chi(W), \chi_1\rangle = np(p-1);$$

$$\langle \chi(W), a_0\rangle = p(n-1)(p-1); \quad \langle \chi(W), a_\infty\rangle = p(j+n)(p-1).$$

Thus $m = \langle \chi(W), \chi(\rho_Y)\rangle / \langle \chi(W), \chi(W)\rangle = j - 1$. □

It follows that the endomorphism ring of $\operatorname{Jac}(Z_C)$ contains the direct sum of $(q-1)/\operatorname{lcm}(n, p-1)$ copies of the field K, which has degree $(p-1)/n_1$ over $\mathbb{Q}$ and is totally ramified over the prime p. More precisely:

Theorem 2.5. *If $r = (q-1)/\operatorname{lcm}(n, p-1)$, we see that $\oplus_r\ \mathbb{Q}(\zeta_p^{(n_1)}) \hookrightarrow \operatorname{End}(\operatorname{Jac}(Z_C)) \otimes \mathbb{Q}$.*

Proof. By Lemma 2.3 and Proposition 2.4, it is sufficient to show that the representation of $(q-1)/(p-1)\ \mathbb{Q}(\zeta_p)$ on W has dimension $q-1$. This follows since W is the regular representation of $\mathbb{F}_q$ viewed as an I-representation. □

2.3. Some corollaries. Let L denote a set of representatives of the $(q-1)/\mathrm{lcm}(n,p-1)$ non-trivial orbits of $\mathbb{F}_q$ under the action of H.

COROLLARY 2.6. *There is an isogeny* $\mathrm{Jac}(Z_C) \sim \times_{\ell \in L} J_\ell$. *Here* J_ℓ *is a Jacobian with dimension* $(p-1)(j-1)/2n_1$ *with* $\mathbb{Q}(\zeta_p^{(n_1)}) \hookrightarrow \mathrm{End}(J_\ell) \otimes \mathbb{Q}$. *Furthermore, if* $j=3$ *and* n_1 *is even (resp. if* $j=2$ *and* n_1 *is odd) then* J_ℓ *has real (resp. complex) multiplication.*

PROOF. Recall that the normalization of the fibre product of $\{\phi_\ell | \ \ell \in L_1\}$ is isomorphic to ϕ'_C. Here L_1 is a set of representatives of the $(q-1)/(p-1)$ orbits of $\mathbb{F}_q^*$ under the action of $\mathbb{F}_p^*$ and $\phi_\ell : Y_\ell \to \mathbb{P}^1_C$ is the $\mathbb{Z}/p$-Galois cover $y^p - y = \ell x_1^j f(x^{-1})$. The fact that there is an isogeny between $\mathrm{Jac}(Y_C)$ and $\prod_{\ell \in L_1} \mathrm{Jac}(Y_\ell)$ follows from [**KR89**, Theorem B] (or see [**vdG99**, Proposition 2.5]).

Since the action of $H = \langle h \rangle$ on $\mathbb{F}_q$ is irreducible, the orbits of H have order $(q-1)/\mathrm{lcm}(n,p-1)$ and the stabilizer of Y_ℓ is h^{n/n_1}. Define Z_ℓ to be the quotient of Y_ℓ by $\langle h^{n/n_1} \rangle$ and J_ℓ to be the Jacobian of Z_ℓ. After taking the quotient by H, it follows that $\mathrm{Jac}(Z_C) \sim \times_{\ell \in L} J_\ell$; that J_ℓ has dimension $(p-1)(j-1)/2n_1$; and that $\mathbb{Q}(\zeta_p^{(n_1)}) \hookrightarrow \mathrm{End}(J_\ell) \otimes \mathbb{Q}$.

The last statement is immediate since J_ℓ will have real (resp. complex) multiplication if its dimension $(j-1)(p-1)/2n_1$ is the same (resp. half) the degree $(p-1)/n_1$ of $\mathbb{Q}(\zeta_p^{(n_1)})$ (and n_1 is even (resp. odd)). □

REMARK 2.7. a. In the case that $j=2$, then the family J_ℓ of such curves with complex multiplication has dimension 0. In the case that $j=3$ and $n=2$ (resp. $n>2$) then the family of such curves with complex multiplication has dimension 1 (resp. 0).

b. This result gives some evidence for the remark of Van der Geer and Oort: "one expects excess intersection of the Torelli locus and the loci corresponding to abelian varieties with very large endomorphism rings; that is, one expects that they intersect much more than their dimensions suggest," [**vdGO99**, Section 5]. The Torelli locus has codimension $g(g+1)/2-(3g-3)$ in $\mathcal{A}_g$. For $a=1$ and n_1 even, the locus of abelian varieties of dimension $g=(p-1)(j-1)/2n_1$ with endomorphism ring containing K (which has degree $(p-1)/n_1$) has dimension $(j+1)g/4$. So by a pure dimension count, one would expect these loci not to intersect at all.

c. It is possible to produce other families of curves whose Jacobians have real multiplication. For example, consider a $\mathbb{Z}/p \rtimes \mathbb{Z}/2$-Galois cover $f: V \to \mathbb{P}^1$ whose branch locus consists of two totally ramified points each with conductor j. Then a similar proof shows that $\mathbb{Q}(\zeta_p^{(2)}) \hookrightarrow \mathrm{End}(\mathrm{Jac}(Z)) \otimes \mathbb{Q}$ where Z has genus $j(p-1)/2$. In particular, if $j=1$ then this family has real multiplication. However, it is less interesting since the p-rank of Z is $(p-1)/2$ and thus Z is ordinary.

COROLLARY 2.8. *Suppose* $p \equiv 1 \bmod j$. *The slopes of the Newton polygon* β *of any geometric fibre of the Jacobian* J_C *of the family* Z_C *are the values* $\{1/j, 2/j, \ldots (j-1)/j\}$ *each with multiplicity* $e=(q-1)/n$. *In other words,* $\beta = e(G_{1,j-1} + G_{2,j-2} + \ldots + G_{j-1,1})$.

PROOF. Since the Newton polygon is invariant under isogeny and $J \sim \times_{\ell \in L} J_\ell$ by Corollary 2.6, it is sufficient to show that the slopes of the Newton polygon of J_ℓ are equally distributed among the values $(j-i)/j$ for $1 \le i \le j-1$. By [**Zhu**, Corollary 1.4], the slopes of the Newton polygon of the Jacobian of Y_ℓ are the values

$(j-i)/j$ for $1 \leq i \leq j-1$ each with multiplicity $p-1$. The result follows by taking the quotient by the action of H. □

Now we will find the $\mathcal{O}_K$-module structure of $H^0(J_\ell, \Omega^1)$.

LEMMA 2.9. *A basis for $H^0(J_\ell, \Omega^1)$ is given by: $\{y^r x^b dx\}$ for $0 \leq r \leq p-2$, $0 \leq b \leq j-2$, $rj + bp \leq pj - j - p - 1$, and $jr + b \equiv -1 \bmod n_1$.*

PROOF. Let Q be the point of Y_ℓ above $x = \infty$. Note that $(x)_\infty = pQ$, and $(y)_\infty = jQ$, and $(dx) = (2g-2)Q$. So $\mathrm{div}(y^r x^b dx) = r(y)_0 + b(x)_0 + (2g - 2 - (rj + pb))Q$. So the divisors $y^r x^b dx$ for $\mathrm{Jac}(Y_\ell)$ are holomorphic when $rj + pb \leq (2g-2) = pj - j - p - 1$ and are independent for r and b as above.

Recall that $J_\ell = \mathrm{Jac}(Z_\ell)$ where Z_ℓ is the quotient of Y_ℓ by $H_1 \simeq \mu_{n_1}$. Now we find the H_1-invariant subspace of $H^0(\mathrm{Jac}(Y_\ell), \Omega^1)$. The generator ζ_{n_1} of μ_{n_1} acts as follows: $x \mapsto \zeta_{n_1} x$, $y \mapsto \zeta_{n_1}^j y$, $dx \mapsto \zeta_{n_1} dx$. So $y^r x^b dx$ is fixed under the action of μ_{n_1} if and only if $jr + b \equiv -1 \bmod n_1$. □

For $1 \leq i \leq j-1$, let V_i be the subspace of $H^0(J_\ell, \Omega^1)$ generated by the basis elements $\{y^r x^b dx\}$ for which $b = j - 1 - i$. Let $e_i = \#\{r |\ 0 \leq r \leq (pi - j - 1)/j,\ j(r+1) \equiv i \bmod n_1\}$. Note that the dimension of V_i is e_i. When $p \equiv 1 \bmod j$, these numbers simplify to: $r \leq -1 + (j - b - 1)(p-1)/j = -1 + i(p-1)/j$ and $e_i = i(p-1)/n_1 j$. Here $n_1 j | (p-1)$ since $\gcd(n, j) = 1$. Let π be a uniformizer for the completion of $\mathcal{O}_K$ above p.

COROLLARY 2.10. *There exists an isomorphism of $\mathcal{O}_K$-modules as follows:*

$$H^0(J_\ell, \Omega^1) \simeq \sum_{i=1}^{j-1} k[\pi]/\pi^{e_i};$$

Note that $H^0(J_\ell, \Omega^1)$ is not a free $\mathcal{O}_K$-module for $j \geq 3$. Thus J_ℓ does not satisfy the Rapoport condition for $j \geq 3$. Also this implies that the canonical principal polarization on J_ℓ is incompatible with the action of $\mathcal{O}_K$.

PROOF. We can choose $\pi = (\zeta_p - 1)^{n_1}$ to be a uniformizer for the completion of $\mathcal{O}_K$ above p. For the first statement, we need to determine the action of π on $H^0(J_\ell, \Omega^1)$. Note that $\zeta_p - 1$ corresponds to the action of the automorphism $\alpha - \mathrm{id}$ on the function field of Y_ℓ where α is the automorphism of order p acting on Y_ℓ via $\alpha(y) = y + 1$, $\alpha(x) = x$. Thus $\zeta_p - 1$ acts by taking $y^r x^b dx \mapsto ((y+1)^r - y^r) x^b dx$. The action of π corresponds to the automorphism $(\alpha - \mathrm{id})^{n_1}$ on the function field of $Z_\ell = Y_\ell / H_1$. In particular, the $j-1$ subspaces V_i are the invariant subspaces under the action of π. The value $e_i = \dim(V_i)$ is the power of the uniformizer π which annihilates V_i. □

3. Example

Recall that k is an algebraically closed field of characteristic p, that $n | (p^a - 1)$, and j is a positive integer such that $\gcd(j, pn) = 1$. For the rest of the paper, we suppose that $p \equiv 1 \bmod j$ which implies $\mu_j \subset \mathbb{F}_p^*$. Let Y be the fibre of the decomposable I-Galois family with equations $x_1^n = x$ and $y^{p^a} - y = x_1^j$ (i.e. $f(x^{-1}) = 1$). In this section we restrict to the fibre Z of Z_C, which is the quotient of this fibre Y by the subgroup $H \simeq \mu_n$.

Under these strong conditions, the Jacobian J of Z is easy to investigate since $H^0(J_\ell, \Omega^1)$ decomposes (as an $\mathcal{O}_K$-module) into blocks fixed by the action of the Cartier operator. Here J_ℓ is one of the factors of $J \sim \times_{\ell \in L} J_\ell$ and $\mathcal{O}_K$ is the integral closure of $K = \mathbb{Q}(\zeta_p^{(n_1)})$. We find the $\mathcal{O}_K$-module structure of $H^0(J_\ell, \Omega^1)/CH^0(J_\ell, \Omega^1)$. This allows us to easily compute the Newton polygon of J and the reduced a-numbers of J_ℓ. We show also how the computation of the Newton polygon follows from Katz's fundamental work [**Kat81**].

3.1. Reduced a-numbers. In this section, we describe the action of the Cartier operator C on $H^0(J_\ell, \Omega^1)$ and use this to find the $\mathcal{O}_K$-module structure of $H^0(J_\ell, \Omega^1)/CH^0(J_\ell, \Omega^1)$. Recall that the dimension of J_ℓ equals $(j-1)(p-1)/2$, that the p-rank of J_ℓ is 0, and that J_ℓ is isomorphic to $\mathrm{Jac}(Z_\ell)$ where Z_ℓ is the quotient of Y_ℓ by the action of $H_1 \simeq \mu_{n_1}$ (and $n_1 = \gcd(n, p-1)$).

For b such that $0 \leq b \leq j-2$, let $h_b = (b+1)(p-1)/j$. Note that $n | h_b$ and that h_b is the unique integer in $[0, p-1]$ such that $h_b \equiv (-1-b)j^{-1} \bmod p$.

Lemma 3.1. *The Cartier operator acts on $H^0(J_\ell, \Omega^1)$ as follows: $C(y^r x^b dx) = 0$ if $r < h_b$ and $C(y^r x^b dx)$ is in the subspace generated by $y^{r-h_b} x^b dx$ if $r \geq h_b$.*

Proof. Consider the action of the Cartier operator on $y^r x^b dx$.

$$C(y^r x^b dx) = C((y^p - x^j)^r x^b dx) = \sum_{h=0}^{r} C((-1)^h \binom{r}{h} y^{p(r-h)} x^{jh} x^b dx).$$

Thus $C(y^r x^b dx) = \sum_{h=0}^{r} c_{rh} y^{r-h} C(x^{jh+b} dx)$ where $c_{rh} = (-1)^h \binom{r}{h}$ (note $c_{rh} \in \mathbb{F}_p$). Recall that $C(x^{jh+b} dx) = 0$ unless $jh + b \equiv -1 \bmod p$. Since $0 \leq h \leq r \leq p-2$, this is possible if and only if $h = h_b$. Thus $C(x^{jh+b}) \neq 0$ if and only if $h = h_b$. Thus, $C(y^r x^b dx) \neq 0$ if and only if $r \geq h_b$.

When $r \geq h_b$ then $C(y^r x^b dx) = c_{rh} y^{r-h_b} x^{\lfloor (h_b j + b)/p \rfloor} dx$. Also $(h_b j + b)/p = ((b+1)(p-1)+b)/p = b+1-1/p$ so $\lfloor (hj+b)/p \rfloor = b$. So $C(y^r x^b dx) = c_{rh} y^{r-h_b} x^b dx$. □

Proposition 3.2. *Write $a_i = \min\{i, j-i\}(p-1)/n_1 j$ for $1 \leq i \leq j-1$. There exists an isomorphism of $\mathcal{O}_K$-modules as follows:*

$$H^0(J_\ell, \Omega^1)/CH^0(J_\ell, \Omega^1) \simeq \sum_{i=1}^{j-1} k[\pi]/\pi^{a_i}.$$

The numbers a_i are the *reduced a-numbers* of J_ℓ.

Proof. Recall that $C(y^r x^b dx) = 0$ if $r < h_b$ and $C(y^r x^b dx)$ is in the subspace generated by $y^{r-h_b} x^b dx$ if $r \geq h_j$. Thus the subspaces V_i are invariant under the action of the Cartier operator. Also $\dim(CV_i) = \max\{\dim(V_i) - (j-i)(p-1)/jn_1, 0\}$. As in the proof of Corollary 2.10, the $j-1$ subspaces V_i are the invariant subspaces under the action of π. The value a_i is the power of the uniformizer π which anniliates V_i/CV_i. Since $\dim(V_i) = i(p-1)/n_1 j$, we see that $a_i = \dim(V_i/CV_i) = \min\{i, j-i\}(p-1)/n_1 j$. □

Recall that the *a-number* of an abelian variety A over an algebraically closed field is $\dim_k \mathrm{Hom}(\alpha_p, A)$ where α_p is the group scheme which is the kernel of Frobenius $\mathrm{Fr} : \mathbb{G}_a \to \mathbb{G}_a$ (written $\alpha_p = \mathrm{Spec}(k[x]/x^p)$). By duality, the a-number equals the dimension of the kernel of the Cartier operator on $H^0(J_\ell, \Omega^1)$. The next corollary shows that the a-number of J is approximately half of its genus.

COROLLARY 3.3. *Suppose $p \equiv 1 \bmod j$. The quotient curve Z_ℓ has a-number $(p-1)j/4n_1$ if j is even and $(p-1)(j-1)(j+1)/4jn_1$ if j is odd.*

PROOF. This is immediate from Proposition 3.2 since the a-number is the sum of the reduced a-numbers or $\sum_{i=1}^{j-1}\min\{i, j-i\}(p-1)/jn_1$; this is because it is the dimension of the kernel of the Cartier operator on $H^0(J_\ell, \Omega^1)$. □

REMARK 3.4. Suppose $p \not\equiv 1 \bmod j$. Define h_b to be the unique integer in $[0, p-1]$ such that $h_b \equiv (-1-b)j^{-1} \bmod p$. Then $C(y^r x^b dx) = 0$ if $r < h_b$ and $C(y^r x^b dx) = c_{rh} y^{r-h_{jb}} x^{\lfloor (hj+b)/p \rfloor} dx$ if $r \geq h_{jb}$. So when $p \not\equiv 1 \bmod j$, the proof of Lemma 3.1 gives the following expression for the a-number α_ℓ of the Jacobian J_ℓ in terms of h_b:

$$\alpha_\ell = \sum_{b=0}^{j-2} \min(h_b, p - \lceil (p+1+bp)/j \rceil).$$

This follows since the dimension of the kernel of the Cartier operator on $H^0(J_\ell, \Omega^1)$ is:

$$\#\{(r,b) | 0 \leq r < h_b,\ 0 \leq b,\ rj + bp \leq jp - j - p - 1\}.$$

For example, if $p \equiv -1 \bmod j$ then $h_b = p-(b+1)(p-1)/j$. Thus $C(y^r x^b dx) = 0$ for all $y^r x^b dx \in H^0(J_\ell, \Omega^1)$. Thus $\alpha_\ell = (j-1)(p-1)/2$ and Z is supersingular. When $n_1 = 1$, this is one direction of the main result of [**IS91**]. We note in passing that for $j = q+1$, the curve Y is isomorphic over $\mathbb{F}_{q^2}$ to the maximal curve studied by [**FGT97**]. In other words, the number of $\mathbb{F}_{q^2}$-points of Y is equal to the Hasse-Weil bound.

3.2. Newton polygons. By Corollary 2.8, the slopes of the Newton polygon β of the Jacobian J_C of Z_C are the values in the set $\{1/j, 2/j, \ldots (j-1)/j\}$ each with multiplicity $e = (q-1)/n$. In other words, $\beta = e(G_{1,j-1} + G_{2,j-2} + \ldots G_{j-1,1})$. In this section, we give two explicit proofs of this result for the Jacobian J of the fibre Z of the family Z_C. The first proof considers the action of the Cartier operator on $H^0(J_\ell, \Omega^1)$ and uses Katz's sharp slope estimate. These are also ingredients of Zhu's proof, which is more complicated since the subspaces V_i are not invariant under the Cartier operator and since it is necessary to lift to characteristic 0. The second proof uses a result of Katz, in which he relates the eigenvalues of Frobenius on this fibre to certain Gauss sums. After these proofs we find some lower bounds for the dimension of the intersection of the Torelli locus with certain Newton polygon strata in $\mathcal{A}_g$ (the moduli space of principally polarized abelian varieties of dimension g).

PROOF. (First proof of Corollary 2.8 for the Jacobian J of the fibre Z): Since the Newton polygon is invariant under isogeny and $J \sim \times_{\ell \in L} J_\ell$ by Corollary 2.6, it is sufficient to show that the slopes of the Newton polygon of J_ℓ are equally distributed among the values $(j-i)/j$ for $1 \leq i \leq j-1$. Recall that $\mathrm{End}(J_\ell) \otimes \mathbb{Q}$ contains the field $K = \mathbb{Q}(\zeta_p^{(n_1)})$ and that $\pi = (\zeta_p - 1)^{n_1}$ is a uniformizer for the completion of K over the prime p.

The subspace V_i of $H^0(Z, \Omega^1)$ is invariant under the action of the Cartier operator. Let $h_i = (j-i)(p-1)/j$. For all $(j-1)/2 \leq i \leq j-1$, we see that h_i/n is the largest exponent ϵ such that $C(V_i) \subset \pi^\epsilon V_i$. This follows since the Cartier operator acts on $y^r x^{j-1-i} dx$ by reducing the exponent r by h_i, by Lemma 3.1. The uniformizer π of $\mathcal{O}_K$ acts by reducing the exponent r by n. Also the Cartier operator has coefficients in $\mathbb{F}_p^*$ since $c_{rh} \in \mathbb{F}_p^*$.

Thus the eigenvalues for C on V_i are divisible exactly by $\pi^{h_i/n}$ for all $(j-1)/2 \leq i \leq j-1$. We see that $\pi^{h_i/n}$ has valuation $v_i = h_i/ne_1 = (j-i)/j$. Thus p^{v_i} exactly divides the Verschiebung endomorphism V on V_i for $(j-1)/2 \leq i \leq j-1$. We can show this implies that $p^{\lceil nv_i \rceil} | V^{n+g}$. By Katz's sharp slope estimate [**Kat79**, Theorem 1.5], the first slope of the Newton polygon on V_i is exactly $(j-i)/j$.

The proof follows from the following two statements. For each slope λ which occurs in the Newton polygon, the slope $1-\lambda$ must occur by duality. Also, (as in [**Yu**, Lemma 3.2]), the slopes must occur with multiplicity at least $e_1 = (p-1)/n_1$ since p is totally ramified in $\mathcal{O}_K$. □

Now here is the proof using Katz's work on eigenvalues of Frobenius and Gauss sums. Note that there is an action of μ_j on Y_ℓ whose generator takes $(x_1, y_\ell) \mapsto (\zeta_j x_1, y_\ell)$.

Proof. (Second proof of Corollary 2.8 for the Jacobian J of the fibre Z): It is sufficient to show that the slopes of the Newton polygon of the curve $Y_\ell : y^p - y = x_1^j$ are equally distributed (with multiplicity $p-1$) among the values $(j-i)/j$ for $1 \leq i \leq j-1$. The general case follows by Corollary 2.6 and by considering the quotient by the action of the subgroup H.

By [**Kat81**, Corollary 2.2], $H^1(Y_\ell) = \oplus (H^1)^{(\psi,\chi)}$. Here the sum is over all pairs (ψ, χ) where ψ is a nontrivial additive character of $\mathbb{F}_p$ and χ is a nontrivial multiplicative character of μ_j. The statement holds for l-adic or crystalline cohomology groups. Furthermore, for each of the $(j-1)(p-1)$ pairs (ψ, χ), the eigenspace $H^1(Y_\ell)^{(\psi,\chi)}$ has dimension 1. Finally, on $H^1(Y_\ell)^{(\psi,\chi)}$ the eigenvalue of Frobenius is the Gauss sum $-g(\psi, \chi, P)$.

There are $\varphi(m)$ non-trivial characters χ of μ_j having order $m|j$. The choice of such a character is equivalent to a choice of prime P of $\mathbb{Q}(\zeta_m)$ containing p. We identify the numbers $1 \leq t < m$ with $\gcd(t, m) = 1$ with elements $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. By the Stickelberger relation, [**IR90**, 14, Theorem 2], $g(\psi, \chi, P)^m = P^{\sum t\sigma_t}$ where the sum is over $1 \leq t < m$ with $\gcd(t, m) = 1$. Thus the valuations v_t of $g(\psi, \chi, P)$ for the $\varphi(m)$ characters with order m are the numbers t/m. The result follows since the number of eigenvalues with slope t/m is the number of non-trivial additive characters ψ of $\mathbb{F}_q$, namely $p-1$. □

Remark 3.5. a. In fact, a third proof of Corollary 2.8 for this fibre should be possible along the following lines. The Artin-Schreier curve $y^p - y = x^j$ is a quotient of the Fermat curve $w^{j(p-1)} + v^{j(p-1)} = 1$, [**Kat81**, pg. 228]. In [**Yui80**, Theorem 2.8], Yui determined the Newton polygon of the Fermat curve. The result would follow by finding which slopes of the Fermat curve match the appropriate characters from the Artin-Schreier curve.

b. We quickly summarize the case when $j = 3$ and $n_1 = \gcd(p-1, n)$ is even. Then J is isogenous to a direct sum of isomorphic Jacobians which have real multiplication by a totally ramified field, $\mathbb{Q}(\zeta_p^{(n_1)})$, by 2.6. Thus the Newton polygon of J can have at most two slopes. If $p \equiv 2 \bmod 3$ then the a-number of Z is $(p-1)/n$, by 3.3. So J is supersingular and its Newton polygon has slopes $1/2$. If $p \equiv 1 \bmod 3$ then the a-number of Z is $2(p-1)/3n$. By Corollary 2.8, its Newton polygon has two slopes: $1/3$ and $2/3$.

c. For Newton polygons of hyperelliptic curves, see [**Yui78**]. For supersingular curves in characteristic 2, see [**vdGvdV92**].

Fix $e \in \mathbb{N}^+$ and $j \in \mathbb{N}^+$ with $j|e$; let $g = e(j-1)/2$. Let P_{je} be the infinite set of all primes p for which $p \equiv 1 \bmod j$ and for which the quotient $n = (p-1)/e$ is relatively prime to j. This set of primes has Dirichlet density $\varphi(j)/\varphi(ej)$ since it consists of all primes $p \equiv n_0 e + 1 \bmod je$ where n_0 is a unit modulo j; (for these p, one can always choose $a = 1$).

Let $\beta = \beta_{ej}$ be the Newton polygon with slopes $1/j, 2/j, \ldots (j-1)/j$ each repeated e times. In other words, $\beta = e(G_{1,j-1} + G_{2,j-2} + \ldots G_{j-1,1})$. Let NP_β be the locus in $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties, with Newton polygon β. Note that $NP_\beta = \emptyset$ without the hypotheses on j and g.

Corollary 3.6. *The intersection in $\mathcal{A}_g$ of the Torelli locus with NP_β has dimension at least $r(I,j) - 2$ for all primes $p \in P_{je}$.*

Here $r(I,j)$ is approximately $2g/(p-1)$.

Proof. This follows immediately from Corollary 2.8 and Lemma 2.2. □

In some special cases, we can give an even better lower bound for the dimension of the intersection of the Newton polygon strata with the Torelli locus.

Corollary 3.7. *Given a genus g, a prime p, and a Newton polygon stratum β in the following table, the dimension d_β of the intersection of the Newton polygon stratum with the Torelli locus is at least the given number.*

g	β	p	$d_\beta \geq$
4	$4G_{1,1}$	$p \equiv 3, 5 \bmod 8$	3
5	$5G_{1,1}$	$p \equiv 11 \bmod 20$	3
6	$6G_{1,1}$	$p \equiv 13 \bmod 24$	3
6	$2G_{1,2} + 2G_{2,1}$	$p \equiv 7, 13 \bmod 18$	6
6	$G_{1,3} + 2G_{1,1} + G_{3,1}$	$p \equiv 5 \bmod 8$	7
7	$7G_{1,1}$	$p \equiv 11, 15, 23 \bmod 28$	2
8	$8G_{1,1}$	$p \equiv 3, 5, 7, 9, 11, 13 \bmod 16$	1
9	$3G_{1,2} + 3G_{2,1}$	$p \equiv 4, 7, 10, 13, 16, 19, 22, 25 \bmod 27$	6
10	$G_{1,4} + G_{2,3} + G_{3,2} + G_{4,1}$	$p \equiv 6, 11, 16, 21 \bmod 25$	9
12	$4G_{1,2} + 4G_{2,1}$	$p \equiv 7, 13 \bmod 18$	3
15	$G_{1,5} + 2G_{1,3} + 3G_{1,1} + 2G_{2,3} + G_{5,1}$	$p \equiv 7, 31 \bmod 36$	8
21	$G_{1,6} + G_{2,5} + G_{3,4} + G_{4,3} + G_{5,2} + G_{6,1}$	$p \equiv 8, 15, 22, 29, 36, 43 \bmod 49$	6

Proof. Suppose β is one of the Newton polygons above with corresponding genus g. Let j be the reciprocal of the first slope of β. Let $e = 2g/(j-1)$. For the primes above, $p \equiv 1 \bmod j$ and an explicit calculation shows that there exists an a so that $n = (p^a - 1)/e$ is relatively prime to j. By Corollary 2.8, the intersection of β with the Torelli locus is non-empty.

Let ℓ_β be the number of Newton polygons α for which $\beta > \alpha \geq gG_{0,1} + gG_{1,0}$. In fact, ℓ_β is the number of lattice points beneath β with x-coordinate $1 \leq x \leq g$. Let d_β be the dimension of the intersection of the Newton polygon strata β with the Torelli locus. By purity of the Newton polygon stratification [**dJO00**], $\dim(\mathcal{M}_g) \leq d_\beta + \ell_\beta$. The proof follows by an explicit calculation of ℓ_β in the cases above, which are the only ones for which $\dim(\mathcal{M}_g) - \ell_\beta$ is positive. □

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