Questions on p-torsion of hyperelliptic curves

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1 Introduction

We describe geometric questions raised by recent work on the p-torsion of Jacobians of curves defined over an algebraically closed field k of characteristic p. These questions involve invariants of the p-torsion such as the p-rank or a-number. Such invariants are well-understood and have been used to define stratifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. A major open problem is to understand how the Torelli locus intersects such strata in $\mathcal{A}_g$. In [3], we show that some of these strata intersect the image of the hyperelliptic locus under the Torelli map. This work relies upon geometric results on the configurations of branch points for non-ordinary hyperelliptic curves and raises some new geometric questions.

2 Notation

Let k be an algebraically closed field of characteristic p. Consider the moduli space $\mathcal{M}_g$ (resp. $\mathcal{H}_g$) of smooth (resp. hyperelliptic) curves of genus g.

The group scheme $\mu_p = \mu_{p,k}$ is the kernel of Frobenius on $\mathbb{G}_m$, so $\mu_p \simeq \text{Spec}(k[x]/(x-1)^p)$. If $\text{Jac}(X)$ is the Jacobian of a k-curve X, the *p-rank*, $\dim_{\mathbb{F}_p} \text{Hom}(\mu_p, \text{Jac}(X))$, of X is an integer between 0 and g. A curve of genus g is said to be *ordinary* if it has p-rank equal to g. In other words, X is ordinary if $\text{Jac}(X)[p] \cong (\mathbb{Z}/p \oplus \mu_p)^g$. Let $V_{g,f}$ denote the sublocus of curves of genus g with p-rank at most f. For every g and every $0 \leq f \leq g$, the locus $V_{g,f}$ has codimension $g-f$ in $\overline{\mathcal{M}}_g$, [2].

The group scheme $\alpha_p = \alpha_{p,k}$ is the kernel of Frobenius on $\mathbb{G}_a$, so $\alpha_p \simeq \text{Spec}(k[x]/x^p)$. The *a-number*, $\dim_k \text{Hom}(\alpha_p, \text{Jac}(X))$, of X is an integer between 0 and g. A generic curve has a-number equal to zero. A supersingular elliptic curve E has a-number equal to one. In this case there is a non-split exact sequence $0 \to \alpha_p \to E[p] \to \alpha_p \to 0$. There is a unique isomorphism type of group scheme for the p-torsion of a supersingular elliptic

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curve, which we denote M. Let $T_{g,a}$ denote the sublocus of curves of genus g with a-number at least a.

Let N be the group scheme corresponding to the p-torsion of a supersingular abelian surface which is not superspecial. By [4, Example A.3.15], there is a filtration $H_1 \subset H_2 \subset N$ where $H_1 \simeq \alpha_p$, $H_2/H_1 \simeq \alpha_p \oplus \alpha_p$ and $N/H_2 \simeq \alpha_p$. Moreover, the kernel G_1 of Frobenius and the kernel G_2 of Verschiebung are contained in H_2 and there is an exact sequence $0 \to H_1 \to G_1 \oplus G_2 \to H_2 \to 0$. Finally, let Q be the group scheme corresponding to the p-torsion of an abelian variety of dimension three with p-rank 0 and a-number 1.

These group schemes can be described in terms of their covariant Dieudonné modules. Consider the non-commutative ring $E = W(k)[F,V]$ with the Frobenius automorphism $\sigma : W(k) \to W(k)$ and the relations $FV = VF = p$ and $F\lambda = \lambda^\sigma F$ and $\lambda V = V\lambda^\sigma$ for all $\lambda \in W(k)$. Recall that there is an equivalence of categories between finite commutative group schemes $\mathbb{G}$ over k (with order p^r) and finite left E-modules $D(\mathbb{G})$ (having length r as a $W(k)$-module). By [4, Example A.5.1-5.4], $D(\mu_p) = k[F,V]/k(V, 1-F)$, $D(\alpha_p) = k[F,V]/k(F,V)$, and $D(N) = k[F,V]/k(F^3, V^3, F^2 - V^2)$. One can also show that $D(Q) = k[F,V]/k(F^4, V^4, F^3 - V^3)$.

The p-rank of a curve X with $\mathrm{Jac}(X)[p] \simeq N$ is zero. To see this, note that $\mathrm{Hom}(\mu_p, N) = 0$ or that F and V are both nilpotent on $D(N)$. The a-number of a curve X with $\mathrm{Jac}(X)[p] \simeq N$ is one since $N[F] \cap N[V] = H_1 \simeq \alpha_p$.

3 Results

Here are the results from [3] on the p-torsion of hyperelliptic curves.

Theorem 3.1. *For all $0 \leq f \leq g$, the locus $V_{g,f} \cap \mathcal{H}_g$ is non-empty of dimension $g-1+f$. In particular, there exists a smooth hyperelliptic curve of genus g and p-rank f.*

The proof follows from the fact that $V_{g,0} \cap \mathcal{H}_g$ is non-empty [2], the purity result of [1], and a dimension count at the boundary of $\mathcal{H}_g$.

For the rest of the paper, suppose $p > 2$. We consider the sublocus $\mathcal{H}_{g,n}$ of the moduli space $\mathcal{M}_g$ consisting of smooth curves of genus g which admit an action by $(\mathbb{Z}/2\mathbb{Z})^n$ so that the quotient is the projective line. We analyze the curves in the locus $\mathcal{H}_{g,n}$ in terms of fibre products of hyperelliptic curves. We extend results of Kani and Rosen [6] to compare the p-torsion of the Jacobian of a curve X in $\mathcal{H}_{g,n}$ to the p-torsion of the Jacobians of its $\mathbb{Z}/2\mathbb{Z}$-quotients (up to isomorphism rather than up to isogeny).

This approach allows us to produce families of Jacobians of (non-hyperelliptic) curves whose p-torsion contains interesting group schemes. The difficulty lies in controling the p-torsion of all of the hyperelliptic quotients of X. This reduces the study of $\mathrm{Jac}(X)[p]$ to the study of the intersection of some subvarieties in the configuration space of branch points. For example, for Corollaries 3.3 and 3.5, we study the geometry of the subvariety defined by Yui corresponding to the branch loci of non-ordinary hyperelliptic curves, [8]. Similarly, we use this method to show that $T_{g,a} \cap \mathcal{M}_g$ is non-empty under certain conditions on g and a.

In special cases, these families of curves intersect $\mathcal{H}_g$. This leads to the following partial results on the existence of hyperelliptic curves with interesting types of p-torsion.

Corollary 3.2. *Let N be the p-torsion of a supersingular abelian surface which is not superspecial. For all $g \geq 2$, there exists a smooth hyperelliptic curve X of genus g so that* $\mathrm{Jac}(X)[p]$ *contains N.*

Corollary 3.2 is proved inductively starting with a curve X of genus 2 with $\mathrm{Jac}(X)[p] = N$. In fact, we expect the p-torsion of the generic point of $V_{g,g-2} \cap \mathcal{H}_g$ (which has dimension $2g-3$) to have group scheme $N \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$. We explain in [3] how this would follow from an affirmative answer to Question 4.2.

Corollary 3.3. *Suppose $g \geq 2$ and $p \geq 5$. There exists a dimension $g-2$ family of smooth hyperelliptic curves of genus g whose fibres have p-torsion $M^2 \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$ (and thus have a-number equal to 2).*

In fact, we expect $T_{g,2} \cap V_{g,g-2} \cap \mathcal{H}_g$ to have dimension $2g-4$.

Corollary 3.4. *Let Q be the p-torsion of an abelian variety of dimension three with p-rank 0 and a-number 1. Suppose $g \geq 3$ is not a power of two. Then there exists a smooth hyperelliptic curve X of genus g so that* $\mathrm{Jac}(X)[p]$ *contains Q.*

Corollary 3.4 is proved inductively starting from the supersingular hyperelliptic curve X of genus 3 and a-number 1 (and thus $\mathrm{Jac}(X)[p] = Q$) from [7]. An affirmative answer for $g = 4$ in Question 4.4 would allow us to remove the restriction on g in Corollary 3.4. We expect the generic point of $V_{g,g-3} \cap \mathcal{H}_g$ (which has dimension $2g-4$) to have group scheme $Q \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-3}$.

Corollary 3.5. *Suppose $g \geq 5$ is odd and $p \geq 7$. There exists a dimension $(g-5)/2$ family of smooth hyperelliptic curves of genus g whose fibres have p-torsion containing M^3 (and thus a-number at least 3).*

To determine the precise form of the group scheme in Corollary 3.5, one could consider Question 4.3. We expect $T_{g,3} \cap V_{g,g-3} \cap \mathcal{H}_g$ to have dimension $2g-7$.

4 Questions

These results raise the following geometric questions. First, the expectations in the preceeding section all rest on the assumption that natural loci such as $\mathcal{H}_g$, $V_{g,f}$ and $T_{g,a}$ should intersect as transversally as possible. This tranversality can be measured both in terms of dimension and tangency. The meaning behind Theorem 3.1 is that the intersection of $\mathcal{H}_g$ and $V_{g,f}$ at least has the appropriate dimension. We could further ask this question.

Question 4.1. Is $V_{g,f} \cap \mathcal{H}_g$ reduced?

This relates to the question of whether $\mathcal{H}_g$ and $V_{g,f}$ are transversal in the strict geometric sense. Our proof of Corollaries 3.3 and 3.5 required us to show that $V_{g,g-1} \cap \mathcal{H}_g$ is not completely non-reduced.

An affirmative answer to the next question would imply by our fibre product construction that for all $g \geq 4$ there exists a smooth hyperelliptic curve X with $\mathrm{Jac}(X)[p] \simeq N \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$. This would then be the p-torsion of the generic point of $V_{g,g-2} \cap \mathcal{H}_g$.

Question 4.2. Given an arbitrary hyperelliptic cover $C \to \mathbb{P}^1_k$, is it possible to deform C to an ordinary hyperelliptic curve by moving only one of the branch points?

The answer to Question 4.2 will be affirmative if the hypersurface studied by [8] does not contain a line parallel to a coordinate axis. To rephase this, consider the branch locus $\Lambda = \{\lambda_1, \dots, \lambda_{2g}\}$ of an arbitrary hyperelliptic curve of genus $g-1$. Does there exist $\mu \in \mathbb{A}^1_k - \Lambda$ so that the hyperelliptic curve branched at $\{\lambda_1, \dots, \lambda_{2g}, \infty, \mu\}$ is ordinary? For a generic choice of Λ, the answer to this question is yes, but this is not helpful for interesting applications.

One would like to strengthen Corollary 3.5 to state that there are hyperelliptic curves of genus g with a-number exactly three. This raises a question which we state here in its simplest case. Recall that λ is supersingular if the elliptic curve branched at $\{0,1,\infty,\lambda\}$ is supersingular. There are $(p-1)/2$ supersingular values of λ by [5].

Question 4.3. Which of the group schemes $(\mathbb{Z}/p \oplus \mu_p)^2$, $M \oplus (\mathbb{Z}/p \oplus \mu_p)$, N, or M^2 occur as the p-torsion of the hyperelliptic curve branched at $\{0,1,\infty,\lambda_1,\lambda_2,\lambda_3\}$ when $\lambda_1,\lambda_2,\lambda_3$ are distinct supersingular values?

We expect that for all p there exist distinct supersingular values $\lambda_1,\lambda_2,\lambda_3$, so that the hyperelliptic curve branched at $\{0,1,\infty,\lambda_1,\lambda_2,\lambda_3\}$ is ordinary. This has been verified by C. Ritzenthaler for $7 \leq p < 100$. An affirmitive answer for any p implies that there exists a smooth hyperelliptic curve of genus 5 in characteristic p, with p-rank 2 and a-number 3. For multiple values of p, it appears that there do not exist distinct supersingular values $\lambda_1,\lambda_2,\lambda_3$, so that the hyperelliptic curve branched at $\{0,1,\infty,\lambda_1,\lambda_2,\lambda_3\}$ has p-rank 0.

Question 4.4. Does there exist a smooth hyperelliptic curve X of genus 4 (resp. 5) so that $\mathrm{Jac}(X)[p] = Q \oplus (\mathbb{Z}/p \oplus \mu_p)$ (resp. $Q \oplus (\mathbb{Z}/p \oplus \mu_p)^2$)?

One would guess that the answer to Question 4.4 is yes, but there does not seem to be much data. If the answers to Questions 4.2 and 4.4 are both affirmative, then the generic point of $V_{g,g-3} \cap \mathcal{H}_g$ has p-torsion of the form $Q \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-3}$ for all $g \geq 3$.

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