Construction of Covers with Formal and Rigid Geometry

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This paper gives a brief introduction to the construction of Galois covers of curves with formal and rigid patching. The first section states the GAGA principle and Grothendieck's Existence Theorem and proves a "Van Kampen" patching statement in the formal topology. These results are used in the second section to prove the existence of covers of a curve defined over a discrete valuation ring and its fraction field. The third section contains a stronger patching theorem which remains valid for arbitrary coherent modules. The fourth section includes applications of this theorem to Galois covers of curves and their inertia groups. Many of the results of this paper can be proven with either rigid or formal geometry. The discussion of Abhyankar's Conjecture is left to later lectures. Appropriate references will be given for complete versions of proofs.

1 Algebraization Theorems

In order to motivate the proofs using formal and rigid patching, one can first consider the classical case. Let X, Y be projective varieties over $\mathbb{C}$ and denote the corresponding complex analytic spaces by X^{an}, Y^{an}.

Theorem 1.1 *GAGA principle:* [Se2] (Prop. 15, Thm. 1-3) . *Every analytic map, $X^{an} \to Y^{an}$, is algebraic. Furthermore, every coherent analytic sheaf over X^{an} is algebraic, every analytic morphism between such sheaves is algebraic, and the analytic cohomology groups correspond to the algebraic ones.*

Given a complex curve, X, the GAGA principle can be used to show that every finite group is the Galois group of a branched cover of X. In particular, one can prove the Riemann Existence Theorem by constructing an analytic cover, $Y \to X$, from given inertia data and patching data. Since the sheaf of holomorphic functions on Y comes from a sheaf over $\mathcal{O}_X$, one can use GAGA to show that the cover is in fact algebraic. The next result generalizes this principle to formal schemes.

Notation 1.2 Let G be a finite group. Let R be a complete discrete valuation ring, with maximal ideal m, residue field $k = R/m$, and fraction field $K = \text{Frac}(R)$. If X is a scheme over R, assume that X is flat and of finite type over R. Let X_k denote the special fibre and X_K the generic fibre of X. A scheme X is a *curve* over R if it is of relative dimension 1 over R.

A finite morphism, $f : Y \to X$, where X is an integral scheme, is a (possibly branched) *cover* if Y is generically separable over X. A *G-Galois cover* is a cover, $f : Y \to X$, together with a group homomorphism, $G \to \text{Aut}_X(Y)$, such that G acts simply transitively on a generic geometric fibre (again allowing branching). If $f : Y \to X$ is a G-Galois cover and $G \subset G'$, define the *induced cover*, $\text{Ind}_G^{G'}(Y) \to X$, to be the disconnected G'-Galois cover consisting of $(G' : G)$ copies of Y indexed by the left cosets of G with the induced action of G'.

Denote the formal completion of X along X_k by $\hat{X}$. If F is a coherent sheaf on X, let $\hat{F}$ be its formal completion which is a formal coherent sheaf on $\hat{X}$.

Theorem 1.3 *Grothendieck's Existence Theorem:* [Gr] (Thm 5.1.6). *Let X be a proper scheme over a complete discrete valuation ring, R. There is an equivalence of categories from the category of coherent sheaves on X to the category of formal coherent sheaves on $\hat{X}$, taking $F \mapsto \hat{F}$.*

The following corollary of Grothendieck's Existence Theorem will be useful in patching covers as in the proof of Riemann's Existence Theorem. Let X be a connected scheme over R. Suppose U_k, V_k are Zariski open connected subsets of X_k such that $X_k = U_k \cup V_k$. If $U_k = X_k - D_k$ then there exists a closed subscheme $D \subset X$ whose closed fibre is D_k. Thus there exist open subsets $U, V \subset X$ whose closed fibres are U_k, V_k respectively. Let $\hat{U}$ and $\hat{V}$ be the formal completions of U and V along their closed fibres. For a scheme S, let $\mathcal{M}(S)$ denote the category of coherent $\mathcal{O}_S$-modules, and let $\mathcal{GA}(S)$ denote the category of coherent $\mathcal{O}_S$-algebras which are G-Galois.

Corollary 1.4 *Let X be a proper scheme over a complete discrete valuation ring, R. The functor*

$$\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(\hat{U}) \times_{\mathcal{M}(\hat{U} \times_X \hat{V})} \mathcal{M}(\hat{V})$$

is an equivalence of categories, as is the corresponding functor for $\mathcal{GA}$.

Proof. Given a pair of coherent modules over $\hat{U}$ and $\hat{V}$, along with an isomorphism between their pullbacks to $\hat{U} \times_X \hat{V}$, one finds a coherent sheaf of modules over $\hat{X}$ by gluing in the Zariski topology on formal schemes. By Grothendieck's Existence Theorem, this yields a unique coherent sheaf of modules over X. The equivalence between the categories of sheaves of modules and algebras allows one to extend this result to the category of G-Galois algebras ([H3] Cor. to Prop. 3). □

Another application of Grothendieck's Existence Theorem to fundamental groups is the following "Van Kampen" theorem which is written for the formal topology without the choice of a base point.

Definition 1.5 Given homomorphisms of profinite groups $\psi_1 : A \to G_1$ and $\psi_2 : A \to G_2$, define the *amalgamated product*, $G_1 *_A G_2$, to be the quotient of the free product of G_1 and G_2 (as profinite groups) by the closure of the normal subgroup generated by $\psi_1(a)\psi_2(a)^{-1}$ for all $a \in A$.

Theorem 1.6 *There is an isomorphism $\pi_1(\hat{U}) *_{\pi_1(\hat{U} \times_X \hat{V})} \pi_1(\hat{V}) \xrightarrow{\sim} \pi_1(X)$.*

Proof. The existence of the map follows from the universal property of the amalgamated product. Note that a G-Galois cover, $U' \to \hat{U}$, corresponds to an $\mathcal{O}_{\hat{U}}$-algebra with an action of G. Thus covers, $U' \to \hat{U}$ and $V' \to \hat{V}$, along with an isomorphism between their pullbacks to $\hat{U} \times_X \hat{V}$, determine a coherent sheaf of G-algebras over $\hat{X}$. By Corollary 1.4, this is given by a unique coherent sheaf of G-algebras over X. This, in turn, determines a G-Galois cover of X. Thus there is a bijection between the finite quotients of the two fundamental groups. The fact that this bijection is compatible with the inverse system of covers proves that the map between fundamental groups is an isomorphism. □

Remark 1.7 The bijection between finite quotients of fundamental groups in Theorem 1.6 will be used in Section 2 in the following way. Let G be a finite group generated by subgroups H_1 and H_2. Suppose there exists an H_1-Galois cover $U'' \to \hat{U}$ and an H_2-Galois cover $V'' \to \hat{V}$. Consider the induced G-Galois covers $U' = \mathrm{Ind}_{H_1}^G(U'') \to \hat{U}$ and $V' = \mathrm{Ind}_{H_2}^G(V'') \to \hat{V}$ and suppose that there is an isomorphism between the pullbacks of these covers to $\hat{U} \times_X \hat{V}$. Then by Theorem 1.6, the given information yields a G-Galois cover, $X' \to X$. Furthermore, if U'' and V'' are connected then so is X'.

There are also rigid versions of the above results. See [R] (3.1) for the connection between the formal and rigid viewpoints. In particular, the following rigid analogue of GAGA can be used to show that covers of rigid analytic spaces are algebraic. If F is a coherent sheaf on X_K, let F^{an} be the associated rigid coherent sheaf on the associated rigid analytic space, X_K^{an} [T].

Theorem 1.8 *Rigid GAGA:* [Ki]. *Let X be a proper scheme over a field, K, which is complete with respect to a non-archimedean valuation. There is an equivalence of categories from the category of coherent sheaves on X_K and coherent analytic sheaves on X_K^{an}, taking $F \mapsto F^{an}$.*

2 Mock Covers and Covers of a p-adic Curve

This section contains the proof that every finite group is the Galois group of an irreducible branched cover of a given curve, X, over a complete discrete valuation ring, R. The plan is to deform reducible (mock) covers of a curve defined over k to irreducible covers over R, by patching together local deformations with Grothendieck's Existence Theorem. As a corollary, one can prove that G is the Galois group of a branched cover of X_K. By taking $R = \mathbb{Z}_p$ or $R = k[[x]]$, one finds covers of X defined over $K = \mathbb{Q}_p$ or $K = k((x))$ respectively.

Definition 2.1 Let X be an irreducible scheme. A finite cover, $f : Y \to X$, is *mock* if the restriction of f to each irreducible component of Y is an isomorphism onto X.

Theorem 2.2 *Let G be a finite group. Let R be a complete discrete valuation ring. Then there exists an irreducible branched G-Galois cover, $\phi : Y \to X_R$, such that the closed fibre is a mock cover.*

Proof. [H2] (Thm 2.3). The first step is to prove the theorem in the case that G is a cyclic group of order $q = p^n$. Choosing a map $X \to \mathbb{P}^1$ and taking fibre products reduces to the case that $X = \mathbb{P}^1_R$. If k contains a primitive qth root of unity, this case is straight forward using Kummer theory. For example, if $R = k[[t]]$, the cover given by $y^q = x^{q-1}(x - t)$ over $\mathrm{Spec}(R)$ satisfies the requirements. If $\mu_q \notin k$, then one constructs a cover over $K(\zeta_q)$ given by $y^q = f(x)$ which can descend to a q-cyclic Galois cover of K as in [Sl]. If p=char(K), one may instead use Witt–Artin–Schreier theory.

For the general case, one can assume by induction that G is generated by two elements, $g_i \in G$ whose orders, q_i, are prime powers ($i = 1, 2$). Let $H_i = \langle g_i \rangle$. One can construct H_i-Galois covers of X as described above such that the branch loci, $B_i \subset X_R$, are disjoint. Inducing these covers from H_i to G, one finds disconnected G-Galois covers, $Y_i \to X_R$ ($i = 1, 2$). Let $U_k = X_k - B_1$, $V_k = X_k - B_2$, and $W_k = U_k \cap V_k$.

The closed fibre of each of the covers, $Y_i \to X_R$, is mock so the covers $Y_{i,k}$ are both trivial over W_k and thus isomorphic. Let $U, V, W \subset X$ be open subsets whose closed fibres are U_k, V_k, W_k respectively. Since the deformation of any étale cover is unique, the restrictions of the covers Y_i to $\hat{U}$ and $\hat{V}$ must be isomorphic over $\hat{W}$. As in Theorem 1.6, the covers Y_i yield a coherent formal sheaf of G-algebras over the formal completion of X_R. By Grothendieck's Existence Theorem, this sheaf is induced by a unique coherent sheaf of G-algebras on X_R. Let $\phi : Y \to X_R$ be the cover determined by this sheaf.

Since the g_i generate G, the closed fibre Y_k is connected. The curve Y is connected since Y_k is connected and X_R is proper over R. Finally, Y is irreducible because each Y_i is locally irreducible over B_i by construction. □

Remark 2.3 Note that all covers of X_R constructed by means of this theorem are reducible on the closed fibre and so have bad reduction. In fact, a given branched cover, $Y_K \to X_K$, has an unramified rational point if and only if there exists a model for the cover over R whose closed fibre is mock.

Corollary 2.4 *Let G be a finite group. Let K be the fraction field of a complete discrete valuation ring. Then there exists an irreducible branched G-Galois cover, $Y \to X_K$.*

Proof. [H2] (Cor 2.6). The result follows immediately by considering the generic fibre of the cover of X_R which was constructed in Theorem 2.2. It is also possible to prove this result directly using rigid patching [Li]. One begins as in the proof of Theorem 2.2 by constructing q-cyclic covers of X_K with an unramified rational point. By induction it is sufficient to prove the corollary is true for G assuming it is true for subgroups H_1 and H_2 which generate G. The given H_i-Galois covers induce G-Galois covers of rigid analytic disks which are trivial over the overlap. Using gluing of rigid analytic spaces [B], one finds an irreducible branched G-Galois cover of X_K. Finally, the rigid analogue of the GAGA principle proves that this cover is algebraic. See also [MM] for more details. □

This shows the existence of G-Galois covers of X_K for fields K such as $\mathbb{Q}_p$ or $k((x))$, although it does not give much control over the branch locus. (See [P2] and [Le] for generalizations of these theorems). Similarly, the following corollary is weaker than Abhyankar's Conjecture since it involves an unspecified branch locus. However, it follows directly from Theorem 2.2.

Corollary 2.5 *Let G be a finite group. Let k be an algebraically closed field (or large field* [P1]*). Then there exists an irreducible branched G-Galois cover, $Y \to X_k$.*

Proof. Taking $R = k[[t]]$, there exists an irreducible G-Galois cover, $Y \to X_R$, by Theorem 2.2. This cover descends to a flat family of covers of X_V parametrized by an algebraic variety, V, of finite type over k. The locus of points of V for which the fibre is degenerate is Zariski closed. Since k is algebraically closed (or large), there exists a k-point, $v \in V$, which is not in this set. The fibre over this point gives the desired cover, $Y_v \to X_k$. □

One can see that it is straightforward to get results about Galois branched covers of X using Grothendieck's Existence Theorem. The disadvantage, however, is that the covers must be isomorphic over Zariski open neighborhoods; in this case, the covers were mock on the close fibre and so were trivial over a Zariski open. On the other hand, there are a wealth of examples of non-trivial irreducible Galois covers constructed through methods such as [Se1]. The techniques of the next section will generalize the above mock cover construction to enable one to modify non-trivial covers in order to find others with different Galois group or inertia group.

3 Stronger Patching Results

The goal of this section is to prove a stronger patching theorem which generalizes Corollary 1.4 and Grothendieck's Existence Theorem for curves and extends patching results found in [H3], [HS]. (The main improvement of this theorem is that one can drop the hypothesis that the modules are projective and that there is more flexibility with the choice of ring R. This result was known to several people but has not yet appeared in print). It is first necessary to prove a variant of a result of [FR] and [A] which will apply in the case that X is a proper curve over an Artin local ring.

Notation 3.1 Let X be a curve, i. e. a noetherian, 1-dimensional scheme. Let $Z \subset X$ be a finite set of closed points. Let $U = X - Z$. For simplicity, assume U is affine (in fact, U is automatically quasi-compact which is sufficient). If ξ is a closed point of X, let $\hat{\mathcal{O}}_\xi$ be the complete local ring of functions of X at ξ. Let $X_\xi = \operatorname{Spec}(\hat{\mathcal{O}}_\xi)$. Let K_ξ be the total ring of fractions of $\hat{\mathcal{O}}_\xi$. Now let $X' = \coprod_{\xi \in Z} X_\xi$ and $U' = \coprod_{\xi \in Z} \operatorname{Spec}(K_\xi)$. Note $U' = U \times_X X'$. Let $i : X' \to X$ and $i' : U' \to U$ be the natural inclusions. For a scheme S, let $\mathcal{M}(S)$ denote the category of coherent $\mathcal{O}_S$-modules and let $\mathcal{GA}(S)$ denote the category of coherent $\mathcal{O}_S$-algebras which are G-Galois.

Theorem 3.2 *With the notation of 3.1, the following functor is an equivalence of categories:*

$$\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(U) \times_{\mathcal{M}(U')} \mathcal{M}(X').$$

Proof. If X is affine, the equivalence of categories is a special case of [FR] (Prop. 4.2) or [A] (Thm. 2.6). (In particular, [FR] states that

$$\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(U) \times_{\mathcal{M}(U' \amalg U)} \mathcal{M}(X' \amalg U)$$

is an equivalence of categories, from which the result follows for X affine).

Now for the general case, consider a triple $(F_{X'}, F_U, u)$ where $F_{X'} \in \mathcal{M}(X')$ and $F_U \in \mathcal{M}(U)$ and $u : i'^* F_U \to F_{X'}|_{U'}$ is an isomorphism of coherent $\mathcal{O}_{U'}$-modules. It is necessary to show that there exists a coherent $\mathcal{O}_X$-module, $F_X \in \mathcal{M}(X)$, unique up to isomorphism, such that $F_X|_U \simeq F_U$ and $i^* F_X \simeq F_{X'}$, compatibly with the isomorphism u. To do so, choose an open affine subset $V \subset X$ such that $U \cup V = X$. (For example, $V = X - S$ where $S \subset U$ is a finite set of closed points, with at least one point from each component of X). Consider $F_{U\cap V} = F_U|_{U\cap V}$ and the isomorphism $u' : i'^* F_{U\cap V} \to F_{X'}|_{U'}$ of coherent $\mathcal{O}_{U'}$-modules induced from u. Using the result for the affine case, the functor

$$\mathcal{M}(V) \xrightarrow{\sim} \mathcal{M}(U \cap V) \times_{\mathcal{M}(U')} \mathcal{M}(X')$$

is an equivalence of categories. It follows that the data $(F_{X'}, F_{U\cap V}, u')$ yields a coherent module, $F_V \subset \mathcal{M}(V)$, unique up to isomorphism. Furthermore, $F_V|_{U\cap V} \simeq F_{U\cap V} \simeq F_U|_{U\cap V}$ and $i^* F_V \simeq F_{X'}$ compatibly with u'. Since U, V are Zariski open subsets of X with $U \cup V = X$, there exists a coherent module $F_X \in \mathcal{M}(X)$ which is unique up to isomorphism and is compatible with F_U, $F_{X'}$, and u as required. Notice that this theorem applies in the case that X is a proper curve over an Artin local ring. □

Notation 3.3 Let X be a proper curve over a complete discrete valuation ring, R, which has uniformizer π. As in 3.1, let $Z \subset X$ be a finite set of closed points and let $U_k = X_k - Z$. Let $X' = \coprod_{\xi \in Z} \operatorname{Spec}(\hat{\mathcal{O}}_{X,\xi})$. Choose $U \subset X$ open such that the closed fibre of U is U_k. Let $U' = U \times_X X'$. Let $\hat{U}$, $\hat{U}'$, $\hat{X}'$ be the formal completions of U, U', X' along their closed fibres (note that these do not depend on the choice of U). One can think of $\hat{U}'$ as the overlap of $\hat{X}'$ and $\hat{U}$; in particular, note that $\hat{U}'_k = \hat{X}'_k - Z$. If S is a scheme over R and $n \geq 1$ is an integer, let S_n be the reduction of S modulo π^n. Note that $\hat{S}_n = S_n$.

Theorem 3.4 *With the above notation, the functor*

$$\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(\hat{U}) \times_{\mathcal{M}(\hat{U}')} \mathcal{M}(\hat{X}')$$

is an equivalence of categories, as is the corresponding functor for $\mathcal{GA}$.

Proof. Since X_n is a proper curve over the Artin local ring, R/π^n, for all $n \geq 1$, it follows from Theorem 3.2 that:

$$\mathcal{M}(X_n) \xrightarrow{\sim} \mathcal{M}(U_n) \times_{\mathcal{M}(U'_n)} \mathcal{M}(X'_n)$$

is an equivalence of categories. The inverse system of these is compatible since the constructions in Theorem 3.2 are unique up to isomorphism. Taking the inverse limit shows that:

$$\mathcal{M}(\hat{X}) \xrightarrow{\sim} \mathcal{M}(\hat{U}) \times_{\mathcal{M}(\hat{U}')} \mathcal{M}(\hat{X}').$$

By Grothendieck's Existence Theorem, there is an equivalence, $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(\hat{X})$, which proves the first statement of the theorem. The result for $\mathcal{GA}(X)$ follows from the equivalence of categories between coherent modules and algebras as in Corollary 1.4. □

The next section includes more refined theorems about fundamental groups of curves. Recall that Theorem 3.4 will be applicable in this situation since a G-Galois cover of a curve, X, corresponds to a coherent sheaf of G-Galois $\mathcal{O}_X$-algebras in $\mathcal{GA}(X)$. The general strategy will be to modify pre-existing covers by patching covers of curves which are isomorphic only over the fraction field of a complete local ring.

4 Applications

One of the most significant applications of formal and rigid patching is the proof of Abhyankar's Conjecture ([H4] [R]) which is discussed in this volume by Saïdi and Chambert-Loir. The first application here is to the problem of enlarging inertia groups. The choice of a base point will be dropped from the notation of fundamental groups. In this section, let k be an algebraically closed field of characteristic $p > 0$. If ξ is a closed point of X_k, let $\hat{\mathcal{O}}_\xi$ be the complete local ring of functions of X_k at ξ. Let $X_\xi = \mathrm{Spec}(\hat{\mathcal{O}}_\xi)$. Let $K_\xi = \mathrm{Frac}(\hat{\mathcal{O}}_\xi)$.

Theorem 4.1 *Let G be a finite quotient of $\pi_1(\mathbb{P}^1_k - \{\xi\})$. Then there exists an irreducible regular G-Galois cover of $\mathbb{P}^1_k$ branched only over ξ, whose inertia groups are the Sylow-p subgroups of G.*

Proof. [H3] (Thm 2). (The full result is much stronger; also see [HS]). By hypothesis, there exists an irreducible, regular G-Galois cover, $f_0 : Y_0 \to \mathbb{P}^1_k$. Its inertia group at a point $\eta_0 \in f_0^{-1}(\xi)$ is of the form $I = P \rtimes C_m$ with $(m,p) = 1$ and P a p-group [Se3]. Consider a C_m-cyclic cover, $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$, such that g is totally ramified over ξ, ξ' and $g(\xi) = \xi$. The normalization of the pullback of the cover, f_0, by g is a cover, $f : Y_k \to \mathbb{P}^1_k$. The curve, Y_k, is irreducible since g and f_0 are linearly disjoint. By Abhyankar's Lemma, the cover, $f : Y_k \to \mathbb{P}^1_k$, is étale over $U_k = \mathbb{P}^1_k - \{\xi\}$ and the inertia group at a point $\eta \in f^{-1}(\xi)$ is of the form $I = P$. Let S be a Sylow-p subgroup containing P.

Consider the induced disconnected S-Galois cover, $f_\xi : Y_\xi = \mathrm{Spec}(\mathrm{Ind}_P^S(\hat{\mathcal{O}}_{Y_k,\eta})) \to X_\xi$. To apply Theorem 3.4, let $R = k[[t]]$, $X = \mathbb{P}^1_R$, $X' = X_\xi \times_k R$, $U = U_k \times_k R$, and $U' = U \times_X X'$. There exists a deformation of this cover to an irreducible S-Galois cover, $f'_\xi : Y'_\xi \to X'$. One proof of this is as follows: f_ξ determines a point, μ, in the fine moduli space, $\mathcal{M}^{\mathrm{loc}}$, of S-Galois covers of $\mathrm{Spec}(K_\xi)$ [H1]; a morphism of $\mathrm{Spec}(R)$ into $\mathcal{M}^{\mathrm{loc}}$ which maps the point $(t = 0)$ to μ and maps the generic point to an irreducible cover determines f'_ξ. There also exists a trivial deformation, $f_U : Y_U \to U$, of the restriction of f_k over U_k. Since this trivial deformation is unique, the pullbacks of the covers f'_ξ and f_U must be isomorphic over U'. Thus these covers can be patched together using Theorem 3.4 to yield an irreducible regular G-Galois cover of X_R. This cover descends to a flat family of covers of $\mathbb{P}^1_V$ parametrized by an algebraic variety, V, of finite type over k. Since k is algebraically closed, one can specialize this cover to the fibre over an appropriate k-point of V to find the desired cover whose inertia groups are the conjugates of S. □

Question 4.2 What are the possible inertia groups for a G-Galois branched cover of a curve, $X_k \not\simeq \mathbb{P}^1_k$, with given group G and branch locus?

The following stronger theorem is an important step in the proof of the Abhyankar Conjecture and can be proven using Theorem 3.4 (where X is chosen so that the generic fibre is a projective line and the special fibre consists of two projective lines intersecting transversally).

Theorem 4.3 ([R] Thm. 2.2.3). *Let G be a finite group and Q a p-subgroup of G. Let G_1 and G_2 be subgroups of G and suppose that $G = \langle G_1, G_2, Q\rangle$. For $i = 1, 2$, let Q_i be a subgroup of $G_i \cap Q$ and suppose that there exists a connected étale G_i-Galois cover of $\mathbb{P}^1_k - \{\infty\}$ such that Q_i is one of the inertia groups above ∞. Then there exists a connected étale G-Galois cover of $\mathbb{P}^1_k - \{\infty\}$ such that Q is one of the inertia groups above ∞.*

Example 4.4 Let G be the group $\mathrm{SL}_2(\mathbb{F}_q)$. Assume $p|(q-1)$ if p is odd and $p^2|(q-1)$ if $p = 2$. Consider a Borel subgroup $B = T \ltimes U$ of G where U is a radical unipotent group invariant under the action of a maximal torus, T. Let Q be a Sylow-p subgroup of T. By the assumption on p, $Q \ltimes U$ is a quasi-p group. Since $Q \ltimes U$ is solvable, it is a quotient of $\pi_1(\mathbb{P}^1_k - \{\infty\})$ [Se1]. By Theorem 4.1, there exists a connected étale $Q \ltimes U$-Galois cover of $\mathbb{P}^1_k - \{\infty\}$ with inertia Q above ∞. Choose two opposite unipotents, U^+ and U^- and let $G_1 = Q \ltimes U^+$ and $G_2 = Q \ltimes U^-$. Since G is generated by G_1 and G_2, one can apply Theorem 4.3 to show that there exists a connected étale G-Galois cover of $\mathbb{P}^1_k - \{\infty\}$ whose inertia group over ∞ is Q.

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