Pries: 460 Information and Coding Theory:
Sample Quiz 3, 2015.

New Reed-Solomon code: Let α be a generator of $\mathbb{F}_q^*$. Choose $2 \leq t \leq q-1$.
Let $g(x) = (x-\alpha)\cdots(x-\alpha^{t-1})$.
Given $\vec{c} = (c_0, \ldots, c_{n-1}) \in \mathbb{F}_q^n$, let $p_{\vec{c}}(x) = c_0 + c_1 x + \cdots c_{n-1}x^{n-1} \in \mathbb{F}_q[x]$.

$$nRS(t,q) = \{\vec{c} \in \mathbb{F}_q^n \mid p_{\vec{c}}(x) = g(x)h(x) \text{ for some } h(x) \in \mathbb{F}_q[x] \text{ with } \deg(h(x)) \leq q-1-t\}.$$

1. Find a polynomial of degree 4 in $(\mathbb{Z}/3)[x]$ that factors but has no root.

2. Suppose α is a root of the irreducible polynomial $f(x) = x^4 + x + 1 \in \mathbb{Z}/2[x]$. Find the multiplicative inverse of α in $(\mathbb{Z}/2)[x]/(f(x))$.

3. Using the generator 3 for $\mathbb{F}_7^*$, find the generator matrix of $nRS(3,7)$.

4. When $t = q-1$, prove that the generator polynomial of $nRS(t,q)$ is

$$g(x) = 1 + x + x^2 + \cdots + x^{q-2}.$$

5. For a general choice of q and t, what is the dimension k of $nRS(t,q)$? Explain your answer.

6. Prove that the minimal distance of $nRS(t,q)$ is $d = t$.

7. Fix a polynomial $g(x) \in (\mathbb{Z}/p)[x]$ with degree d such that $d < n$. Let $C \subset (\mathbb{Z}/p)^n$ be the set of vectors $(c_0, \ldots, c_{n-1})$ such $c_0 + c_1 x + \cdots c_{n-1}x^{n-1}$ is a multiple of $g(x)$. Write down the definition of a cyclic code. Under what condition on $g(x)$ is C a cyclic code? Explain your answer.

8. Mirage codes: Let α be a root of a monic irreducible polynomial $p(x) \in \mathbb{Z}/2[x]$ of degree n. Fix an isomorphism

$$I : (\mathbb{Z}/2)^n \to \mathbb{F}_{2^n},\ (a_0, \ldots, a_{n-1}) \mapsto a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}.$$

   For $\beta \in \mathbb{F}_{2^n}$, define

$$C_\beta = \{\vec{c} = (\vec{v}, \vec{\beta v}) \mid \vec{v} \in (\mathbb{Z}/2)^n\}.$$

   (Technically, $\vec{\beta v}$ should be $I^{-1}(\beta \cdot I(\vec{v}))$.) If $\beta = \alpha^i$ for $1 \leq i \leq n-1$, what is the weight of $\vec{c} = (\vec{v}, \vec{\beta v})$? Explain why these choices of β produce a lousy code.

9. Review concatenated codes and low density parity check codes. Extra credit: send me a good problem about one of these topics.