Imaging in Random Media

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PART I: Formulation of the problem

- Wave equation (sound waves).
- Setup for imaging with sensor arrays.
- Model of the medium. What can we estimate?

Inverse problem for the wave equation

• **Sound wave:** acoustic pressure $p(t, \vec{\boldsymbol{x}})$ and particle velocity $\vec{\boldsymbol{u}}(t, \vec{\boldsymbol{x}})$ in medium with mass density $\varrho(\vec{\boldsymbol{x}})$ and bulk modulus $K(\vec{\boldsymbol{x}})$

$$\varrho(\vec{\boldsymbol{x}})\partial_t \vec{\boldsymbol{u}}(t, \vec{\boldsymbol{x}}) + \nabla p(t, \vec{\boldsymbol{x}}) = \vec{\boldsymbol{\mathfrak{F}}}(t, \vec{\boldsymbol{x}})$$
$$\partial_t p(t, \vec{\boldsymbol{x}}) + K(\vec{\boldsymbol{x}})\nabla \cdot \vec{\boldsymbol{u}}(t, \vec{\boldsymbol{x}}) = 0, \quad t \in \mathbb{R}, \quad \vec{\boldsymbol{x}} \in \mathbb{R}^3$$

• Work with 2nd order form:

$$\partial_t^2 p(t, \vec{\boldsymbol{x}}) - K(\vec{\boldsymbol{x}})\nabla \cdot \left[\frac{1}{\varrho(\vec{\boldsymbol{x}})}\nabla p(t, \vec{\boldsymbol{x}})\right] = -K(\vec{\boldsymbol{x}})\nabla \cdot \left[\frac{1}{\varrho(\vec{\boldsymbol{x}})}\vec{\boldsymbol{\mathfrak{F}}}(t, \vec{\boldsymbol{x}})\right]$$

Model medium by wave speed $c(\vec{\boldsymbol{x}}) = \sqrt{\frac{K(\vec{\boldsymbol{x}})}{\varrho(\vec{\boldsymbol{x}})}}$ and the impedance $\zeta(\vec{\boldsymbol{x}}) = c(\vec{\boldsymbol{x}})\varrho(\vec{\boldsymbol{x}})$.

• In the case of a constant mass density (often assumed) we get the usual wave equation:

Inverse problem for the wave equation

$$\left[\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2}{\partial t^2} - \Delta\right] p(t,\vec{\boldsymbol{x}}) = F(t,\vec{\boldsymbol{x}}), \quad t \in \mathbb{R}, \quad \vec{\boldsymbol{x}} \in \mathbb{R}^3,$$
$$p(t,\vec{\boldsymbol{x}}) \equiv 0, \quad t \ll 0.$$

- Source term $F = -\nabla \cdot \vec{\mathfrak{F}}$ is determined by the exerted force $\vec{\mathfrak{F}}(t,\vec{\boldsymbol{x}})$ and is compactly supported in t and $\vec{\boldsymbol{x}}$.

- The wave propagates in the whole space $\rightsquigarrow$ using causality and finite wave speed we can impose outgoing conditions.

Inverse problem: One or more known or unknown sources emit waves which are measured at sensors. Can we determine the medium and/or the source from these measurements?

Sensor arrays

• Sensors located in a compact set $\mathcal{A}$ on some measuring surface.

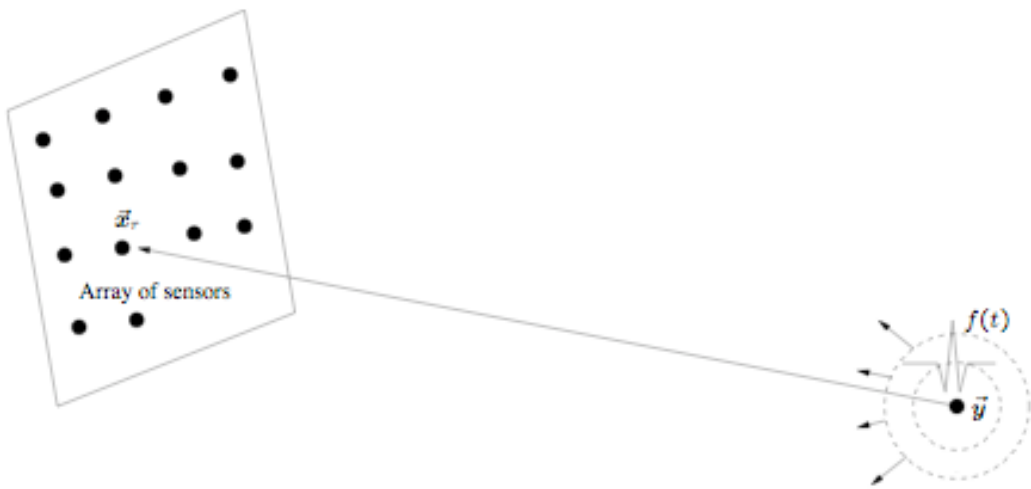
• To form images we sum the measurements at these sensors (after some other operations). If the sensor separation in small, these sums are approximated by integrals over the set $\mathcal{A}$.

Sensors behave like a collective entity the array with aperture $\mathcal{A}$.

• Depending on the application we may have a passive array, an active array or a synthetic array.

Passive array imaging

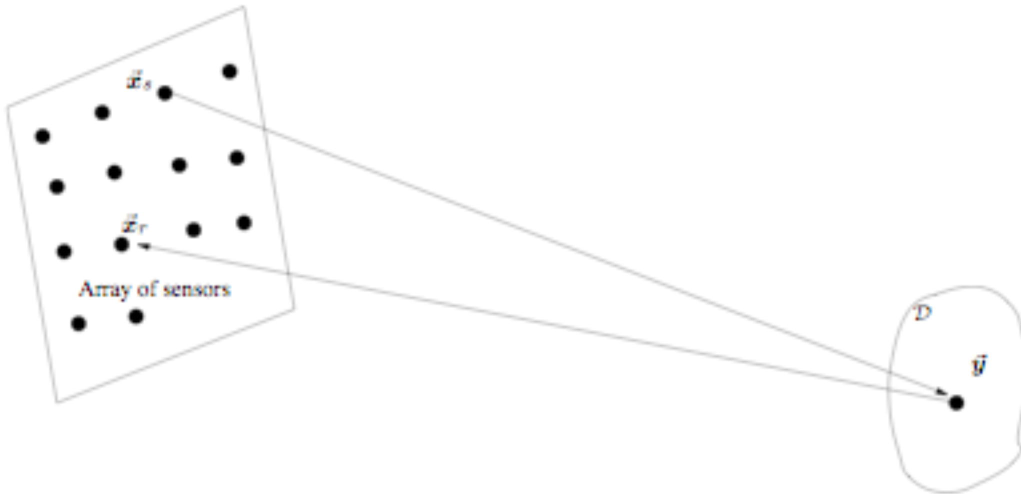
Passive array data: vector $\{p(t, \vec{x}_r)\}_{r=1,\ldots,N}$ for $t \in (0, T]$ is collected by N receiver sensors



• The goal here is to localize the source of waves, which is far from the array. We need to know $c(\vec{x})$ to do a good job!

• There are other imaging modalities with passive arrays, for imaging heterogeneous media, using controlled or noise sources.

Active array data: response matrix $\{p(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)\}$ for $t \in (0, T]$ and $r = 1, \ldots, N$ and $s = 1, \ldots N_s \leq N$.



We control excitations & seek to estimate the medium.

Synthetic arrays: One source/receiver sensor that moves (Synthetic Aperture Radar (SAR)) or a small active array that moves.

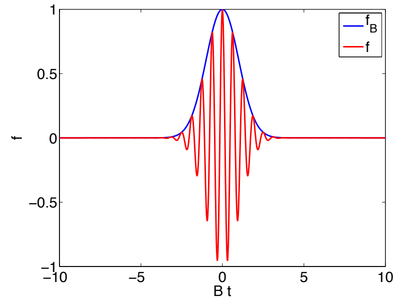
Probing signals

• Source term is $F(t,\vec{\boldsymbol{x}}) = \delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_s)f(t)$ or a superposition of such functions.

• The signal $f(t)$ may be the same for all the sources if we control the illumination, or it may vary from source to source.

• Often, it is a pulse $f(t) = e^{-i\omega_o t} B\varphi(Bt)$ with envelope φ

display of Real$[f(t)]$



It oscillates at central frequency ω_o and is supported at $t \sim 1/B$, where $B =$ bandwidth

$$\widehat{f}(\omega) = \int_{\mathbb{R}} dt\, f(t) e^{i\omega t} = \widehat{\varphi}\Big(\frac{\omega - \omega_o}{B}\Big)$$

Why a pulse?

Length scale relations are important:

- Central wavelength $\lambda_o = \frac{2\pi c_o}{\omega_o}$, for c_o reference wave speed.
- Distance (range) L between array and imaging scene.
- Linear size a of array aperture (may be synthetic).
- Distance c_o/B traveled by waves over pulse duration.

In most applications: $L \gtrsim a \gg c_o/B \gg \lambda_o$

$\rightsquigarrow$ high frequency (small wavelength) regime.

As a rule, the smaller λ_o and c_o/B are, the better the imaging (under some conditions). The ratio a/L also plays a role.

• Antennas have limited instantaneous power: $|f(t)|^2 \leq P_{\text{max}}$. For a signal of duration T, the emitted energy is $\leq TP_{\text{max}}$.

• The received energy is a fraction of this (partial reflection, geometrical spreading). This energy should be large to distinguish from noise $\rightsquigarrow$ Use more antennas or increase duration T.

• Chirped (linear frequency modulated) signal

$$f(t) = e^{-i\omega_o t + i\gamma t^2} \varphi\Big(\frac{t}{T}\Big)$$

Let us calculate $\widehat{f}(\omega)$ for the case $\sqrt{\gamma}\, T \gg 1$:

$$\widehat{f}(\omega) = \int_{\mathbb{R}} dt\, e^{i(\omega-\omega_o)t + i\gamma t^2} \varphi\Big(\frac{t}{T}\Big) = \int_{\mathbb{R}} \frac{dw}{2\pi} \widehat{\varphi}(w) \int_{\mathbb{R}} e^{i\gamma t^2 + i(\omega-\omega_o)t - iw\frac{t}{T}}$$

- Rewrite the phase:

$$\gamma t^2 + (\omega - \omega_o)t - w\frac{t}{T} = \left[\sqrt{\gamma}\left(t + \frac{\omega - \omega_o}{2\gamma}\right) - \frac{w}{2\sqrt{\gamma}T}\right]^2 + \frac{w(\omega - \omega_o)}{2\gamma T} - \frac{(\omega - \omega_o)^2}{4\gamma} - \frac{w^2}{4\gamma T^2}$$

and use $\int_{\mathbb{R}} dt\, e^{i\left[\sqrt{\gamma}\left(t+\frac{\omega-\omega_o}{2\gamma}\right)-\frac{w}{2\sqrt{\gamma}T}\right]^2} = \sqrt{i\pi/\gamma}$

- We get for $\sqrt{\gamma}T \gg 1$,

$$\widehat{f}(\omega) = \sqrt{\frac{i\pi}{\gamma}} e^{-i\frac{(\omega-\omega_o)^2}{4\gamma}} \int_{\mathbb{R}} \frac{dw}{2\pi} \widehat{\varphi}(w) e^{i\frac{w(\omega-\omega_o)}{2\gamma T} - i\frac{w^2}{4\gamma T^2}}$$
$$\approx \sqrt{\frac{i\pi}{\gamma}} e^{-i\frac{(\omega-\omega_o)^2}{4\gamma}} \varphi\left(\frac{\omega_o - \omega}{2\gamma T}\right) \rightsquigarrow \quad B = \gamma T.$$

- Note that $T \gg \frac{1}{\sqrt{\gamma}} = \sqrt{\frac{T}{B}} \rightsquigarrow \quad T \gg 1/B$. Long signal can be compressed to get a pulse of duration $\sim 1/B$.

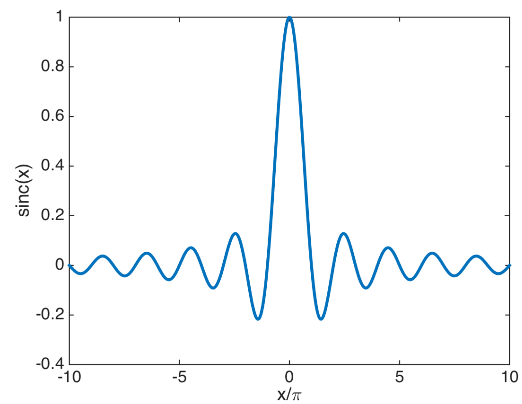
Chirped signals and pulse compression*

• Pulse compression realized by convolving the echoes with $f(-t)$:

$$f_c(t) = f(t) \star f(-t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} |\hat{f}(\omega)|^2 e^{-i\omega t} \approx \frac{T}{2B} \int_{\mathbb{R}} d\omega \left|\varphi\left(\frac{\omega_o - \omega}{2B}\right)\right|^2 e^{-i\omega t}$$

Example: if $\varphi(s) = 1_{[-1/2,1/2]}(s)$ we get

$$f_c(t) = Te^{-i\omega_o t}\text{sinc}(Bt)$$



• We have transformed the signal with duration $T \gg 1/B$ to a pulse oscillating at frequency ω_o and support $t \sim 1/B$.

*For compression of noise signals: "Passive Imaging with Ambient Noise" by J. Garnier and G. Papanicolaou, Cambridge University Press, 2016.

- Consider a simple case of a medium with planar interface:

$$c(\vec{\boldsymbol{x}}) = \begin{cases} c_-, & z < 0 \\ c_+, & z > 0, \end{cases} \qquad \zeta(\vec{\boldsymbol{x}}) = \begin{cases} \zeta_-, & z < 0 \\ \zeta_+, & z > 0, \end{cases} \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z).$$

- Incoming plane wave propagating along $z \rightsquigarrow$ problem is 1−D

$$\begin{aligned} \partial_t\, p(t,z) + \zeta(z)c(z)\partial_z\, u(t,z) &= 0 \\ \frac{\zeta(z)}{c(z)}\partial_t\, u(t,z) + \partial_z\, p(t,z) &= 0 \end{aligned}$$

- In each half space we can write the solution as:

$$\begin{aligned} p_\pm(t,\vec{\boldsymbol{x}}) &= \zeta_\pm^{\frac{1}{2}}\Big[A_\pm\Big(t - \frac{z}{c_\pm}\Big) - B_\pm\Big(t + \frac{z}{c_\pm}\Big)\Big] \\ u_\pm(t,\vec{\boldsymbol{x}}) &= \zeta_\pm^{-\frac{1}{2}}\Big[A_\pm\Big(t - \frac{z}{c_\pm}\Big) + B_\pm\Big(t + \frac{z}{c_\pm}\Big)\Big] \end{aligned}$$

Model of the medium: What scatters?

- At $z = 0$ the pressure and velocity must be continuous:

$$\zeta_-^{\frac{1}{2}}\Big[A_-(t) - B_-(t)\Big] = \zeta_+^{\frac{1}{2}}\Big[A_+(t) - B_+(t)\Big]$$
$$\zeta_-^{-\frac{1}{2}}\Big[A_-(t) + B_-(t)\Big] = \zeta_+^{-\frac{1}{2}}\Big[A_+(t) + B_+(t)\Big]$$

- For wave coming from $z < 0$, we know A_-. Wave must be outgoing at $z > 0$ so $B_+ = 0$:

$$B_-(t) = RA_-(t), \quad R = \frac{\zeta_- - \zeta_+}{\zeta_- + \zeta_+} = \text{reflection coefficient}$$

$$A_+(t) = \frac{\zeta_+^{\frac{1}{2}}}{\zeta_-^{\frac{1}{2}}}(1 + R)A_-(t) = \text{transmission}$$

- Reflection is due to change in impedance*. If mass density is constant this is due to change in velocity.

*Beylkin, Burridge, *Linearized inverse scattering problems in acoustics and elasticity*, Wave Motion, 12(1990), pp.15-52.

Model of the medium

- There is separation of scales in this problem:

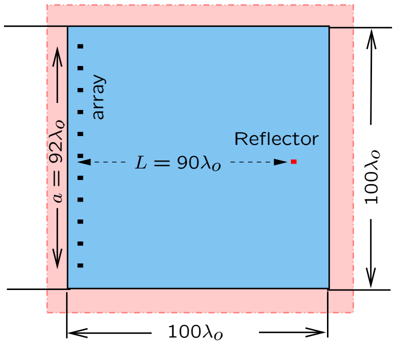
$$\frac{1}{c^2(\vec{x})} = \frac{1}{c_o^2(\vec{x})}[1 + \rho(\vec{x}) + \mu(\vec{x})]$$

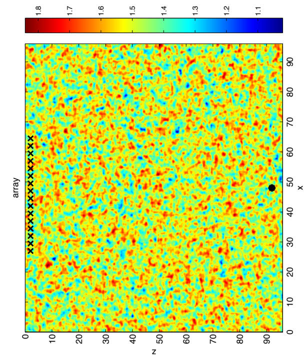
$c_o(\vec{x})$ = smooth, determines kinematics of waves (travel times). In many applications this is known and constant. In seismic inversion $c_o(\vec{x})$ is not known and must be estimated.

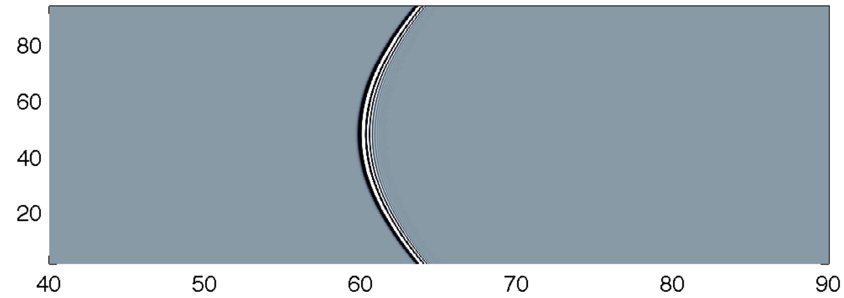
$\rho(\vec{x})$ = rough part, is the reflectivity that we wish to determine in active array imaging. This causes the reflections of the waves that are captured at the array.

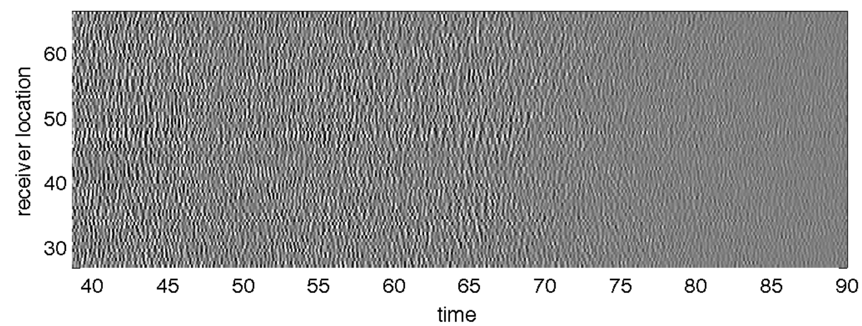
$\mu(\boldsymbol{x})$ models small variations at small scale (clutter), that may have a cumulative scattering effect on the wave.

Numerical simulations (Tsogka) of active array data









- Travel time $\tau_{r,s} = \frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}| + |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}|}{c_o}$ for reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, y_3)$ and sensors at $\vec{\boldsymbol{x}}_r = (\boldsymbol{x}_r, 0) \rightsquigarrow$ arrivals form hyperbola

$$\frac{c_o^2}{y_3^2}\left(\tau_{r,s} - \frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}|}{c_o}\right)^2 - \frac{|\boldsymbol{x}_r - \boldsymbol{y}|^2}{y_3^2} = 1$$

- Fuctuations of $c(\vec{\boldsymbol{x}})$ cause echoes that mask useful signal from the reflector that we wish to find.

What can we estimate?

- Smooth $c_o(\boldsymbol{x})$ (velocity analysis):

This is difficult because the signals are oscillatory and a small perturbation of c_o causes a large change in the wave (travel time error exceeds small period of oscillation).

Full wave inversion approaches (data fitting in least squares sense) are computationally intensive and suffer from lack of convexity (if low frequencies are present in the data things improve).

Travel time tomography (many applied papers, theory of Uhlmann, Stefanov, Vasy). Differential semblance optimization (Symes).

We will let $c_o =$ known constant.

- Reflectivity ρ (imaging problem).
- Clutter cannot be estimated.

Because clutter $\mu(\vec{x})$ cannot be estimated $\rightsquigarrow$ uncertainty.

Stochastic model: the clutter is a realization of a random medium (a set of possible media described statistically).

• The random model incorporates what we know (or can estimate) such as: length scale of fluctuations (correlation length), typical amplitude of fluctuations (standard deviation), autocorrelation of the fluctuations.

• Mapping from statistical distribution of $\mu(\vec{x})$ to statistical distribution of $p(t, \vec{x})$ is highly nonlinear and difficult to determine.

• Computational (UQ) approaches are unlikely to succeed as we have a high dimensional parametrization. We need scaling regimes and asymptotic stochastic theory.

• Pursuit of statistically stable results (which depend only on statistics of the random medium, not the particular realization).

PART II: Basic imaging

- Array data model.
- Formulation of the inverse problem based on data fitting.
- Basic resolution results.
- Dealing with noise.
- Random medium effects.

- Using Green's theorem, we can write the pressure wave as

$$p(t,\vec{\boldsymbol{x}}) = \int_0^t ds \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\, F(s,\vec{\boldsymbol{y}})G(t-s,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}),$$

where $G(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ is the causal Green's function satisfying

$$\left[\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2}{\partial t^2} - \Delta\right] G(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = \delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})\delta(t), \quad t>0, \quad \vec{\boldsymbol{x}}\in\mathbb{R}^3$$

$$G(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = 0, \quad t<0.$$

- F and G are causal $\rightsquigarrow$ extend the s interval to $\mathbb{R}$ so we have a time convolution $\rightsquigarrow$ work in frequency domain

$$p(t,\vec{\boldsymbol{x}}) = \int_{\mathbb{R}} \frac{dt}{2\pi}\widehat{p}(\omega,\vec{\boldsymbol{x}})e^{-i\omega t}, \quad \widehat{p}(\omega,\vec{\boldsymbol{x}}) = \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\,\widehat{F}(\omega,\vec{\boldsymbol{y}})\widehat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}).$$

Passive array data model

• The wave propagation between the point $\vec{\boldsymbol{y}}$ in the support of the source and $\vec{\boldsymbol{x}}$ is via the Green's function

$$\left[\frac{\omega^2}{c^2(\vec{\boldsymbol{x}})} + \Delta\right] \widehat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) = -\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}), \qquad \frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1 + \mu(\vec{\boldsymbol{x}})}{c_o^2},$$

that satisfies radiation conditions as $|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}| \to \infty$.

• The array measures $D(t, \vec{\boldsymbol{x}}_r) = p(t, \vec{\boldsymbol{x}}_r) + \mathcal{N}(t, \vec{\boldsymbol{x}}_r)$, for $t \in (0, T)$ and $\vec{\boldsymbol{x}}_r \in \mathcal{A}$, where $\mathcal{N}$ is noise.

• Mapping from source F to wave $p(t, \vec{\boldsymbol{x}}_r)$ is linear but unknown because of μ. We replace it by

$$F \to \int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\, \widehat{F}(\omega, \vec{\boldsymbol{y}}) \widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r),$$

where $\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r) = \frac{e^{ik(\omega)|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}|}}{4\pi|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}|}$ and $k(\omega) = \frac{\omega}{c_o}$ is the wavenumber.

Passive array data model

- Simplifying assumption: $\widehat{F}(\omega,\vec{\boldsymbol{y}}) = \widehat{f}(\omega)\rho(\vec{\boldsymbol{y}})$ with known $\widehat{f}(\omega)$.

- The forward map is

$$[\mathcal{M}\rho](t,\vec{\boldsymbol{x}}_r) = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\,\widehat{f}(\omega)\int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\,\rho(\vec{\boldsymbol{y}})\frac{e^{ik(\omega)|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|-i\omega t}}{4\pi|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|}$$

- The source imaging problem can be formulated as

$$\min \|\mathcal{M}\rho - D\|_2^2 = \int_{\mathbb{R}} dt \sum_{r=1}^{N} |[\mathcal{M}\rho](t,\vec{\boldsymbol{x}}_r) - D(t,\vec{\boldsymbol{x}}_r)|^2$$

with some penalty on L^2 or L^1 norm of ρ, for regularization.

Note: data $D(t,\vec{\boldsymbol{x}}_r)$ are random because of noise and because of the medium. How can we get a robust estimate of ρ?

Active array data model

- The sound wave due to a source at $\vec{\boldsymbol{x}}_s \in \mathcal{A}$ is

$$p(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_s) = \int_{\mathbb{R}} \frac{dt}{2\pi} e^{-i\omega t} \widehat{p}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_s),$$

$$\Big[k^2(\omega)\Big(1 + \mu(\vec{\boldsymbol{x}})\Big) + \Delta\Big] \widehat{p}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_s) = -k^2(\omega)\rho(\vec{\boldsymbol{x}})\widehat{p}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_s) \\ -\widehat{f}(\omega)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_s)$$

- Array data: $D(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s) = p(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s) + \mathcal{N}(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)$, for $t \in (0, T)$ and $\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s \in \mathcal{A}$, where $\mathcal{N}$ is noise.

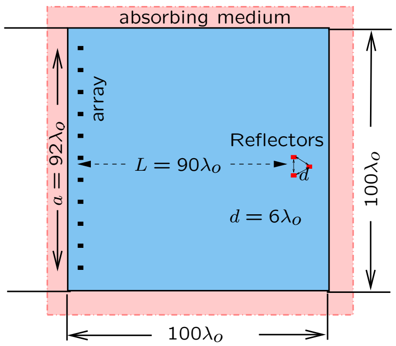
- Mapping between ρ and $p(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)$ is nonlinear. Most imaging is based on linearization assumption

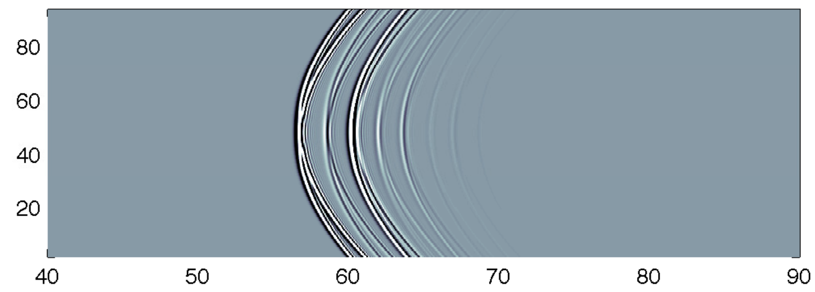
$$\widehat{p}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s) = \widehat{f}(\omega)\widehat{G}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s) + \widehat{p}^{\text{sc}}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)$$

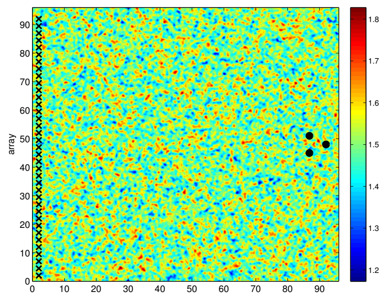
with Born (single scattering) approximation of scattered wave

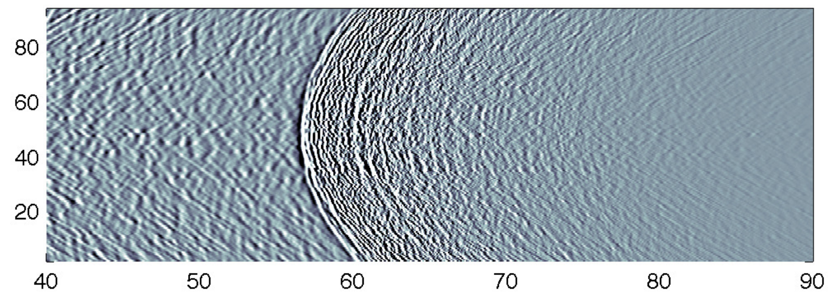
$$\widehat{p}^{\text{sc}}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s) \approx k^2(\omega)\widehat{f}(\omega) \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\, \widehat{G}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}})\rho(\vec{\boldsymbol{y}})\widehat{G}(\omega, \vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}})$$

Active array simulations (Tsogka)









• The multiply scattered echoes are visible in the homogeneous medium but are weaker than the primaries.

• The multiple scattering effects in the medium are much stronger than those from the 3 small reflectors.

Active array data model

- By time windowing, data are determined by

$$\widehat{p}^{\text{SC}}(t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_s) \approx \int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{-i\omega t} k^2(\omega)\widehat{f}(\omega) \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\, \widehat{G}(\omega,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}})\rho(\vec{\boldsymbol{y}})\widehat{G}(\omega,\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}})$$

- We don't know $\widehat{G}$ in the random medium. Define forward map:

$$[\mathcal{M}\rho](t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_s) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} k^2(\omega)\widehat{f}(\omega)\int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}\, \rho(\vec{\boldsymbol{y}}) \frac{e^{ik(\omega)(|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|+|\vec{\boldsymbol{y}}-\vec{\boldsymbol{x}}_s|)-i\omega t}}{(4\pi)^2|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}||\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}|}$$

- The imaging problem can be formulated as

$$\min \|D-\mathcal{M}\rho\|_2^2 = \int_{\mathbb{R}} dt \sum_{r,s=1}^{N} |[\mathcal{M}\rho](t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_s) - D(t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_s)|^2$$

with some penalty on L^2 or L^1 norm of ρ, for regularization.

- The least squares minimizer satisfies the normal equations

$$[\mathcal{M}^\star \mathcal{M} \rho](\vec{\boldsymbol{y}}) = \text{real}\,[\mathcal{M}^\star D\,](\vec{\boldsymbol{y}})$$

where $\vec{\boldsymbol{y}}$ varies in the imaging region.

- Here $\mathcal{M}^\star$ is the adjoint of the forward map given by

$$[\mathcal{M}^\star D\,](\vec{\boldsymbol{y}}) = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\,\widehat{f}(\omega) \sum_{r=1}^{N} \overline{\widehat{D}(\omega, \vec{\boldsymbol{x}}_r)}\,\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r)$$

for the passive array case and

$$[\mathcal{M}^\star D\,](\vec{\boldsymbol{y}}) = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\,\widehat{f}(\omega) \sum_{r,s=1}^{N} \overline{\widehat{D}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)}\,\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r)\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_s)$$

for the active array case.

- The adjoint operator

$$[\mathcal{M}^\star D](\vec{\boldsymbol{y}}) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \widehat{f}(\omega) \sum_{r=1}^{N} \overline{\widehat{D}(\omega, \vec{\boldsymbol{x}}_r)} \widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r)$$

- The reverse time migration operator

$$\begin{aligned}[\mathcal{T} D](\vec{\boldsymbol{y}}) &= \int_{\mathbb{R}} \frac{d\omega}{2\pi} \sum_{r=1}^{N} \overline{\widehat{D}(\omega, \vec{\boldsymbol{x}}_r)} \widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r) \\ &= \int_{\mathbb{R}} dt\, \overline{D(-(0-t), \vec{\boldsymbol{x}}_r)} G_o(t, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r)\end{aligned}$$

Data is conjugated and time reversed. It is propagated in the reference medium to point $\vec{\boldsymbol{y}}$ and result is evaluated at time 0.

- Adjoint has extra $\widehat{f}(\omega)$ which is good for pulse compression. Data $\widehat{D} \approx \widehat{f}\widehat{G}$ are matched filtered to $\widehat{f}\widehat{G}_o \rightsquigarrow$ peaks at $\vec{\boldsymbol{y}} \in \mathrm{supp}\rho$

- Similar interpretation for active array imaging.

The normal operator

- The least squares solution satisfies: $[\mathcal{M}^\star\mathcal{M}\rho]\,(\vec{\boldsymbol{y}}) = \text{real}\,[\mathcal{M}^\star D\,]\,(\vec{\boldsymbol{y}})$.

- The normal operator $\mathcal{M}^\star\mathcal{M}$ in the case of passive array is:

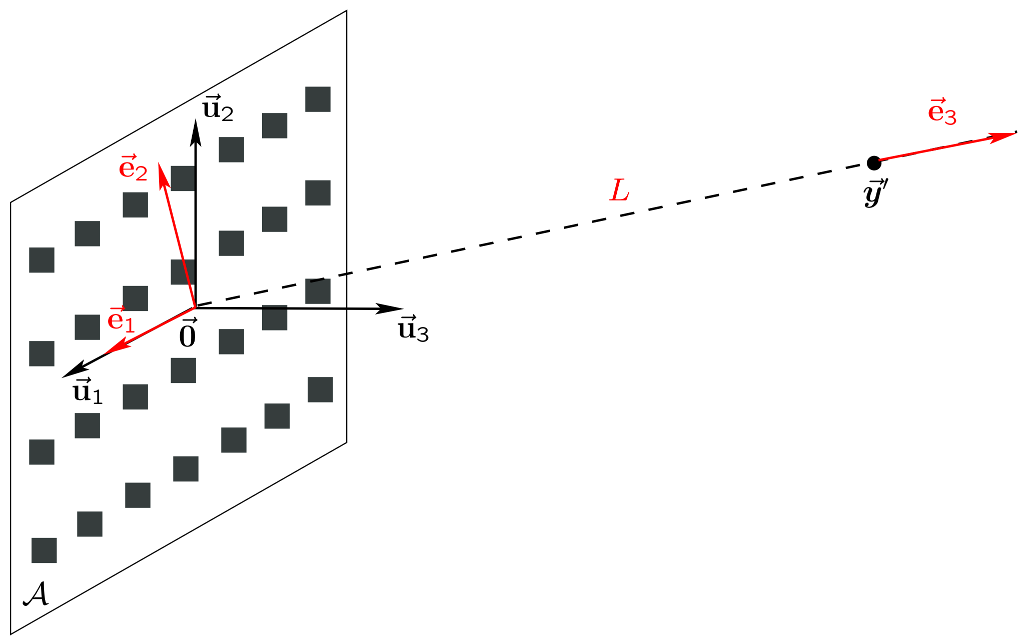
$$[\mathcal{M}^\star\mathcal{M}\rho\,]\,(\vec{\boldsymbol{y}}) = \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}'\rho(\vec{\boldsymbol{y}}')\mathcal{K}(\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}')$$

with kernel given by

$$\mathcal{K}(\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}') = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\,|\widehat{f}(\omega)|^2 \sum_{r=1}^{N} \overline{\widehat{G}_o(\omega,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}')}\widehat{G}_o(\omega,\vec{\boldsymbol{y}},\vec{\boldsymbol{x}}_r)$$
$$\approx \frac{N}{(4\pi)^2|\mathcal{A}|}\int_{\mathbb{R}} \frac{d\omega}{2\pi}|\widehat{f}(\omega)|^2 \int_{\mathcal{A}} d\vec{\boldsymbol{x}}\,\frac{e^{ik(\omega)(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|)}}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}||\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|}$$

- If the kernel where close to $\delta(\vec{\boldsymbol{y}}-\vec{\boldsymbol{y}}')$, we could estimate ρ using the adjoint operator.

Analysis of kernel of normal operator



- Range direction $\vec{\mathbf{e}}_3$ points from the center of the array to $\vec{y}'$
- $\mathcal{A}$ is square of side length a with normal at angle θ from $\vec{\mathbf{e}}_3$

$$\forall \vec{x} \in \mathcal{A}, \quad \vec{x} = a s_1 \vec{\mathbf{e}}_1 + a s_2 (\cos\theta \vec{\mathbf{e}}_2 + \sin\theta \vec{\mathbf{e}}_3)$$

- The search point is $\vec{y} = \vec{y}' + \sum_{j=1}^{3} \xi_j \vec{\mathbf{e}}_j$ and we let $\vec{y}' = L\vec{\mathbf{e}}_3$

Analysis of kernel of normal operator

- The kernel is

$$\mathcal{K}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}') \approx \frac{N}{(4\pi)^2} \int_{\mathbb{R}} \frac{d\omega}{2\pi} |\widehat{f}(\omega)|^2 \iint_{-\frac{1}{2}}^{\frac{1}{2}} ds_1 ds_2 \, \frac{e^{ik(\omega)(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|)}}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}||\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|}$$

where $\widehat{f}(\omega) = \widehat{\varphi}\left(\frac{\omega - \omega_o}{B}\right)$ and

$$|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}'| = \sqrt{(as_1)^2 + (as_2 \cos\theta)^2 + (L - as_2 \sin\theta)^2}$$

$$|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}| = \sqrt{(as_1 - \xi_1)^2 + (as_2 \cos\theta - \xi_2)^2 + (L + \xi_3 - as_2 \sin\theta)^2}$$

- To analyze $\mathcal{K}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}')$ and understand image resolution we need scaling assumptions.

Analysis of normal operator: Scaling assumptions

Typically: $\omega_o \gg B$ and $L \gg a \gg |\xi_{1,2,3}| \gg \lambda_o$:

- Geometrical spreading factors: $\frac{1}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}||\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|} \approx \frac{1}{L^2}$
- Phase is:

$$k(\omega)\Big(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}| - |\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|\Big) \approx k(\omega)\Big[\xi_3 - s_1\frac{a\xi_1}{L} - s_2\frac{a\xi_2\cos\theta}{L} - \frac{\xi_3\big[(as_1)^2 + (as_2\cos\theta)^2\big]}{2L^2} + \dots\Big]$$

for $k(\omega) = \frac{\omega_o}{c_o} + \frac{\omega-\omega_o}{c_o} = k_o + O(B/c_o)$

- Kernel becomes

$$\mathcal{K}(\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}') \approx \frac{N}{(4\pi L)^2}\int_{\mathbb{R}}\frac{d\omega}{2\pi}\left|\widehat{\varphi}\Big(\frac{\omega-\omega_o}{B}\Big)\right|^2 e^{ik(\omega)\xi_3}\,\mathcal{U}\Big(\frac{k_o a\xi_1}{L},\frac{k_o a^2\xi_3}{L^2}\Big) \times\mathcal{U}\Big(\frac{k_o a\cos\theta\xi_2}{L},\frac{k_o(a\cos\theta)^2\xi_3}{L^2}\Big)$$

where $\mathcal{U}(\alpha,\gamma) = \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, \exp(-i\alpha s - i\gamma s^2/2)$ = Fresnel integral.

- If we set $\xi_{2,3} = 0$, kernel is proportional to

$$\mathcal{U}\Big(\frac{k_o a \xi_1}{L}, 0\Big) = \text{sinc}\Big(\frac{k_o a \xi_1}{2L}\Big) = \text{sinc}\Big(\frac{\pi a \xi_1}{\lambda_o L}\Big)$$

and is large for $|\xi_1| \lesssim \lambda_o L/a$.

- If we set $\xi_{1,3} = 0$, kernel is proportional to

$$\mathcal{U}\Big(\frac{k_o a \cos\theta \xi_2}{L}, 0\Big) = \text{sinc}\Big(\frac{\pi a \cos\theta \xi_2}{\lambda_o L}\Big)$$

and is large for $|\xi_2| \lesssim \lambda_o L/(a\cos\theta)$.

- If we set $\xi_{1,2} = 0$, kernel is proportional to

$$\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{i(\omega-\omega_o)\xi_3/c_o} \Big|\widehat{\varphi}\Big(\frac{\omega-\omega_o}{B}\Big)\Big|^2 \mathcal{U}\Big(0, \frac{k_o a^2 \xi_3}{L^2}\Big)\mathcal{U}\Big(0, \frac{k_o (a\cos\theta)^2 \xi_3}{L^2}\Big)$$

and is large for $|\xi_3| \lesssim \min\Big\{\frac{c_o}{B}, \frac{\lambda_o L^2}{a^2}\Big\}$.

• The kernel of the normal operator is approximately an identity in the sense that it almost does not move the singularities of ρ. It is expensive to compute in practice and it is not invertible.

• In the normal equations $[\mathcal{M}^\star\mathcal{M}\rho](\vec{\boldsymbol{y}}) = \text{real}\,[\mathcal{M}^\star D\,](\vec{\boldsymbol{y}})$ we replace it by the identity operator.

• Matched field imaging function for passive arrays:

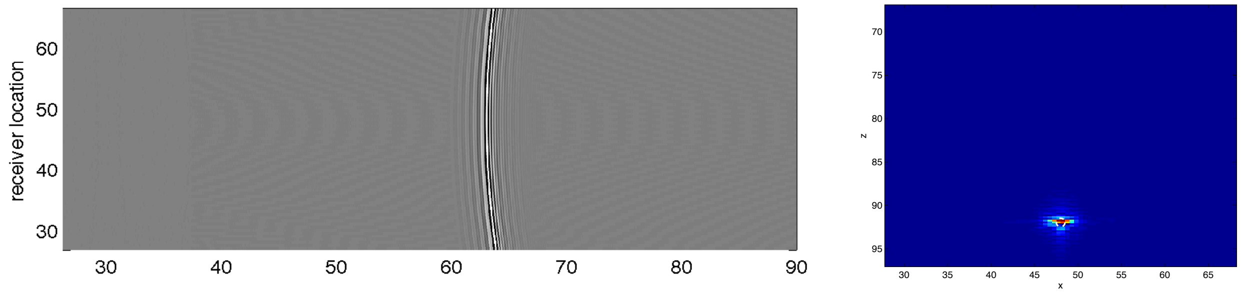
$$\mathcal{I}(\vec{\boldsymbol{y}}) = \text{real}\,[\mathcal{M}^\star D\,](\vec{\boldsymbol{y}}) = \text{real}\int_{\mathbb{R}} \frac{d\omega}{2\pi}\,\widehat{f}(\omega)\sum_{r=1}^{N} \overline{\widehat{D}(\omega,\vec{\boldsymbol{x}}_r)}\frac{e^{ik(\omega)|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|}}{4\pi|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|}$$

• Kirchhoff migration imaging*

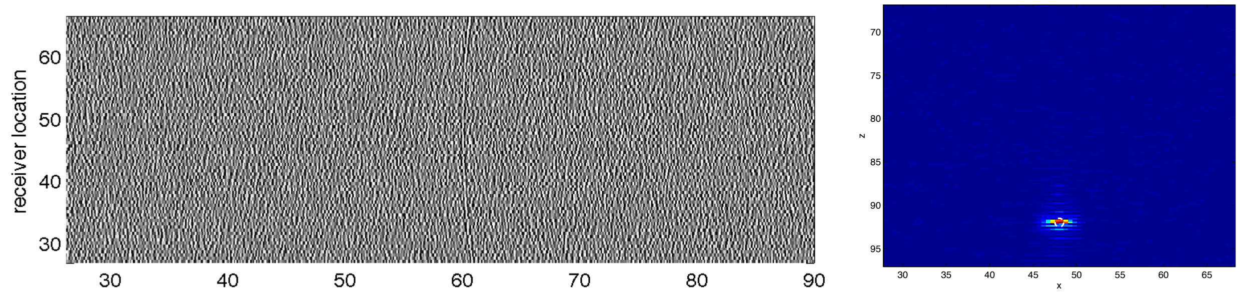
$$\mathcal{I}^{\text{KM}}(\vec{\boldsymbol{y}}) = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\sum_{r=1}^{N}\widehat{D}(\omega,\vec{\boldsymbol{x}}_r)e^{-ik(\omega)|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|} = \sum_{r=1}^{N} D\Big(\frac{|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}|}{c_o},\vec{\boldsymbol{x}}_r\Big)$$

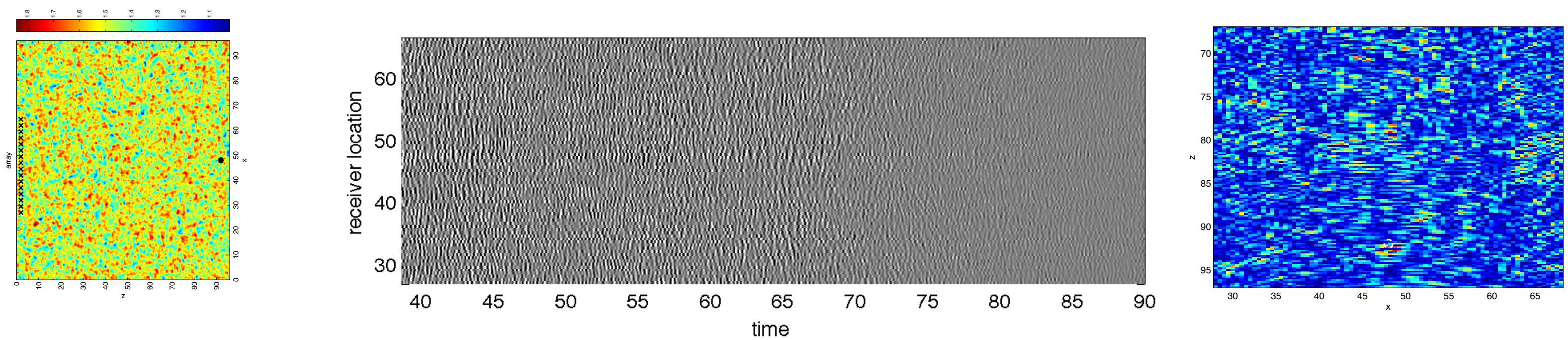
and similar for active array.

*Here we used that $1/|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}| \approx L$.



Noise

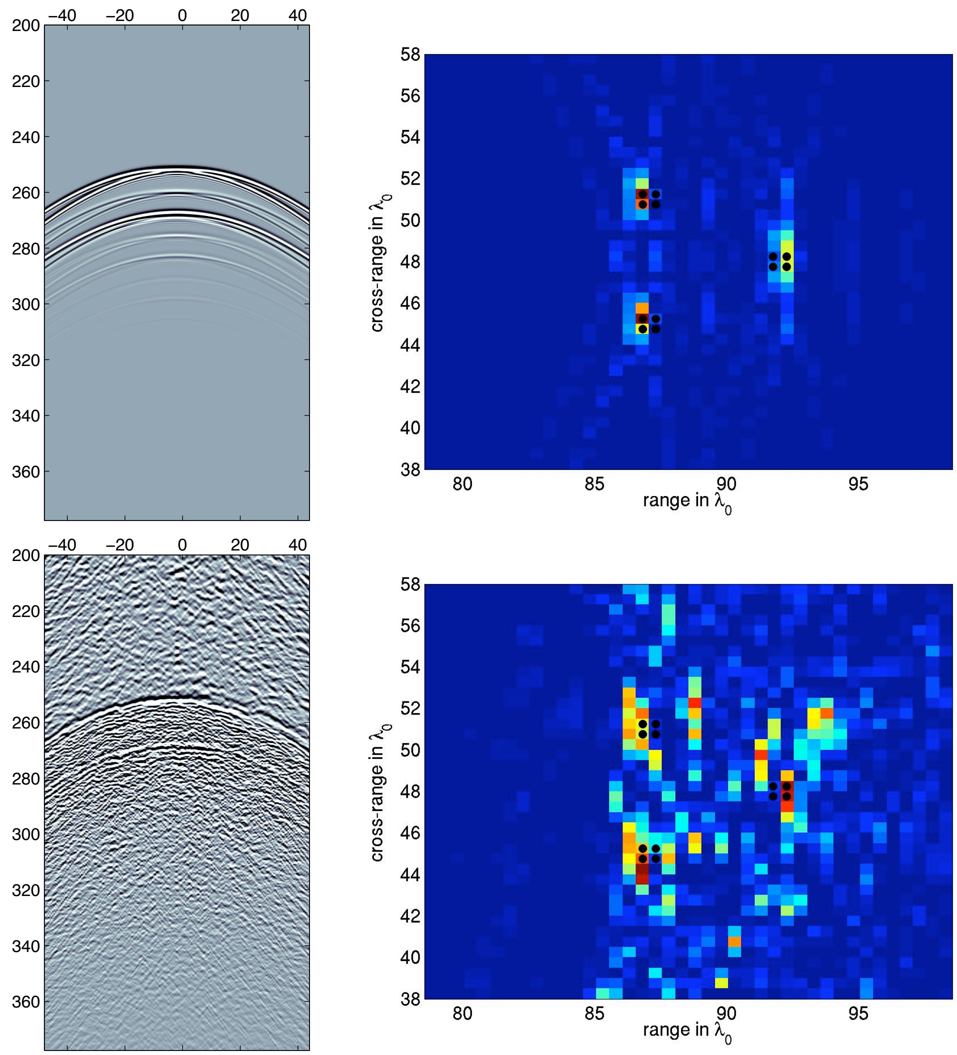




Noise is averaged out in migration. Clutter is harder to deal with.

*Simulations by Chrysoula Tsogka.

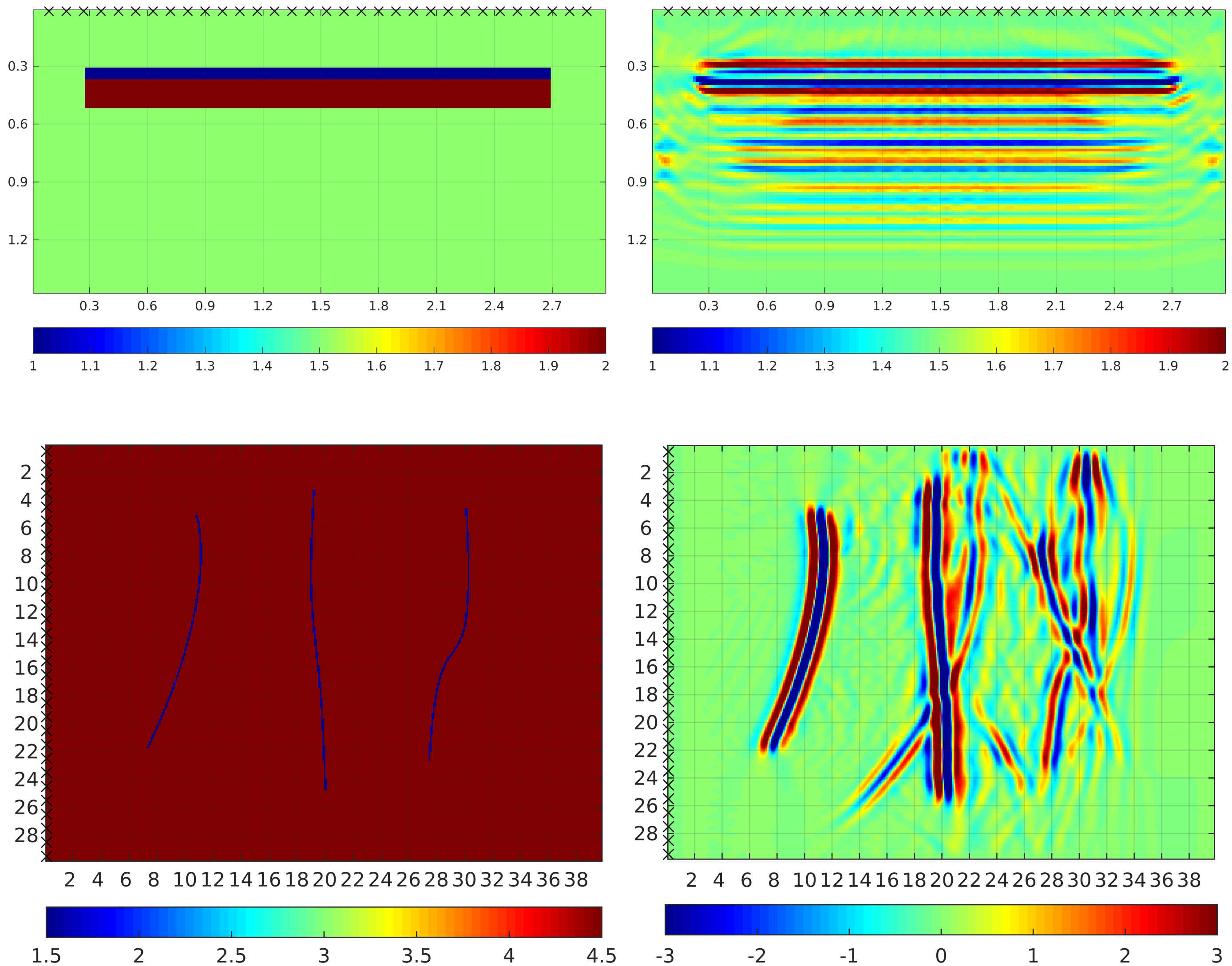
Born approximation vs. clutter effects *



Multiple scattering in clutter is a bigger issue than the Born approximation in this case.

*Simulations by Chrysoula Tsogka.

Multiple scattering effects in media with strong reflectors*



*Simulations by Alexander Mamonov

Connection to Bayesian inversion

• To see why migration deals well with noise, consider a single, time harmonic point source at $\vec{\boldsymbol{y}}$.

• The data model is $\widehat{D}(\omega, \vec{\boldsymbol{x}}_r) = \widehat{f}(\omega)\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r) + \widehat{\mathcal{N}}(\omega, \vec{\boldsymbol{x}}_r)$

• The noise $\widehat{\mathcal{N}}(\omega, \vec{\boldsymbol{x}}_r)$ is complex, i.i.d. Gaussian with mean zero and variance σ^2 i.e., real and imaginary parts are Gaussian, i.i.d., with variance $\sigma^2/2$.

• Define data and noise vectors

$$\widehat{\mathbf{d}}(\omega) = \Big(\widehat{D}(\omega, \vec{\boldsymbol{x}}_r)\Big)_{1 \le r \le N} \quad \text{and} \quad \widehat{\boldsymbol{n}}(\omega) = \Big(\widehat{\mathcal{N}}(\omega, \vec{\boldsymbol{x}}_r)\Big)_{1 \le r \le N}$$

and normalized vector of Green's functions

$$\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}}) = \Big[\sum_{s=1}^{N} |\widehat{G}_o(\omega, \vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}})|^2\Big]^{-1/2} \Big(\widehat{G}_o(\omega, \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_r)\Big)_{1 \le r \le N}$$

Connection to Bayesian inversion

- The data model becomes

$$\widehat{\mathbf{d}}(\omega) = \widehat{\phi}(\omega)\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}}) + \widehat{\boldsymbol{n}}(\omega)$$

with

$$\widehat{\phi}(\omega) = \widehat{f}(\omega)\Big[\sum_{s=1}^{N} |\widehat{G}_o(\omega, \vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}})|^2\Big]^{1/2} \approx \frac{\sqrt{N}\widehat{f}(\omega)}{4\pi L}.$$

- Define two random vectors: $\boldsymbol{\chi}$ with realizations $(\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})$ (the unknowns) and $\boldsymbol{\Delta}$ with realizations $\widehat{\mathbf{d}}(\omega)$.

- Bayes' theorem:

$$p_\chi\Big((\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})|\boldsymbol{\Delta} = \widehat{d}(\omega)\Big) = \frac{p_\Delta(\widehat{\mathbf{d}}|\boldsymbol{\chi} = (\vec{\boldsymbol{y}}, \sigma, \widehat{\phi}))p_\chi\Big((\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})\Big)}{p_\Delta(\widehat{\mathbf{d}})}$$

Connection to Bayesian inversion

- Since the noise is Gaussian

$$p_{\Delta}(\widehat{\mathbf{d}}|\boldsymbol{\chi} = (\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})) = \frac{1}{(2\pi\sigma^2)^N} \exp\Big[-\frac{\|\widehat{\mathbf{d}}(\omega) - \widehat{\phi}(\omega)\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}})\|^2}{2\sigma^2}\Big]$$

- Jeffreys prior (non-informative distribution on parameter space which is proportional to the square root of the determinant of the Fisher information matrix) gives prior for the standard deviation of noise proportional to $1/\sigma$. Uniform distribution for $\widehat{\phi}$ and $\vec{\boldsymbol{y}}$.

- The likelihood function is:

$$\mathfrak{L}\big((\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})|\boldsymbol{\Delta} = \widehat{d}(\omega)\big) = \frac{1}{\sigma^{2N+1}} \exp\Big[-\frac{\|\widehat{\mathbf{d}}(\omega) - \widehat{\phi}(\omega)\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}})\|^2}{2\sigma^2}\Big]$$

- The maximum likelihood estimate:

$$(\vec{\boldsymbol{y}}^\star, \sigma^\star, \widehat{\phi}^\star) = \operatorname{argmax} \mathfrak{L}\big((\vec{\boldsymbol{y}}, \sigma, \widehat{\phi})|\boldsymbol{\Delta} = \widehat{d}(\omega)\big)$$

Connection to Bayesian inversion

- The estimate of the noise standard deviation

$$\partial_\sigma \mathfrak{L} = 0 \rightsquigarrow \sigma^\star = \frac{\|\widehat{\mathbf{d}}(\omega) - \widehat{\phi}^\star(\omega)\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}}^\star)\|}{(2N+1)^{1/2}}$$

- Estimation of source pulse

$$\partial_{\widehat{\phi}} \mathfrak{L} = 0 \rightsquigarrow \widehat{\phi}^\star(\omega) = \overline{\widehat{\mathbf{g}}^T(\omega, \vec{\boldsymbol{y}}^\star)}\widehat{\mathbf{d}}(\omega)$$

- Estimate of source location:

$$\vec{\boldsymbol{y}}^\star = \mathrm{argmin}\|\widehat{\mathbf{d}}(\omega) - \widehat{\phi}^\star(\omega)\widehat{\mathbf{g}}(\omega, \vec{\boldsymbol{y}})\|^2 = \mathrm{argmax}\left|\overline{\widehat{\mathbf{g}}^T(\omega, \vec{\boldsymbol{y}})}\widehat{\mathbf{d}}(\omega)\right|^2$$

Thus, $\vec{\boldsymbol{y}}^\star$ is the maximizer of the migration imaging function

$$\mathcal{I}(\vec{\boldsymbol{y}}) = \overline{\widehat{\mathbf{g}}^T(\omega, \vec{\boldsymbol{y}})}\widehat{\mathbf{d}}(\omega) = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \widehat{D}(\omega, \vec{\boldsymbol{x}}_r) e^{-ik(\omega)|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}|}$$

Connection to Bayesian inversion

Conclusion: migration is optimal for additive, Gaussian noise.

• Result extends to signals with some bandwidth, assuming independent noise. The same analysis applies to active arrays.

Note: We used a strong prior: We assumed a single point source and that waves propagate in homogeneous medium.

None of this applies to clutter! Effects are not additive noise!

If we had data over many realizations of clutter and averaged result would not be like in the homogeneous medium!

Average is small $\rightsquigarrow$ randomization of wave (loss of coherence).

PART III: Imaging in random media

- Coherent Interferometric Imaging.
- Analysis in a weak scattering regime in open environments.
- Stronger scattering regimes require an in depth study of waves in random media.
- We only consider coherent imaging (at distances that do not exceed a transport mean free path). Incoherent imaging is very different.
- Boundaries introduce additional difficulties, especially if they are random. Studies in random waveguides: (Alonso, B., Garnier, Issa, Solna, Tsogka; Kohler, Papanicolaou).

• Coherent interferometric (CINT) imaging:

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \sum_{r,r'=1}^{N} \iint_{\mathbb{R}} \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} \widehat{\Phi}\left(\frac{\omega-\omega'}{\Omega}\right) \Psi\left(\frac{\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}}{X}\right) \times \widehat{D}(\omega, \vec{\boldsymbol{x}}_r)\overline{\widehat{D}(\omega', \vec{\boldsymbol{x}}_{r'})} e^{-i\omega\tau_o(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}})+i\omega'\tau_o(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}})}$$

$\widehat{\Phi}$ and Ψ are windows with support Ω and X; $\tau_o(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}) = \frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}|}{c_o}$.

• In general $X = X(\omega)$. If X is constant in bandwidth,

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \sum_{r,r'=1}^{N} \Psi\left(\frac{\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}}{X}\right) \int_{\mathbb{R}} dt\, \Omega\, \Phi\left(t\Omega\right) \times D\left(\tau_o(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}) - t, \vec{\boldsymbol{x}}_r\right) \overline{D\left(\tau_o(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}) - t, \vec{\boldsymbol{x}}_{r'}\right)}$$

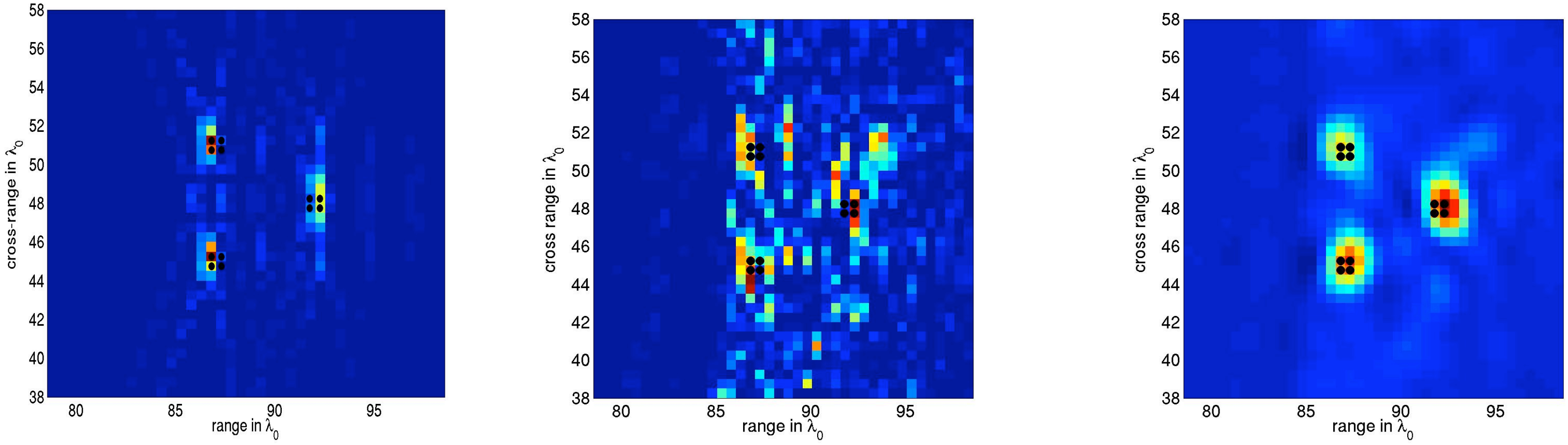
We cross-correlate migrated data over the duration $1/\Omega$ of the time window Φ and over the nearby sensors in the array.

• The imaging function is similar except that it also involves migration from the sources to $\vec{\boldsymbol{y}}$

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \sum_{r,r';s,s'} \iint_{\mathbb{R}} \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} \widehat{\Phi}\left(\frac{\omega-\omega'}{\Omega}\right) \Psi\left(\frac{\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}}{X}\right) \Psi\left(\frac{\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}}{X}\right)$$

$$\widehat{D}(\omega, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)\overline{\widehat{D}(\omega', \vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{x}}_{s'})} e^{-i\omega[\tau_o(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}})+\tau_o(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}})]+i\omega'[\tau_o(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}})+\tau_o(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}})]}$$

Comparison between migration and CINT images



Left: migration in homogeneous medium. Middle: migration in clutter. Right: CINT in clutter (optimal X and Ω).

- Why does this work?
- How should we choose Ω and X?

• Model of the random medium:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_o^2}\left[1 + \sigma\mu\left(\frac{\vec{\boldsymbol{x}}}{\ell}\right)\right]$$

$\mu(\vec{\boldsymbol{x}})$ is mean zero, statistically homogeneous, bounded almost surely, with autocorrelation $\mathcal{R}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \mathbb{E}\left\{\mu(\vec{\boldsymbol{x}})\mu(\vec{\boldsymbol{x}}')\right\}$ satisfying

$$\mathcal{R}(\mathbf{0}) = 1 \quad \text{and} \quad \int_{\mathbb{R}^{n+1}} d\vec{\boldsymbol{x}}\, \mathcal{R}(\vec{\boldsymbol{x}}) = O(1)$$

• The strength of the fluctuations is quantified by σ, and the correlation length is given by ℓ.

• Assume isotropic fluctuations and take for convenience:

$$\mathcal{R}(\vec{\boldsymbol{x}}) = e^{-|\vec{\boldsymbol{x}}|^2/2}$$

CINT analysis in random medium

• We use a simple random travel time model of the random medium effect on the data. We derive and justify it using scaling.

Model applies to clear air turbulence and captures only wave front distortions.

This model is assumed in the adaptive optics methodology.

• For simplicity we focus on the passive array case. The extension to active arrays involves longer calculations.

• Stronger scattering regimes with delay spread are considered in references [2,3,7]. Previous numerical results are in such regime.

The random travel time model

- The travel time is given by Fermat's principle

$$\tau(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = \min_{\Gamma} \int_{\Gamma} \frac{ds}{c(\vec{\boldsymbol{x}}(s))}$$

where Γ are paths from $\vec{\boldsymbol{y}}$ to $\vec{\boldsymbol{x}}$, parametrized by the arclength s.

- It fluctuates randomly about $\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = \frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|}{c_o}$ the travel time in the homogeneous background, used for imaging.

Model: $\widehat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) \approx \frac{e^{i\omega[\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})+\delta\tau(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})]}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|}$ with

$$\delta\tau(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = \frac{\sigma\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})}{2} \int_0^1 ds\, \mu\left(\frac{(1-s)\vec{\boldsymbol{y}}+s\vec{\boldsymbol{x}}}{\ell}\right)$$

Derivation of the random travel time model

- For $\lambda \ll \ell \ll L$ seek solution of Helmholtz's eq. as: $u = \alpha e^{i\omega\tau}$ with excitation modeled as initial condition.

  ◦ Travel time τ solves eikonal equation $\quad |\nabla\tau|^2 = \frac{1}{c^2(\vec{x})}$

  ◦ Amplitude α solves transport equation $\quad 2\nabla\alpha\cdot\nabla\tau + \alpha\Delta\tau = 0$

- Since $\sigma \ll 1$ we can expand formally

$$\alpha = \alpha_0 + \sigma\alpha_1 + \sigma^2\alpha_2 + \ldots \quad \text{and} \quad \tau = \tau_0 + \sigma\tau_1 + \sigma^2\tau_2 + \ldots$$

- Matching powers of σ:

$$|\nabla\tau_0| = \frac{1}{c_o}, \qquad \nabla\tau_0\cdot\nabla\tau_1 = \frac{\mu(\vec{x})}{2c_o^2}, \qquad |\nabla\tau_1|^2 + 2\nabla\tau_0\cdot\nabla\tau_2 = 0$$

$$2\nabla\alpha_0\cdot\nabla\tau_0 + \alpha_0\Delta\tau_0 = 0$$

$$2\nabla\alpha_0\cdot\nabla\tau_1 + 2\nabla\alpha_1\cdot\nabla\tau_0 + \alpha_0\Delta\tau_1 + \alpha_1\Delta\tau_0 = 0$$

Plane wave propagating along range x_3

- To leading order $\alpha_0 = 1$ (normalized constant) and $\tau_0 = \frac{x_3}{c_o}$

- The first order corrections satisfy

$$\partial_{x_3}\tau_1 = \frac{\mu}{2c_o} \quad \text{and} \quad \partial_{x_3}\alpha_1 = -\frac{c_o}{2}\Delta\tau_1 = -\frac{1}{4}\partial_{x_3}\mu - \frac{c_o}{2}\Delta_\perp\tau_1$$

We obtain for $\vec{\boldsymbol{x}} = L\vec{\mathrm{e}}_3$ that

$$\begin{aligned}
\tau_1 &= \frac{1}{2c_o}\int_0^L \mu\left(\frac{s\vec{\boldsymbol{e}}_3}{\ell}\right) ds \\
\alpha_1 &= \frac{1}{4}\mu(\mathbf{0}) - \frac{1}{4}\mu\left(\frac{L\vec{\boldsymbol{e}}_3}{\ell}\right) - \frac{1}{4\ell^2}\int_0^L (L-s)\Delta_\perp\mu\left(\frac{s\vec{\boldsymbol{e}}_3}{\ell}\right) ds
\end{aligned}$$

- $\tau_2 = -\frac{1}{8c_o}\int_0^L \mu^2\left(\frac{s\vec{\boldsymbol{e}}_3}{\ell}\right)ds - \frac{1}{8c_o\ell^2}\int_0^L \left|\int_0^s \nabla_\perp\mu\left(\frac{s'\vec{\boldsymbol{e}}_3}{\ell}\right)ds'\right|^2 ds$

Gaussian statistics of first order corrections

The first order corrections τ_1 and α_1 satisfy:

1. τ_1 has Gaussian statistics with mean $\mathbb{E}[\tau_1] = 0$ and variance

$$\mathbb{E}[\tau_1^2] \approx \frac{\ell L}{2c_o^2}\sqrt{\frac{\pi}{2}} = O\left(\tau_0^2 \frac{\ell}{L}\right)$$

2. α_1 has Gaussian statistics with mean $\mathbb{E}[\alpha_1] = 0$ and variance

$$\mathbb{E}[\alpha_1^2] \approx \frac{2\sqrt{2\pi}}{3}\left(\frac{L}{\ell}\right)^3$$

These formulas are for the autocorrelation $\mathcal{R}(\vec{x}) = e^{-|\vec{x}|^2/2}$. Other expressions of $\mathcal{R}(\vec{x})$ give qualitatively similar results.

Proof

- Begin with

$$\tau_1 = \frac{1}{2c_o} \int_0^L ds\, \mu\left(\frac{s\vec{e}_3}{\ell}\right) = \frac{\sqrt{\ell L}}{2c_o} T(L)$$

where we introduced

$$T(z) = \frac{1}{\sqrt{L\ell}} \int_0^z ds\, \mu\left(\frac{s\vec{e}_3}{\ell}\right)$$

- Note that $T(z)$ satisfies the ODE:

$$\frac{d}{dz} T(z) = \frac{1}{\sqrt{L\ell}} \mu\left(\frac{z\vec{e}_3}{\ell}\right), \qquad T(0) = 0.$$

- Scaled version, with $z = Lz'$ and $\ell = \varepsilon L$,

$$\frac{d}{dz'} T(z') = \frac{1}{\sqrt{\varepsilon}} \mu\left(\frac{z'}{\varepsilon}\vec{e}_3\right), \qquad T(0) = 0.$$

Proof

- Since $\mathbb{E}[\mu] = 0$ we get $\mathbb{E}[T(L)] = 0$.

- For the variance we obtain

$$\mathbb{E}[T^2(L)] = \frac{1}{\ell L} \int_0^L \int_0^L \mathcal{R}_0 \left(\frac{s' - s}{\ell} \right) ds ds'$$

where $\mathcal{R}_0(s' - s) = \mathbb{E}\Big[\mu(s\vec{e}_3)\mu(s'\vec{e}_3)\Big] = e^{-\frac{(s-s')^2}{2}}$.

- Changing variables $s' - s = \tilde{s} = \ell h$ and using that R_0 is even

$$\mathbb{E}[T(L)^2] \;=\; \frac{2}{\ell L} \int_0^L \int_{\tilde{s}}^L \mathcal{R}_0 \left(\frac{\tilde{s}}{\ell} \right) ds \, d\tilde{s} = 2 \int_0^{L/\ell} \mathcal{R}_0(h) \left(1 - h\frac{\ell}{L} \right) dh.$$

- Integrand is bounded by integrable $\mathcal{R}_o(h) \rightsquigarrow$

$$\mathbb{E}[T(L)^2] \stackrel{L/\ell \to \infty}{\longrightarrow} 2 \int_0^\infty \mathcal{R}_o(s) ds = \sqrt{2\pi} \qquad \checkmark$$

Proof that τ_1 is Gaussian

- Gaussianity of τ_1 is straightforward if μ is a Gaussian process.

But Gaussianity holds even when μ is not Gaussian:

Central limit theorem for:

$$\frac{d}{dz'}T(z') = \frac{1}{\sqrt{\varepsilon}}\mu\left(\frac{z'}{\varepsilon}\vec{e}_3\right), \qquad T(0) = 0.$$

Recall that $z' = L/z$ and drop primes for convenience.

Toy model: $\mu(z\vec{\mathbf{e}}_3) = \nu(z) = \sum_{j=1}^{\infty} \nu_j 1_{[j-1,j]}(z)$ with ν_j independent, random variables with mean zero and std deviation σ_ν.

$$T(z) = \sqrt{\varepsilon}\int_0^{z/\varepsilon} ds\, \nu(s) = \sqrt{\varepsilon}\left(\sum_{j=1}^{\lfloor z/\varepsilon \rfloor} \nu_j\right) + \sqrt{\varepsilon}\int_{\lfloor z/\varepsilon \rfloor}^{z/\varepsilon} ds\, \nu(s).$$

Proof that τ_1 is Gaussian

- By central limit theorem,

$$\frac{1}{\sqrt{\lfloor z/\varepsilon \rfloor}} \sum_{j=1}^{\lfloor z/\varepsilon \rfloor} \nu_j \to \mathcal{N}(0, \sigma_\nu^2)$$

in distribution. Moreover,

$$\sqrt{\varepsilon}\sqrt{\lfloor z/\varepsilon \rfloor} \to \sqrt{z}$$

so the first term in $T(z)$ converges in distribution* to $(\sigma_\nu W_z)_{z \geq 0}$, where W_z is Brownian motion (Gaussian process with mean zero and covariance $\mathbb{E}[W_z W_{z'}] = \min(z, z')$.)

- For the second term in $T(z)$ we have

$$\sqrt{\varepsilon} \int_{\lfloor z/\varepsilon \rfloor}^{z/\varepsilon} ds\, \nu(s) = \sqrt{\varepsilon}\Big(z/\varepsilon - \lfloor z/\varepsilon \rfloor\Big)\nu_{\lfloor z/\varepsilon \rfloor} \to 0$$

*Note that $z = 1$ at the range of interest and σ_ν can be chosen to match the variance calculated previously.

Proof

• The result extends to fairly general fluctuations: stationary, mean zero, ergodic in z (mixing).

Diffusion limit theorem: Kohler, Papanicolaou, Varadhan.

• Variance calculation of α_1 and proof of Gaussianity is similar.

The second order travel time correction

- The second order correction τ_2 of the travel time

$$\tau_2 = -\frac{1}{8c_o}\int_0^L \mu^2\Big(\frac{s\vec{e}_3}{\ell}\Big)ds - \frac{1}{8c_o\ell^2}\int_0^L \Big|\int_0^s \nabla_\perp\mu\Big(\frac{s'\vec{e}_3}{\ell}\Big)ds'\Big|^2 ds.$$

- It satisfies:

$$\mathbb{E}[\tau_2] \approx -\frac{\sqrt{2\pi}}{16}\tau_0\frac{L}{\ell}$$

and

$$\mathrm{Var}(\tau_2) \approx \frac{\pi}{48}\tau_0^2\Big(\frac{L}{\ell}\Big)^2$$

Validity of the random travel time model

- We can approximate the phase as $\omega\tau \approx \omega\tau_0 + \omega\sigma\tau_1$ if

$$\omega\sigma^2\tau_2 \sim \sigma^2\frac{L^2}{\lambda l} \ll \omega\sigma\tau_1 \sim \sigma\frac{\sqrt{L\ell}}{\lambda} \quad \rightsquigarrow \quad \sigma \ll \left(\frac{\ell}{L}\right)^{3/2}$$

and if $\sigma^2\frac{L^2}{\lambda\ell} \ll 1$.

- Can approximate amplitude $\alpha \approx \alpha_o$ because: $\sigma\alpha_1 \sim \sigma\left(\frac{L}{\ell}\right)^{3/2} \ll 1$

- The travel time fluctuations are visible if

$$\omega\sigma\tau_1 \sim \omega\sigma\tau_0\sqrt{\frac{\ell}{L}} \sim \sigma\frac{\sqrt{L\ell}}{\lambda} \gtrsim 1$$

- Waves generated from point source analyzed similarly with a more involved derivation of α_o.

Model of Green's function

$$\widehat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \approx \frac{e^{i\omega[\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})+\sigma\tau_1(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})]}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|} = \widehat{G}_o(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) e^{i\omega\sigma\tau_1(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})}$$

Random phase:

$$\omega\sigma\tau_1(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) = \frac{\sigma k(\omega)|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|}{2} \int_0^1 ds\, \mu\left(\frac{(1-s)\vec{\boldsymbol{y}}+s\vec{\boldsymbol{x}}}{\ell}\right)$$

is approximately Gaussian with mean 0 and standard deviation

$$\mathfrak{S} = \frac{(2\pi)^{1/4}}{2} \sigma k(\omega) \sqrt{\ell|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|} \sim \sigma \frac{\sqrt{\ell L}}{\lambda} \gg 1$$

Mean of the Green's function for large distortion

- The expectation of the random factor of $\widehat{G}$ is

$$\mathbb{E}\Big[e^{i\omega\sigma\tau_1(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})}\Big] = \int_{\mathbb{R}} ds\, e^{is}\frac{e^{-\frac{s^2}{2\mathfrak{S}^2}}}{\sqrt{2\pi}\mathfrak{S}} = e^{-\frac{\omega^2\mathfrak{S}^2}{2}}$$

- This gives

$$\mathbb{E}\Big[\widehat{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},\omega)\Big] \approx \widehat{G}_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},\omega)e^{-\frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}|}{\mathcal{S}(\omega)}} \approx 0$$

- Scattering mean free path $\mathcal{S}(\omega) = \frac{8}{\sqrt{2\pi}\sigma^2 k^2(\omega)\ell} \ll L$

Moments of the Green's function for large distortion

• Note: $\mathbb{E}[\widehat{G}] \approx 0$ but $\mathbb{E}[|\widehat{G}|^2] = |\widehat{G}_o|^2$ so the wave is randomized.

• The decay of the second moments over location and frequency offsets describes statistical decorrelation

$$\mathbb{E}\left[\widehat{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},\omega)\overline{\widehat{G}(\vec{\boldsymbol{x}}',\vec{\boldsymbol{y}}',\omega')}\right] \approx \widehat{G}_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},\omega)\overline{\widehat{G}_o(\vec{\boldsymbol{x}}',\vec{\boldsymbol{y}}',\omega')} \times e^{-\frac{|y_3-y_3'|}{\mathcal{S}(\omega)}-\frac{(\omega-\omega')^2}{2\Omega_d^2}-\frac{|\boldsymbol{y}-\boldsymbol{y}'|^2+(\boldsymbol{y}-\boldsymbol{y}')\cdot(\boldsymbol{x}-\boldsymbol{x}')+|\boldsymbol{x}-\boldsymbol{x}'|^2}{2X_d(\omega)^2}}$$

Decoherence frequency: $\Omega_d = \frac{2c_o}{(2\pi)^{1/4}\sigma\sqrt{\ell L}} = \omega_o\sqrt{\frac{\mathcal{S}(\omega_o)}{2L}} \ll \omega_o,$

Decoherence length: $X_d(\omega) = \sqrt{3}\ell\,\frac{\Omega_d}{\omega} \ll \ell.$

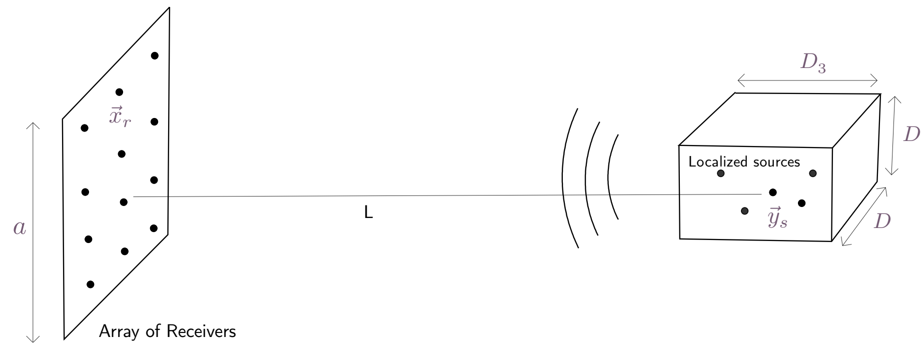
PART IV: Analysis of imaging functions

• We analyze both migration and CINT using the random travel time model with large travel time fluctuations.

• To understand the focusing, we study the mean of the imaging functions.

• To quantify robustness, we calculate SNR at peaks: (mean divided by standard deviation).

Setup for analysis of CINT and migration imaging



- Array is orthogonal to range direction and centered at origin.

- Paraxial regime (as in analysis of the normal operator):

$$k(\omega)|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}| \approx k(\omega) y_3 + k_o \left(\frac{|\boldsymbol{x}|^2}{2L} - \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{L} + \frac{|\boldsymbol{y}|^2}{2L} \right)$$

$$\frac{1}{4\pi|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}|} \approx \frac{1}{4\pi L}$$

Analysis of Kirchhoff migration (KM) with passive arrays

- To study resolution we estimate

$$\mathbb{E}[\mathcal{I}^{\mathsf{KM}}(\vec{\boldsymbol{y}})] = \frac{a^2}{N}\sum_{r=1}^{N} D(\tau_o(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}),\vec{\boldsymbol{x}}_r) \approx \int \frac{d\omega}{2\pi}\int_{\mathcal{A}} d\boldsymbol{x}\,\mathbb{E}[\widehat{D}(\omega,\vec{\boldsymbol{x}})]e^{-i\omega\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})}$$

where $\vec{\boldsymbol{x}} = (\boldsymbol{x},0) \in \mathcal{A}$.

- Data $\widehat{D}(\omega,\vec{\boldsymbol{x}}) = \widehat{f}(\omega)\int_{\mathbb{R}} d\vec{\boldsymbol{y}}\,\rho(\vec{\boldsymbol{y}})\widehat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ (we neglect noise)

$$\mathbb{E}[\mathcal{I}^{\mathsf{KM}}(\vec{\boldsymbol{y}})] \approx \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}'\,\rho(\vec{\boldsymbol{y}}')\int\frac{d\omega}{2\pi}\widehat{f}(\omega)\int_{\mathcal{A}} d\boldsymbol{x}\,\widehat{G}_o(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}')e^{-i\omega\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})-\frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}'|}{\mathcal{S}(\omega)}}$$

- To compute SNR we need the variance.

- Calculation like that of kernel of the normal operator gives:

$$\mathbb{E}[\mathcal{I}^{\mathrm{KM}}(\vec{\boldsymbol{y}})] \approx \int_{\mathbb{R}^3} d\vec{\boldsymbol{y}}' \, \rho(\vec{\boldsymbol{y}}')\mathcal{K}^{\mathrm{KM}}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}')$$

with kernel

$$\mathcal{K}^{\mathrm{KM}}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}') \approx \frac{e^{-\frac{L}{\mathcal{S}(\omega_o)}}}{4\pi L} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \widehat{f}(\omega) e^{ik(\omega)(y_3'-y_3)} \times \mathcal{U}\Big(\frac{k_o a(y_1'-y_1)}{L}, \frac{k_o a^2}{L^2}(y_3'-y_3)\Big) \times \mathcal{U}\Big(\frac{k_o a \cos\theta (y_2'-y_2)}{L}, \frac{k_o (a\cos\theta)^2 (y_3'-y_3)}{L^2}\Big)$$

where $\mathcal{U}(\alpha,\gamma) = \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \, \exp(-i\alpha s - i\gamma s^2/2) =$ Fresnel integral.

- Focusing as in homogeneous medium but very small $\sim e^{-L/\mathcal{S}(\omega_o)}$

Analysis of Kirchhoff migration (KM) with passive arrays

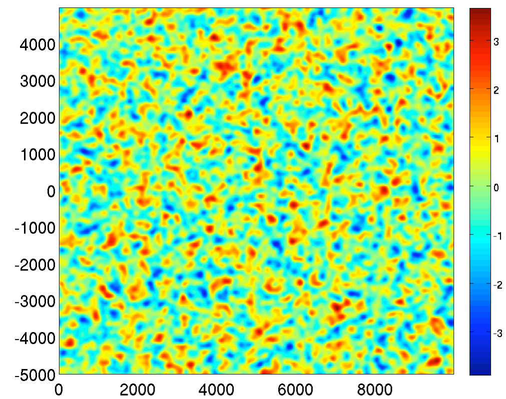
• The variance of the imaging function involves second moments $\mathbb{E}[\widehat{G}\overline{\widehat{G}}]$ which do not decay:

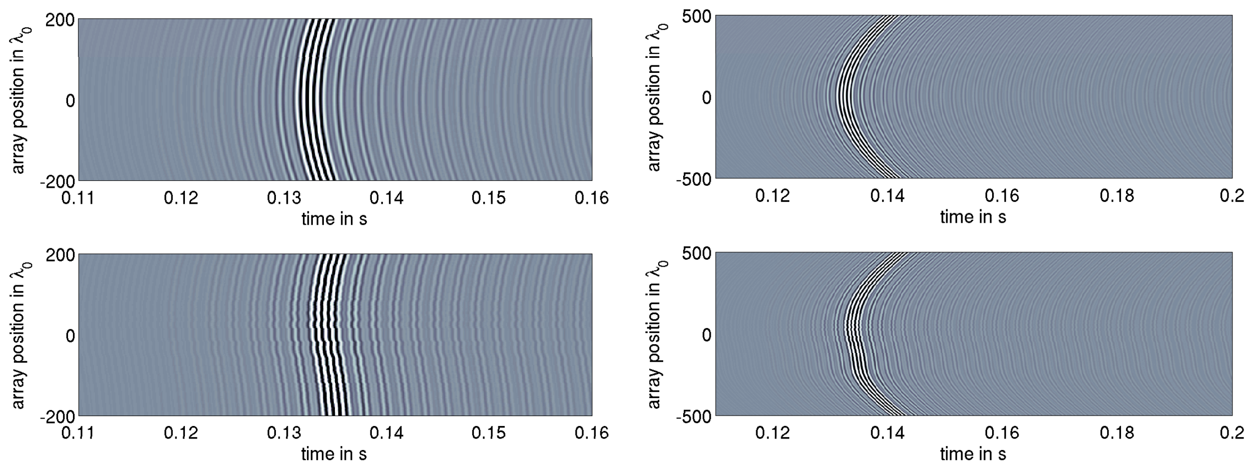
$$\mathrm{SNR}^{\mathrm{KM}}(\vec{\boldsymbol{y}}) = \frac{\left|\mathbb{E}\left[\mathcal{I}^{\mathrm{KM}}(\vec{\boldsymbol{y}})\right]\right|}{\mathrm{StD}\left[\mathcal{I}^{\mathrm{KM}}(\vec{\boldsymbol{y}})\right]} \sim e^{-L/\mathcal{S}(\omega_o)} \ll 1.$$

• The migration function is not statistically stable: In practice, we will see that the location of the peak dances around.

We do not see speckle with the random travel time model. We need stronger fluctuations for that.

KM: Illustration with numerical simulations*



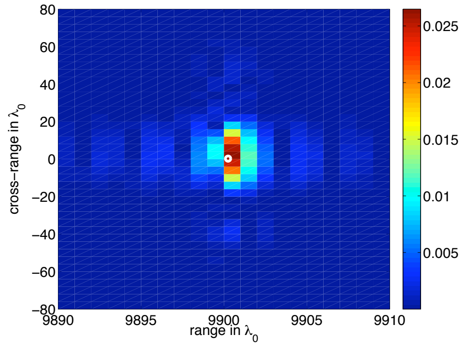


Top: traces in homogeneous medium. Bottom: traces with wavefront distortions. Left: the recordings over the array of aperture $a = 4\ell$.

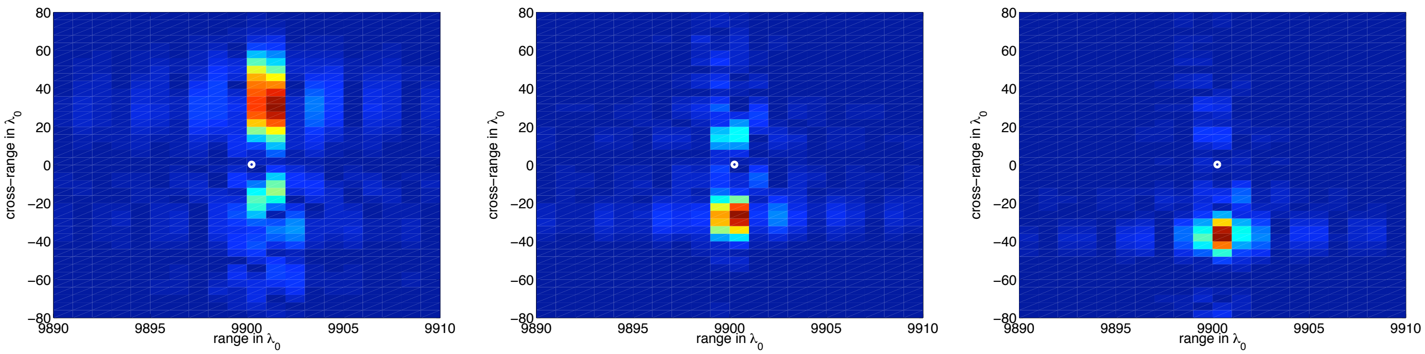
*$N = 101$, $a = 4\ell$, $\lambda_o = 2$cm, $\ell = 100\lambda_o$, $L = 99\ell$, bandwidth $[125, 175]$kHz.

KM: Illustration with numerical simulations

	$\sigma = 4e-4$
mean(KM)	0.68
std(KM)	25.79
SNR(KM)	0.0265



- The mean is not observable. The image peak dances around.



The CINT imaging function is

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \widehat{\Phi}\left(\frac{\omega-\omega'}{\Omega}\right) \int_{\mathcal{A}} d\boldsymbol{x} \int_{\mathcal{A}} d\boldsymbol{x}' \Psi\left(\frac{\boldsymbol{x}-\boldsymbol{x}'}{X}\right)$$
$$\widehat{D}(\omega,\vec{\boldsymbol{x}})\overline{\widehat{D}(\omega',\vec{\boldsymbol{x}}')} e^{-i\omega\tau_o(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})+i\omega'\tau_o(\vec{\boldsymbol{x}}',\vec{\boldsymbol{y}})}$$

with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, 0) \in \mathcal{A}$ and similar for $\vec{\boldsymbol{x}}'$.

- Simplify: single point source at $\vec{\boldsymbol{y}}^\star = (\mathbf{0}, L)$
- To obtain nicer expressions, we assume Gaussian windows

$$\begin{aligned} \Phi(t) &= e^{-t^2/2} \quad \rightsquigarrow \widehat{\Phi}(h) = \sqrt{2\pi} e^{-h^2/2} \\ \Psi\big(\mathbf{r}\big) &= e^{-|\mathbf{r}|^2/2} \end{aligned}$$

and Gaussian pulse: $f(t) = e^{-i\omega_o t} e^{-\frac{B^2 t^2}{2}} \rightsquigarrow \widehat{f}(\omega) = \frac{\sqrt{2\pi}}{B} e^{-\frac{(\omega-\omega_o)^2}{2B^2}}$

CINT as smoothed Wigner transform

With the center and difference variables $\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}+\boldsymbol{x}'}{2}$, $\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}'$, $\bar{\omega} = \frac{\omega+\omega'}{2}$ and $\widetilde{\omega} = \omega - \omega'$ we get by evaluating Gaussian integrals:*

The CINT function is given by Wigner transform

$$W(\bar{\omega}, \bar{\boldsymbol{x}}; \boldsymbol{\kappa}, T) = \int_{\mathbb{R}^2} d\widetilde{\boldsymbol{x}} \int_{\mathbb{R}} d\widetilde{\omega}\, e^{i\boldsymbol{\kappa}\cdot\widetilde{\boldsymbol{x}} - i\widetilde{\omega}T} \widehat{D}\left(\bar{\omega} + \frac{\widetilde{\omega}}{2}, \left(\bar{\boldsymbol{x}} + \frac{\widetilde{\boldsymbol{x}}}{2}, 0\right)\right) \times \overline{\widehat{D}\left(\bar{\omega} - \frac{\widetilde{\omega}}{2}, \left(\bar{\boldsymbol{x}} - \frac{\widetilde{\boldsymbol{x}}}{2}, 0\right)\right)}$$

smoothed over all its arguments

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \frac{\Omega X^2}{(2\pi)^4} \int_{\mathcal{A}} d\bar{\boldsymbol{x}} \int_{\mathbb{R}} d\bar{\omega} \int_{\mathbb{R}^2} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} dT\; W(\bar{\omega}, \bar{\boldsymbol{x}}; \boldsymbol{\kappa}, T) \, \Phi\left[\Omega(T - \tau((\bar{\boldsymbol{x}}, 0), \vec{\boldsymbol{y}}))\right] \widehat{\Psi}\left[X\left(\boldsymbol{\kappa} - k_o \frac{(\boldsymbol{y} - \boldsymbol{x})}{L}\right)\right]$$

*For small enough bandwidth so that $X = X(\omega_o)$. See reference [1].

- Using random travel time model and Gaussian pulse,

$$W(\bar{\omega}, \bar{\boldsymbol{x}}; \boldsymbol{\kappa}, T) \approx \frac{e^{-\frac{(\omega-\omega_o)^2}{B^2}}}{8\pi B^2 L^2} \int_{\mathbb{R}^2} d\tilde{\boldsymbol{x}}\, e^{i\tilde{\boldsymbol{x}}\cdot\boldsymbol{\kappa} + i\bar{\omega}[\tilde{\tau}(\bar{\boldsymbol{x}},\tilde{\boldsymbol{x}}) + \delta\tilde{\tau}(\bar{\boldsymbol{x}},\tilde{\boldsymbol{x}})]} \int_{\mathbb{R}} d\tilde{\omega}\, e^{-\frac{\tilde{\omega}^2}{4B^2} + i\tilde{\omega}[\bar{\tau}(\bar{\boldsymbol{x}},\tilde{\boldsymbol{x}}) + \delta\bar{\tau}(\bar{\boldsymbol{x}},\tilde{\boldsymbol{x}}) - T]}.$$

where $\delta\tau(\boldsymbol{x}) = \sigma\tau_1(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}^\star)$ for $\vec{\boldsymbol{x}} = (\boldsymbol{x}, 0) \in \mathcal{A}$ and

$$\begin{aligned}
\delta\tilde{\tau}(\bar{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) &= \delta\tau(\bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}}/2) - \delta\tau(\bar{\boldsymbol{x}} - \tilde{\boldsymbol{x}}/2) \\
\delta\bar{\tau}(\bar{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) &= \frac{1}{2}\left[\delta\tau(\bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}}/2) + \delta\tau(\bar{\boldsymbol{x}} - \tilde{\boldsymbol{x}}/2)\right] \\
\tilde{\tau}(\bar{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) &= \tau_o((\bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}}/2, 0), \vec{\boldsymbol{y}}_\star) - \tau_o((\bar{\boldsymbol{x}} - \tilde{\boldsymbol{x}}/2, 0), \vec{\boldsymbol{y}}_\star) \\
\bar{\tau}(\bar{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) &= \frac{1}{2}\left[\tau_o((\bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}}/2, 0), \vec{\boldsymbol{y}}_\star) + \tau_o((\bar{\boldsymbol{x}} - \tilde{\boldsymbol{x}}/2, 0), \vec{\boldsymbol{y}}_\star)\right].
\end{aligned}$$

- We would like to expand in $\tilde{\boldsymbol{x}}$ but we need to restrict it first.

- Smoothing $\int_{\mathbb{R}^n} d\boldsymbol{\kappa}\, W(\bar{\omega}, \bar{\boldsymbol{x}}; \boldsymbol{\kappa}, T) e^{-\frac{X^2}{2}\left|\boldsymbol{\kappa} - k_o \frac{(\boldsymbol{y}-\bar{\boldsymbol{x}})}{L}\right|^2}$ restricts $\widetilde{\boldsymbol{x}}$:

$$\int_{\mathbb{R}^2} d\boldsymbol{\kappa}\; e^{i\boldsymbol{\kappa}\cdot\widetilde{\boldsymbol{x}} - \frac{X^2}{2}\left|\boldsymbol{\kappa} - k_o \frac{(\boldsymbol{y}-\bar{\boldsymbol{x}})}{L}\right|^2} = \frac{2\pi}{X^2} e^{-\frac{|\widetilde{\boldsymbol{x}}|^2}{2X^2} + ik_o \frac{\widetilde{\boldsymbol{x}}\cdot(\boldsymbol{y}-\bar{\boldsymbol{x}})}{L}}$$

- This allows expansion in $\widetilde{\boldsymbol{x}}$. Integrating in $\widetilde{\boldsymbol{x}}$ and $\widetilde{\omega}$ $\rightsquigarrow$

$$\int_{\mathbb{R}^2} d\boldsymbol{\kappa}\; W(\bar{\omega}, \bar{\boldsymbol{x}}; \boldsymbol{\kappa}, T) e^{-\frac{X^2}{2}\left|\boldsymbol{\kappa} - \frac{\omega_o}{c_o} \frac{(\boldsymbol{y}-\bar{\boldsymbol{x}})}{L}\right|^2} \approx \frac{\pi^{3/2} e^{-\frac{(\bar{\omega}-\omega_o)^2}{B^2}}}{BL^2}$$
$$e^{-\frac{1}{2}\left(\frac{k_o X}{L}\right)^2 |\boldsymbol{y} + c_o L \nabla_{\boldsymbol{x}} \delta\tau(\boldsymbol{x})|^2 - B^2[\tau_o((\bar{\boldsymbol{x}},0),\vec{\boldsymbol{y}}_\star) + \delta\tau(\bar{\boldsymbol{x}}) - T]^2}$$

Note: This is random, with peak dancing around in $\boldsymbol{y}$ and T. To stabilize it, we integrate in CINT over $\bar{\boldsymbol{x}}$ and T. The integral over $\bar{\omega}$ not so important, because there is no coda in the model.

Straightforward integration and use of moment formulas give

$$\mathbb{E}[\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}})] \approx \frac{a^4}{8\pi L^2}\left(\frac{a^2}{X_d^2}+\frac{a^2}{X^2}\right)^{-1}\frac{\Omega/B}{\sqrt{2+2(\Omega/\Omega_d)^2+(\Omega/B)^2}}$$

$$\times e^{-\frac{B^2}{2\left[1+2(B/\Omega_d)^2+2(B/\Omega)^2\right]}\left(\frac{y_3-L}{c_o}\right)^2-\frac{X_d^2X^2}{2(X_d^2+X^2)}\left(k_o\frac{|\boldsymbol{y}|}{L}\right)^2}$$

Range resolution is $|y_3-L| \lesssim \frac{c_o}{B}\left[1+2(B/\Omega_d)^2+2(B/\Omega)^2\right]^{\frac{1}{2}}$

∘ If we choose $\Omega \gg \Omega_d$, it will make no difference in the focus but the SNR will be lower (smoothing of $W(\omega, x; \boldsymbol{\kappa}, T)$ in T is by integration over window $\sim 1/\Omega$).

∘ If we choose $\Omega \ll \Omega_d$ we get too much blur $|y_3-L| \lesssim \frac{c_o}{\Omega}$.

The mean of CINT

Cross-range resolution is worse than for $\mathbb{E}(\mathcal{I}^{\mathrm{KM}})$

$$|\boldsymbol{y}| \lesssim \frac{\lambda_o L}{a}\sqrt{\frac{a^2}{X_d^2} + \frac{a^2}{X^2}}$$

◦ If we choose $X \gg X_d$ it plays no role in resolution and SNR is low, because we do not smooth enough over vector $\boldsymbol{\kappa}$.

◦ If we choose $X \ll X_d$ we smooth too much and blur.

Optimal choices are $X = X_d$ and $\Omega = \Omega_d$.

- The SNR follows after a straightforward but long calculation*

$$\mathrm{SNR}^{\mathrm{CINT}}(\vec{\boldsymbol{y}}_\star) = \frac{\left|\mathbb{E}\left[\mathcal{I}^{\mathrm{CINT}}(\vec{\boldsymbol{y}}_\star)\right]\right|}{\mathrm{StD}\left[\mathcal{I}^{\mathrm{CINT}}(\vec{\boldsymbol{y}}_\star)\right]} \approx \frac{\sqrt{2}\left(1 + \frac{1}{(\Omega/\Omega_d)^2} + \frac{1}{2(B/\Omega_d)^2}\right)}{C_{\mathcal{A},\ell}}$$

where

$$C^2_{\mathcal{A},\ell} = \frac{\pi}{2a^4}\int_{\mathcal{A}} d\boldsymbol{x} \int_{\mathcal{A}} d\boldsymbol{x}' \left[\frac{\mathrm{erf}\left(\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{\sqrt{2}\ell}\right)}{(|\boldsymbol{x}-\boldsymbol{x}'|/\ell)}\right]^2 = O\left(\frac{\ell^2}{a^2}\right)$$

∘ As random medium effect grows i.e., $\Omega_d = \omega_o\sqrt{\frac{\mathcal{S}}{2L}} \to 0$, SNR becomes independent of Ω_d if $\Omega \sim \Omega_d$.

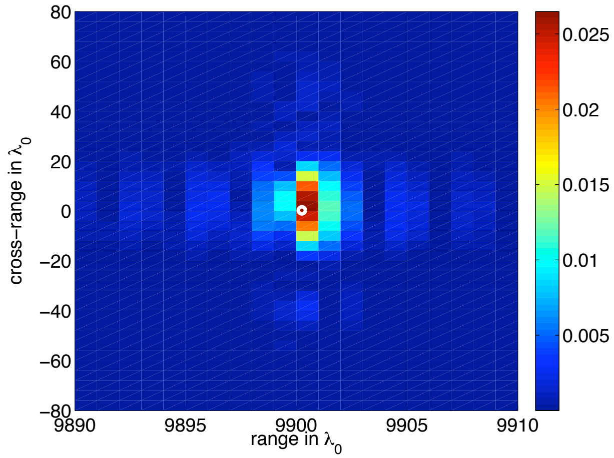
∘ The aperture plays a role in the SNR which is large for $a \gtrsim \ell$.

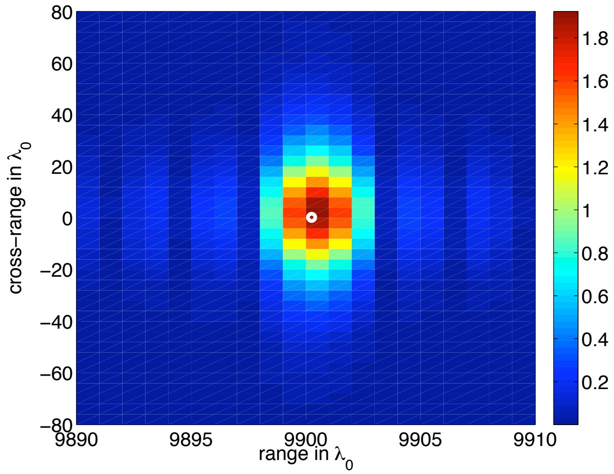
∘ Bandwidth doesn't play a role in this regime. It is very important in stronger regimes, with significant delay spread.

*The calculation assumes $\Omega \lesssim \Omega_d$ and $X \lesssim X_d$. See reference [1].

CINT: Illustration with numerical simulations

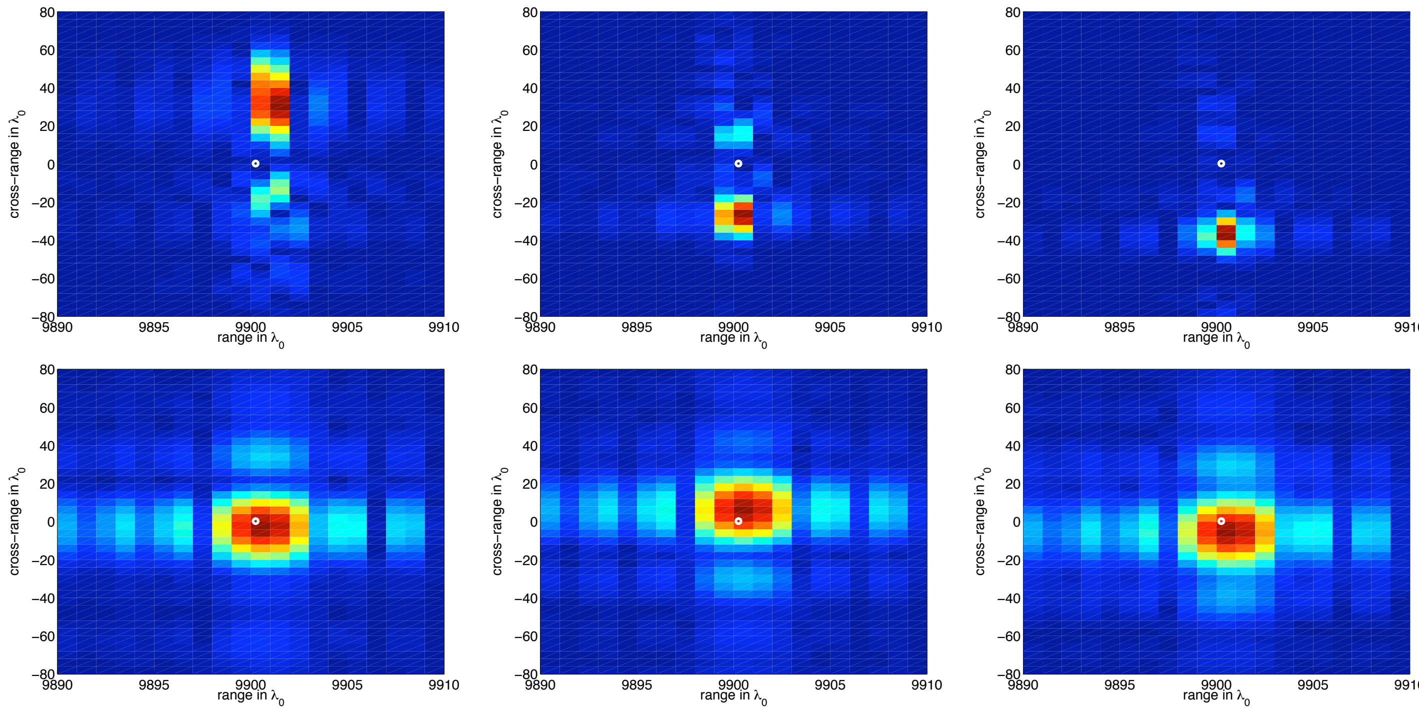
	$\sigma = 4e-4$	$\sigma = 1e-3$
mean(CINT)	0.0019	6.5e-4
std(CINT)	9.87e-4	6.16e-4
SNR(CINT)	1.92	1.06





Left $\mathbb{E}[\mathcal{I}^{\mathrm{KM}}]$ (not observable) and right $\mathbb{E}[\mathcal{I}^{\mathrm{CINT}}]$. The CINT image is blurry but the SNR is large.

Illustration of stability of CINT and instability of KM



Top $\mathcal{I}^{\text{KM}}$ in three realizations of the medium. Bottom $\mathcal{I}^{\text{CINT}}$ in the same realizations of the medium

CINT improvements

- Deblur the CINT images using optimization. See reference [5]. Sparsity enhancing optimization can give sharp images but the kernel must be known via prior calibration and the sources cannot be closer than the CINT resolution.

- Use data processing for partial removal of random media effects. See reference [7] and other works by Aubry, Derode.

- A slightly changed version of CINT shows that we can get some more info (reference [6]).

- Assume decoherence length constant in the bandwidth. Recall:

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \sum_{r,r'=1}^{N} \Psi\left(\frac{\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}}{X}\right) \mathcal{C}\Big(\bar{\tau}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}}), \tilde{\tau}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r'\Big)$$

with cross-correlations

$$\mathcal{C}(\bar{t}, \tilde{t}, \vec{\boldsymbol{x}}_t, \vec{\boldsymbol{x}}_r') = \int_{\mathbb{R}} dt\, \Omega\, \Phi\,(t\Omega)\, D\left(\bar{t} - \tilde{t}/2 - t, \vec{\boldsymbol{x}}_r\right) \overline{D\left(\bar{t} + \tilde{t}/2 - t, \vec{\boldsymbol{x}}_{r'}\right)}$$

and

$$\bar{\tau}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}}) = \frac{1}{2}\Big[\tau_o(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}) + \tau_o(\vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}})\Big]$$
$$\tilde{\tau}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}}) = \tau_o(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}) - \tau_o(\vec{\boldsymbol{x}}_r', \vec{\boldsymbol{y}})$$

- We analyzed this for one point source. If we have N_s sources,

$$\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \sum_{s,s'=1}^{N_s} \rho(\vec{\boldsymbol{y}}_s)\rho(\vec{\boldsymbol{y}}_{s'})\mathcal{K}^{\text{CINT}}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}_s, \vec{\boldsymbol{y}}_{s'})$$

- Consider instead:

$$\mathcal{I}(\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}') = \sum_{r,r'=1}^{N} \Psi\Big(\frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}|}{X}\Big)\mathcal{C}(\bar{\tau}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}'),\tilde{\tau}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}'),\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'})$$

with

$$\bar{\tau}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_r',\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}') = \frac{1}{2}\Big[\tau_o(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}) + \tau_o(\vec{\boldsymbol{x}}_r',\vec{\boldsymbol{y}}')\Big]$$
$$\tilde{\tau}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_r',\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}') = \tau_o(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}) - \tau_o(\vec{\boldsymbol{x}}_r',\vec{\boldsymbol{y}}')$$

- This gives

$$\mathcal{I}(\vec{\boldsymbol{y}}) = \sum_{s,s'=1}^{N_s} \rho(\vec{\boldsymbol{y}}_s)\rho(\vec{\boldsymbol{y}}_{s'})\mathcal{K}(\vec{\boldsymbol{y}},\vec{\boldsymbol{y}}',\vec{\boldsymbol{y}}_s,\vec{\boldsymbol{y}}_{s'})$$

- Continuum aperture approximation and Gaussian apodization: $\exp\Big[-\frac{|\vec{\boldsymbol{x}}|^2}{2(a/6)^2}\Big]$ for $\vec{\boldsymbol{x}} \in \mathcal{A}$.

- Let $\vec{\boldsymbol{y}} = (\boldsymbol{y}, y_3)$ and similar for $\vec{\boldsymbol{y}}'$. Introduce

$$\bar{\boldsymbol{y}} = \frac{\boldsymbol{y}+\boldsymbol{y}'}{2}, \quad \tilde{\boldsymbol{y}} = \boldsymbol{y}-\boldsymbol{y}', \quad \bar{y}_3 = \frac{y_3+y_3'}{2}, \quad \tilde{y}_3 = y_3 - y_3'.$$

- Let $\vec{\boldsymbol{y}}_s = (\boldsymbol{y}_s, y_{s,3})$ and define $\bar{\boldsymbol{y}}_{ss'}$, $\tilde{\boldsymbol{y}}_{ss'}, \bar{y}_{ss',3}$ and $\tilde{y}_{ss',3}$.

- With $\frac{1}{\Omega_e^2} = \frac{1}{\Omega^2} + \frac{1}{\Omega_d^2} + \frac{1}{4B^2}$; $\frac{1}{X_e^2} = \frac{1}{X^2} + \frac{1}{X_d^2} + \frac{1}{4(a/6)^2}$; $\gamma, \gamma_1 = O(1)$:

$$|\mathcal{K}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}', \vec{\boldsymbol{y}}_s, \vec{\boldsymbol{y}}_{s'})| \sim \exp\left\{ -\frac{|\tilde{\boldsymbol{y}}_{ss'}|^2}{2\gamma X_d^2} - \frac{|\tilde{y}_{ss',3} - \tilde{y}_3|^2}{2(c_o/B)^2} - \frac{|\tilde{\boldsymbol{y}}_{ss'} - \tilde{\boldsymbol{y}}|^2}{2[\gamma_1 L/(k_o a)]^2} \right.$$
$$\left. - \frac{|\bar{\boldsymbol{y}}_{ss'} - \bar{\boldsymbol{y}}|^2}{2[L/(k_o X_e)]^2} - \frac{(|(\bar{\boldsymbol{y}}_{ss'}, \bar{y}_{ss',3})| - |(\bar{\boldsymbol{y}}, \bar{y}_3)|)^2}{2(c_o/\Omega_e)^2} \right\},$$

- Excellent resolution for offsets and CINT resolution for centers. Only nearby sources interact.

CINT with shape recognition

Step 1: Calculate $\mathcal{I}^{\text{CINT}}(\vec{\boldsymbol{y}}) = \mathcal{I}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}})$ to identify the peaks. This can be done on coarse mesh, according to CINT resolution.

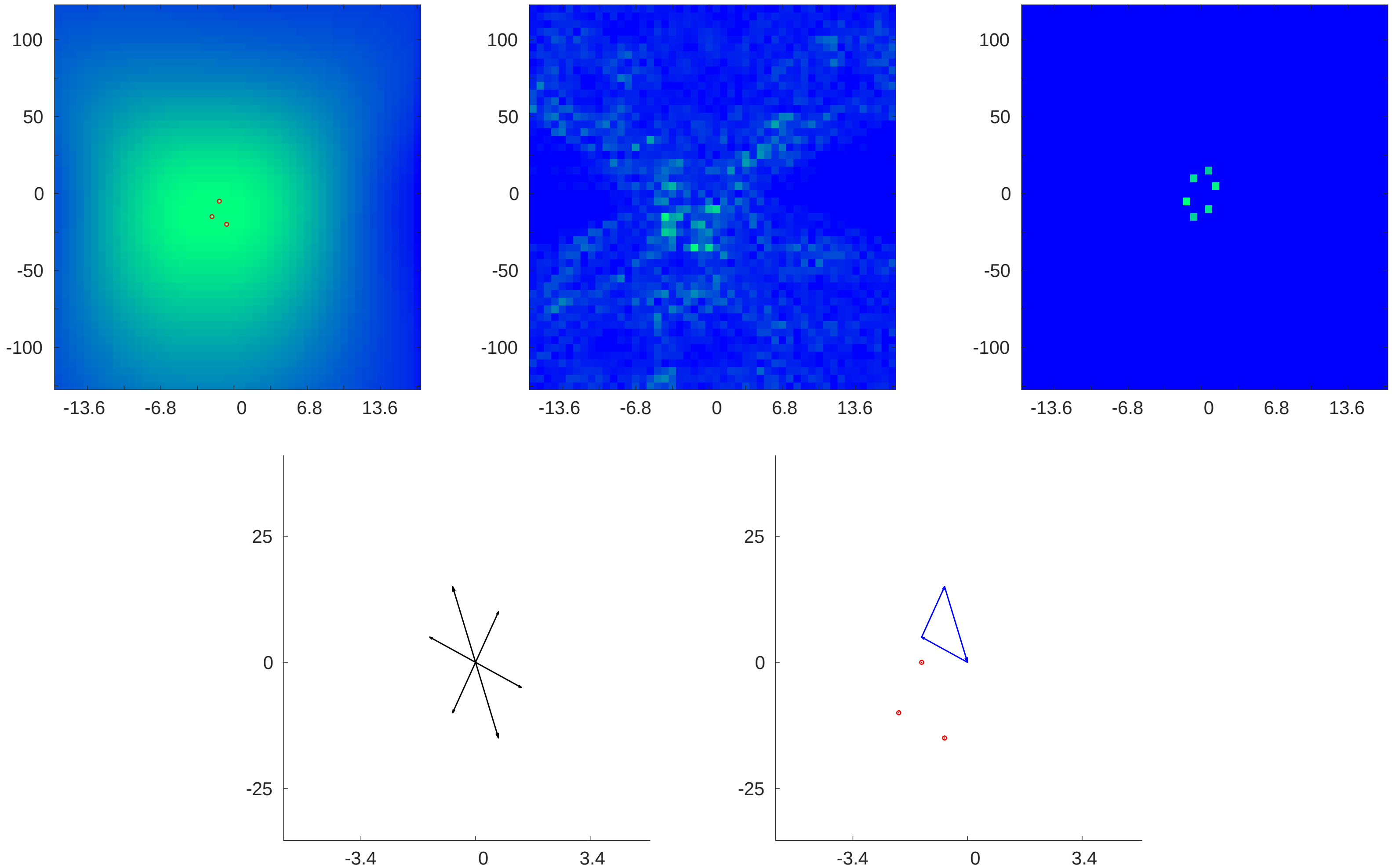
Step 2: Choose one peak, centered at $\vec{\boldsymbol{z}}_o$ and calculate $\mathcal{I}(\vec{\boldsymbol{z}}_o, \vec{\boldsymbol{y}})$ for $\vec{\boldsymbol{y}}$ in the support of the peak on mesh with pixel size $\lambda_o L/a$ in cross-range and c_o/B in range.

Step 3: Identify the peaks of $\mathcal{I}(\vec{\boldsymbol{z}}_o, \vec{\boldsymbol{y}})$. These are points $\vec{\boldsymbol{z}}_j$ satisfying

$$\vec{\boldsymbol{z}}_j - \vec{\boldsymbol{z}}_o \in \{\vec{\boldsymbol{y}}_s - \vec{\boldsymbol{y}}_{s'} : s, s' = 1, \ldots, N_s\}$$

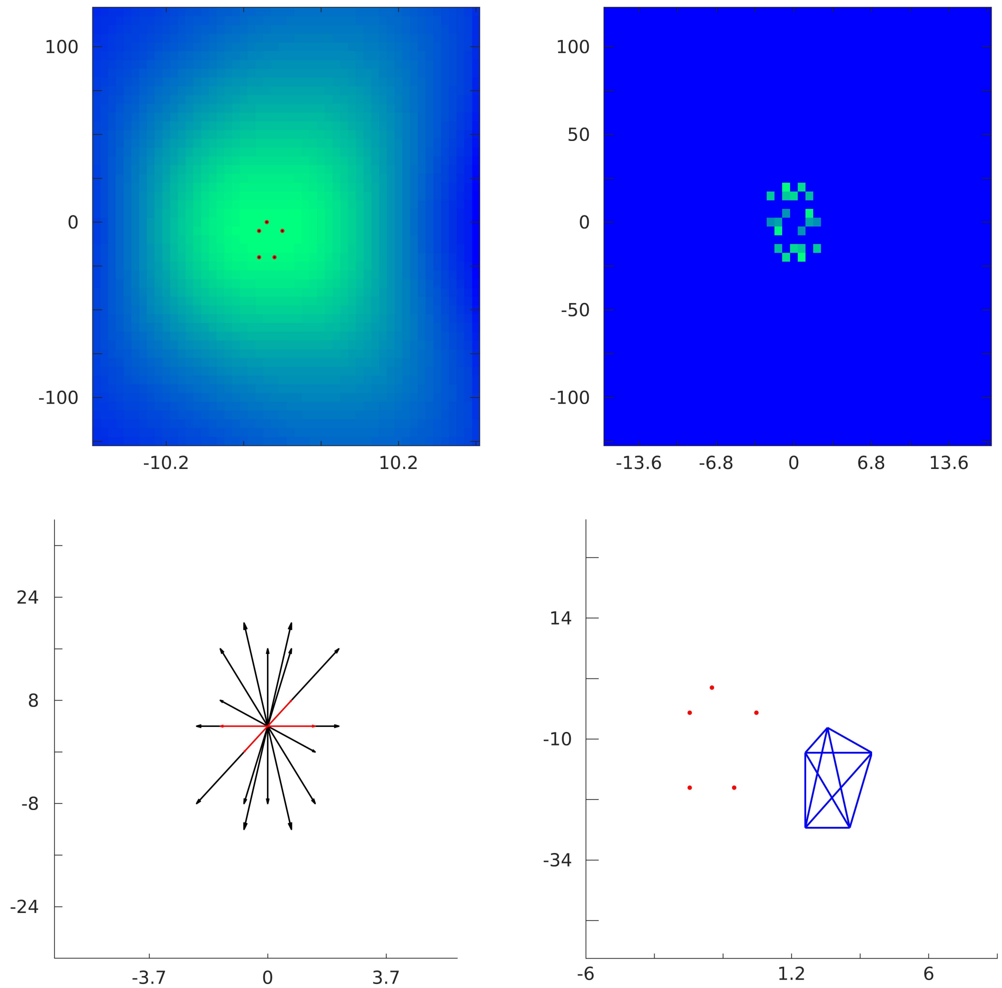
Step 4: Search algorithm for identifying the source locations, relative to $\vec{\boldsymbol{z}}_o$. See reference [6].

CINT with shape recognition



Top: CINT image, KM image, $\mathcal{I}(\vec{z}_o, \vec{y})$. Bottom: Set of edges $\vec{z}_j - \vec{z}_o$ corresponding to peaks of $\mathcal{I}(\vec{z}_o, \vec{y})$. Final reconstruction.

CINT with shape recognition



CINT with shape recognition

- This idea should apply to active array data. Extended targets often appear as a bunch of bright spots (corner diffractors) so the approach can be used for shape recognition.

- But multiple scattering between these can play a role and may lead to spurious peaks!

Summary

We discussed the main steps involved in the imaging process:

1. Modeling:
   (a) Definition of forward map. What we invert for?
   (b) Modeling of uncertainty with random medium.
   (c) Determination of important scales.
2. Asymptotics:
   (a) Separation of scales is important in both homogeneous and random media.
   (b) Limit theorems.
3. Imaging methodologies and resolution analysis.

Summary

• Random media are a serious impediment to imaging.

• If scattering is not too strong i.e., wave has some residual coherence, then CINT methodology is useful.

• CINT can be combined with optimization to get improved imaging. The last imaging method uses a slight modification of CINT that proves useful.

• What if scattering is stronger?

- Process the data to remove clutter effects as much as possible.

- Use different measurement setups.

- Clever ideas based on cross-correlations: ghost imaging, imaging using passive arrays placed near the region of interest.

References: CINT

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References: Related CINT imaging

5. L Borcea, I Kocyigit, *Imaging in random media with convex optimization* SIAM Journal on Imaging Sciences 10.1 (2017): 147-190.
6. L Borcea, I Kocyigit, *Passive array imaging in random media*, arXiv preprint arXiv:1712.04934 (2017).
7. R Alonso, L Borcea, G Papanicolaou, C Tsogka, *Detection and imaging in strongly backscattering randomly layered media* Inverse Problems, 27.2 (2011):025004.

Other references for imaging in random waveguides (with Alonso, Garnier, Tsogka).

Incoherent imaging (based on transport equations satisfied by the Wigner transform) in much stronger scattering regimes: Bal, Pinaud, Ren; Garnier, Papanicolaou, Solna.

Imaging with noise sources: Garnier, Papanicolaou and many references in geophysics.