Second Harmonic Imaging in Random Media

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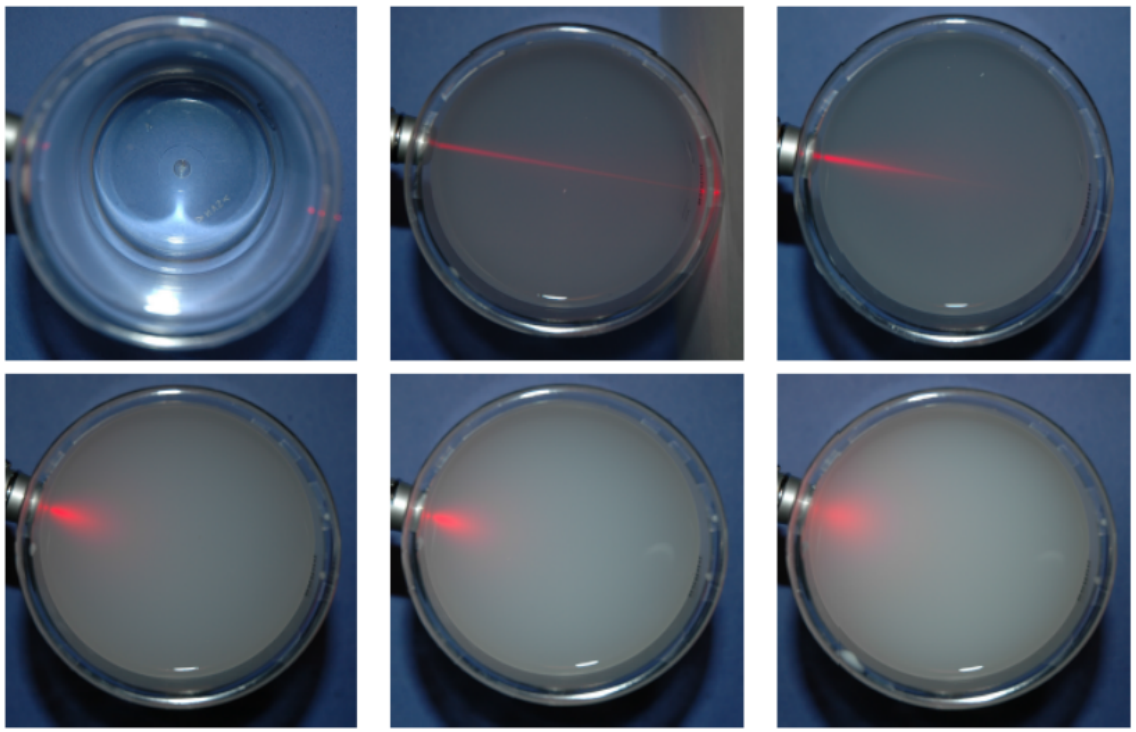
The imaging problem

- Goal: Locate targets in a scattering medium.
- Scattering media has many unknown inhomogeneities, time reversal would not work (human tissues, the atmosphere, construction materials, etc).
- Model the media as realizations of a random process. Wave equations contain random coefficients.
- Each sample is one realization. Predictions has to be statistically stable, characterized by a big SNR$= \frac{\mathbb{E}[\cdot]}{\sqrt{\mathsf{Var}[\cdot]}}$.

Length scales of a scattering medium

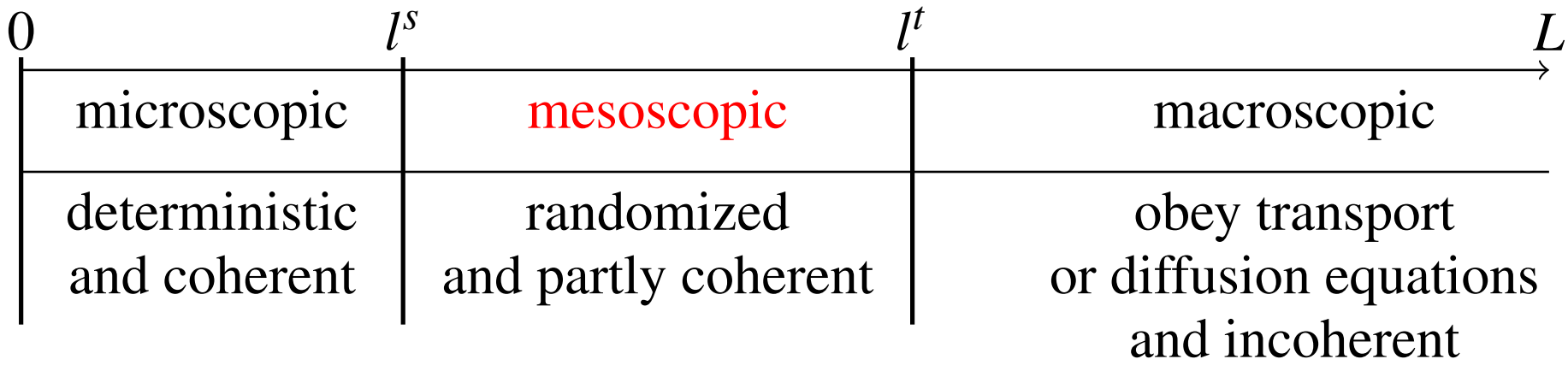
- The scattering mean free path / mean free path l^s: the average distance between two scattering events of a photon.
- The transport mean free path / transport length l^t: the average distance after which the travel direction of a photon is randomized (the wave does not have a well defined propagation direction).
- Propagation distance L: the size of the medium.

Scattering regimes



Scattering regimes for different scattering media (fixed L).
Image credit: John Schotland (and daughter).

Scattering regimes and imaging methods



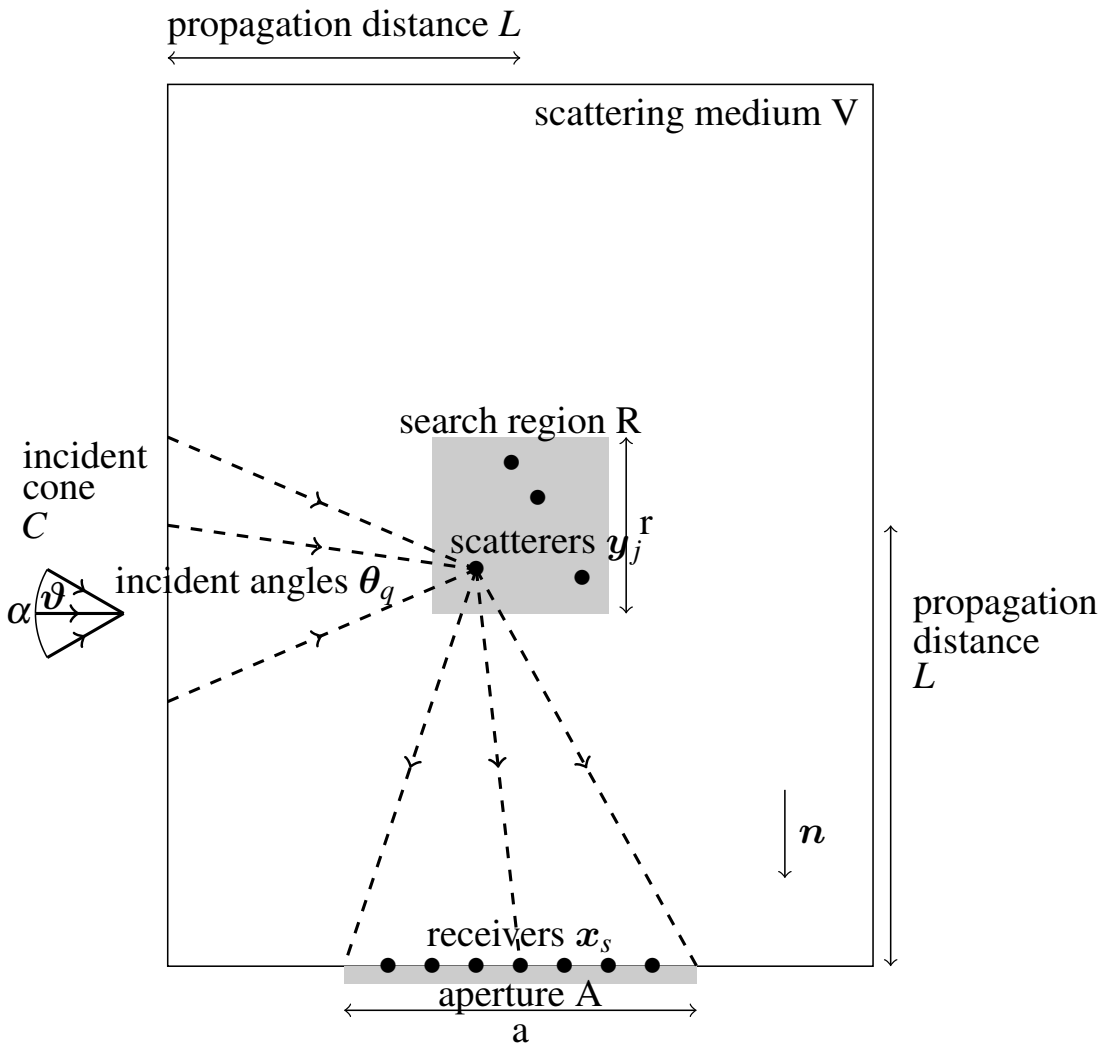
Scattering regimes for L's (fixed medium).

Imaging methods for mesoscopic problems

- Incoherent methods (intensity is measured):
  - confocal microscopy
  - optical projection tomography
  - single-scattering optical tomography
- Coherent methods (both phase and intensity are measured):
  - optical coherence microscopy
  - interferometric synthetic aperture microscopy
  - coherent interferometry (CINT)
- Works on imaging of nonlinear scatterers in the mesoscopic regime
  - experimental: Brown-McKee-diTomaso-Pluen-Boucher-Jain '03, SeedHsieh-Grange-Pu-Psaltis '09
  - theoretical: Habib-Garnier-Millien '14
- Works on CINT Borcea-Papanicolaou-Tsogka '05, '06, '07, Borcea-Garnier-Papanicolaou '11

Set up of imaging

- Scalar wave equation
- 2D
- Linear medium (η)
- Second-order nonlinear scatterers (η_1 and η_2)
- Illumination fields: plane waves ($k = \omega/c$ and θ_q:)
- Second-harmonic generation: fields at ω and 2ω are present.



Wave equations

- Incident field $u_{1,0}^{(i)}(x;\theta_q) = e^{ik\theta_q \cdot x}$:

$$\Delta u_{1,0}^{(i)}(x;\theta) + k^2 u_{1,0}^{(i)}(x;\theta) = 0.$$

- $u_1(x;\theta)$ and $u_2(x;\theta)$: the fields at frequency ω and 2ω.
- Wave equations:

$$\begin{aligned}
&\Delta u_1(x;\theta) + k^2[1 + 4\pi\eta(x)]u_1(x;\theta) \\
&= -4\pi k^2[\eta_1(x)u_1(x;\theta) + 2\eta_2(x)u_2(x;\theta)u_1^*(x;\theta)] \ ,
\end{aligned}$$

$$\begin{aligned}
&\Delta u_2(x;\theta) + (2k)^2[1 + 4\pi\eta(x)]u_2(x;\theta) \\
&= -4\pi(2k)^2[\eta_1(x)u_2(x;\theta) + \eta_{2(x)}u_1^2(x;\theta)].
\end{aligned}$$

Measured fields and data

- Measured fields at the receivers x_s

$$u_1(x_s, \theta_q),\ u_2(x_s, \theta_q), \quad s = 1, \ldots, N_x, \quad q = 1, \ldots, N_\theta.$$

- Data:

$$d_1(x_s, \theta_q) = u_1(x_s; \theta_q) - u_{1,0}^{(i)}(x_s; \theta_q),$$
$$d_2(x_s, \theta_q) = u_2(x_s; \theta_q).$$

Analysis: ideas and results

- The forward model
  - Kinematics: The propagation of the wave is in the scaling regime of geometric optics with randomized phases.
  - Dynamics: The scattering off the targets are approximated by the Born approximation (waves are scattered once by the scatterers).
- The reconstruction
  - The migration imaging functions are superpositions of the backpropagated data.
  - The coherent-interferometry (CINT) imaging functions are superpositions of the backpropagated crosscorrelations of the data.
- The performance
  - CINT is more trustworthy (stable) than migration because CINT mitigates the random phases (at the expense of resolution).
  - The image of the nonlinear η_2 has no spurious peaks.

Forward model of the data

- Born approximations:

$$u_1(x;\theta) \approx u_1^{(i)}(x;\theta) + k^2 \int dy \eta_1(y) G(x,y;\omega) u_1^{(i)}(y;\theta),$$

$$u_2(x;\theta) \approx (2k)^2 \int dy \eta_2(y) G(x,y;2\omega) [u_1^{(i)}(y;\theta)]^2,$$

where G is the Green function in the random medium:

$$\Delta G(x,y;\omega) + k^2[1 + 4\pi\eta(x)]G(x,y;\omega) = -4pi\delta(x),$$

and $u_1^{(i)}$ the randomized incident field:

$$\Delta u_1^{(i)}(x;\theta) + k^2[1 + 4\pi\eta(x)]u_1^{(i)}(x;\theta) = 0.$$

Migration in optically thin media

- The standard Migration imaging function (whose peaks are the estimated locations of the scatterers) is

$$I_j^M(y^R) = \sum_{s=1}^{N_x} \sum_{q=1}^{N_\theta} G_0^\star(y^R, x_s; j\omega) e^{-ijk\theta_q \cdot y^R} d_j(x_s, \theta_q) =: (jk)^2 \int_V dy\, \eta_j(y) K_j^M(y^R, y).$$

$$G_0(x, y; \omega) = \frac{e^{ik|x-y|}}{|x-y|}.$$

- If the medium is optically thin:

$$u_1^{(i)}(x_s; \theta_q) \approx u_{1,0}^{(i)}(x_s; \theta_q) \quad \text{and} \quad G(x_s, y; j\omega) \approx G_0(x_s, y; j\omega),$$

  then the point spread function of the imaging function is

$$K_j^M(y^R) = \sum_{s=1}^{N_x} \sum_{q=1}^{N_\theta} G_0^\star(y^R, x_s; j\omega) e^{-ijk\theta_q \cdot y^R} G_0(x_s, y; j\omega) e^{ijk\theta_q \cdot y}.$$

- It peaks at $y^R \approx y$ due to constructive superposition with resolution $\lambda L/(ja)$.
- It breaks down when the cumulative scattering is not negligible.

Model of the scattering background

- Suppose that the random medium consists of small, mean zero fluctuations which lacks long range correlations and has an integrable autocorrelation function:
$$4\pi\eta(x) = \sigma\mu\left(\frac{x}{l}\right).$$
- μ is a bounded, mean zero and stationary random process, which lacks long range correlation.
- l is the correlation length, the spatial scale over which the autocorrelation decays.
- σ is the amplitude of the fluctuations.
- For convenience (elementary formulas), suppose the autocorrelation takes the form
$$\mathbb{E}[\mu(h)\mu(0)] = e^{-\frac{|h|^2}{2}}.$$

Geometrical optics in random media

- Random geometrical optics wave propagation model

$$\lambda \ll l \ll L,$$

$$\sigma \ll (l/L)^{3/2}, \qquad \sigma \ll \sqrt{\lambda l}/L.$$

the waves propagate along straight lines, the variance of the amplitude of the Green function is negligible.

Geometrical optics in random media

- The Green function is

$$G(x,y;j\omega) = G_0(x,y;j\omega)e^{ijk\nu(x,y)}, \quad \text{for } x \in A \text{ and } y \in R,$$

where

$$\nu(x,y) = \frac{\sigma|x-y|}{2}\int_0^1 dt\,\mu\Big(\frac{(1-t)y}{l} + \frac{tx}{l}\Big).$$

- The randomized incident wave is

$$u_1^{(i)}(x;\theta) = e^{ik\theta\cdot x + ik\gamma(x,\theta)}, \quad \text{for } x \in A \text{ and } \theta \in C,$$

where

$$\gamma(x,\theta) = \frac{\sigma|x-x^{(i)}(\theta)|}{2}\int_0^1 dt\,\mu\Big(\frac{(1-t)x}{l} + \frac{tx^{(i)}(\theta)}{l}\Big).$$

- $\nu(x,y)$ and $\gamma(x,\theta)$ are approximately Gaussian.

Randomization of the waves

- Large wavefront distortions

$$\sigma \gg \frac{\lambda}{\sqrt{lL}}.$$

  consistent when $l \gg \sqrt{\lambda L}$.

- Paraxial regime

$$a \ll (\lambda L^3)^{1/4} \ll L,$$

  and

$$r \ll \frac{\lambda L^2}{a^2} \ll a.$$

Randomization of the waves

- The expectation of the Green function is

$$\begin{aligned}\mathbb{E}\left[G(x,y;j\omega)\right] &= G_0(x,y;j\omega)\mathbb{E}\left[\exp\left[ijk\nu(x,y)\right]\right] \\ &\approx G_0(x,y;j\omega)\exp\left[-\frac{(jk)^2\mathbb{E}[\nu^2(x,y)]}{2}\right] \\ &= G_0(x,y;j\omega)\exp\left[-\frac{|x-y|}{S_j}\right],\end{aligned}$$

  where S_j are the mean scattering free paths:

$$S_j = \frac{8}{\sqrt{2\pi}\sigma^2(jk)^2 l} \ll L.$$

- The expectation of the norm square of the Green function is

$$\mathbb{E}[|G(x,y))|^2] = \mathbb{E}[|G_0(x,y))|^2].$$

- Thus the SNR is exponentially small for $|x-y| \sim L$.

Migration is unstable

- The signal-to-noise ratio (SNR) of the Migration imaging functions are

$$SNR(I_j^M(y^R)) := \frac{\mathbb{E}[|I_j^M(y^R)|^2]}{\mathrm{SD}[I_j^M(y^R)]} \propto \exp\left[-\frac{|x-y|}{S_j}\right].$$

  This is exponentially small near the location of the scatterers.

- It fails to focus or the peaks change unpredictably with the realizations of the random medium.
- This is caused by the large random phases of the Green function (and the incident field).

CINT imaging functions

The CINT imaging function is

$$I_j^{CINT}(y^R)(\overline{x}_{\overline{s}},\overline{\theta}_{\overline{q}},y^R)=\sum_{\widetilde{s}=1}^{\widetilde{N}_x}\sum_{\widetilde{q}=1}^{\widetilde{N}_\theta}\Phi\Big(\frac{\widetilde{x}_{\widetilde{s}}}{X_j}\Big)\Phi\Big(\frac{\widetilde{\theta}_{\widetilde{q}}}{\Theta_j}\Big)$$
$$\times b_j\Big(\overline{x}_{\overline{s}}+\frac{\widetilde{x}_{\widetilde{s}}}{2},\overline{\theta}_{\overline{q}}+\frac{\widetilde{\theta}_{\widetilde{q}}}{2},y^R\Big)b_j^\star\Big(\overline{x}_{\overline{s}}-\frac{\widetilde{x}_{\widetilde{s}}}{2},\overline{\theta}_{\overline{q}}-\frac{\widetilde{\theta}_{\widetilde{q}}}{2},y^R\Big),$$

which uses local correlations to eliminate random phases and extract deterministic phases. Here Φ is a window function,

$$b_j(x_s,\theta_q,y^R)=d_j(x_s,\theta_q)G_0^\star(y^R,x_s;j\omega)e^{-ijk\theta_q\cdot y^R},$$

the center-offsets coordinates for the receivers are

$$\overline{x}_{ss'}=(x_s+x_{s'})/2,\qquad \widetilde{x}_{ss'}=x_s-x_{s'},\qquad s,s'=1,\ldots,N_x,$$

and those for the incident directions are

$$\overline{\theta}_{qq'}=(\theta_q+\theta_{q'})/2,\qquad \widetilde{\theta}_{qq'}=\theta_q-\theta_{q'},\qquad q,q'=1,\ldots,N_\theta.$$

Decorrelation of the waves and choice of window widths

- The second moment of the Green function is

$$\mathbb{E}\left[G(x,y;j\omega)G^{\star}(x',y;j\omega)\right] \approx \frac{1}{L^2}\exp\left[ijk\left(L+\frac{|x_\perp - y_\perp|^2}{2L}\right) - \frac{|x'_\perp - x_\perp|^2}{2X_{d,j}^2}\right],$$

  where

$$X_{d,j} = l\sqrt{\frac{3S_j}{2L}} = O\left(\frac{\lambda\sqrt{l}}{\sigma\sqrt{L}}\right) \ll l.$$

- The second moment of the randomized incident wave is

$$\mathbb{E}\left[u_1^d(x,\theta)\overline{u_1^d(x',\theta')}\right] \approx e^{ik(x\cdot\theta - x'\cdot\theta') - \frac{3|P_{\vartheta}\widetilde{x}|^2 - 3|x-x^{(i)}(\theta)|\widetilde{x}\cdot P_{\vartheta}\widetilde{\theta} + |x-x^{(i)}(\theta)|^2|P_{\vartheta}\widetilde{\theta}|^2}{2X_{d,1}^2}},$$

  where $\widetilde{x} = x - x'$, $\widetilde{\theta} = \theta - \theta'$.

Decorrelation of the waves and choice of window widths

- The second moments of the waves impinging on the scatterer at y are

$$\mathbb{E}\left[u_1^d(y,\theta)\overline{u_1^d(y,\theta')}\right] \approx e^{iky\cdot\widetilde{\theta}-\frac{|P_{\vartheta}\widetilde{\theta}|^2}{2\Theta_d^2}},$$

  where

$$\Theta_d = \frac{X_{d,1}}{|y-y^{(i)}(\vartheta)|} \ll \frac{l}{L}.$$

- The incident waves and the scatterers waves decorrelate:

$$a \gg (\lambda L^2)^{1/3}.$$

- These second moments are exponentially small over receiver location offsets comparable to the decorrelation length $X_{d,j}$, or over incident angle offsets comparable to the decorrelation angle Θ_d.

  Choose $X \approx X_j$ and $\Theta_j \approx \Theta_d/j$.

Stability of CINT

- CINT is statistically stable when

$$a \gg l.$$

- Note that all the above scaling constraints are consistent

$$\lambda \ll \sqrt{\lambda L} \ll l \ll (\lambda L^2)^{1/3} \ll a \ll (\lambda L^3)^{1/4} \ll L,$$

$$\frac{\lambda}{\sqrt{lL}} \ll \sigma \ll \frac{\sqrt{\lambda l}}{L}.$$

The CINT image of the quadratic susceptibility

- The expectation of the CINT image of η_2 is

$$\mathbb{E}\left[I_2^{CINT}(y^R)\right] \approx \frac{(2\pi)^3}{2}\left(4k^2\langle\eta_2\rangle^2\alpha\Theta_e\frac{aX_e}{L^2}\right)^2$$
$$\times\exp\left[-\frac{1}{2}\left(\frac{2kX_e|y_\perp - y_\perp^R|}{L}\right)^2 - \frac{1}{2}\left(2k\Theta_e|P_\vartheta(y-y^R)|\right)^2\right].$$

- The Variance of the CINT image of η_2 is $(l/a)^2\mathbb{E}\left[I_2^{CINT}(y^R)\right]^2$
- The SNR of the imaging function evaluated at the scatterer location is of the order $a/l \gg 1$.
- The resolution is $\frac{L\lambda}{X_e}$, which is worse than the resolution of the migration images in homogeneous media $\frac{L\lambda}{2a}$.
- The peak of the CINT image can be observed in the search region R with linear size r since $\frac{\lambda L/X_e}{\lambda L^2/a^2} \ll 1$.

The CINT image of η_1

- The CINT image of η_1 has big spurious peaks near the receiver array.
- Recall that

$$d_1(x_s, \theta_q) = u_1(x_s; \theta_q) - u_{1,0}^{(i)}(x_s; \theta_q),$$
$$d_2(x_s, \theta_q) = u_2(x_s; \theta_q).$$

- It's caused by the uncompensated distorted incident wave in the data.
- The image of η_2 does not suffer from this problem.

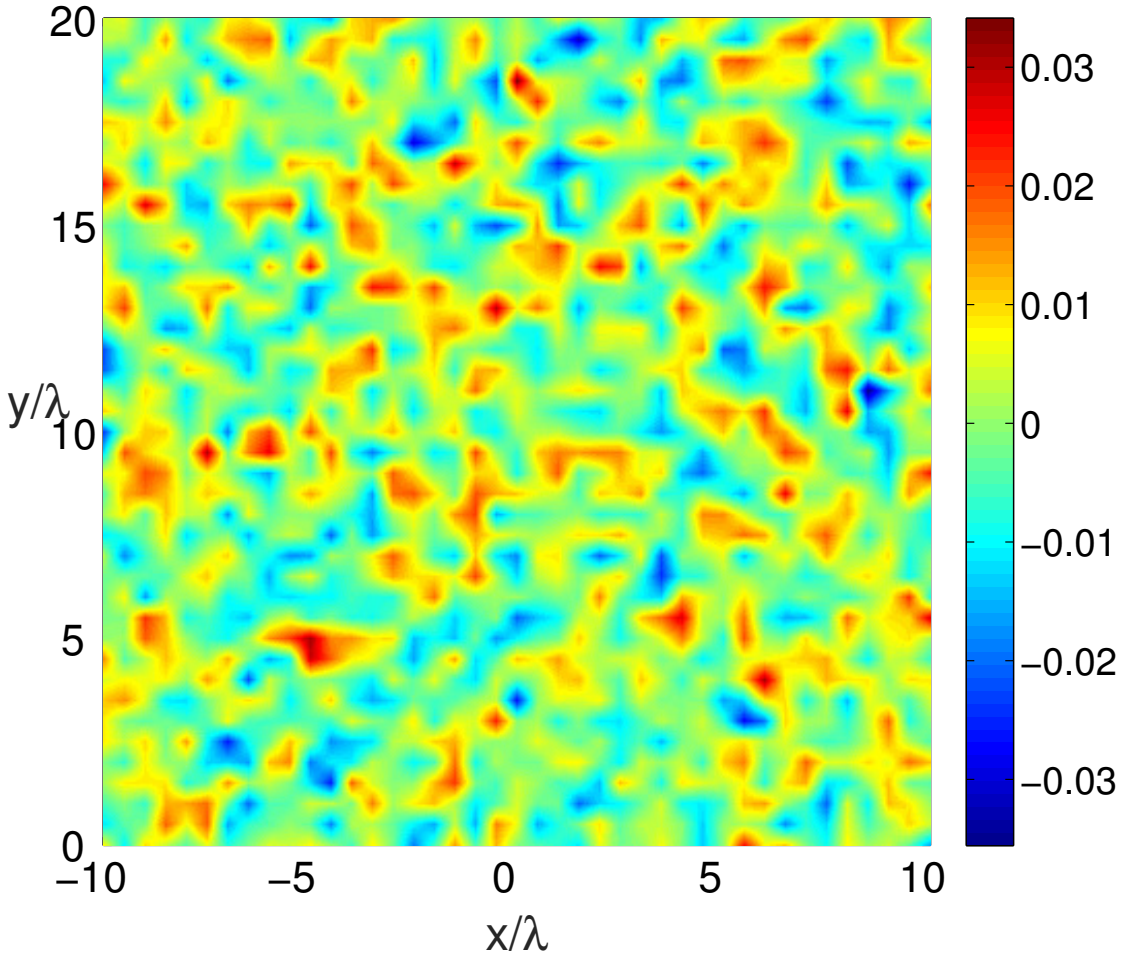
Numerics: another scaling regime

- The scalings for the numerics is different from geometric optics in random media:

$$L = 20\lambda, \quad l = 0.3\lambda, \quad a = 20\lambda, \quad \sigma = 0.01 * 4\pi,$$
$$X_1/2 = X_2 = 7\lambda, \quad \Theta = \frac{\pi}{5}.$$

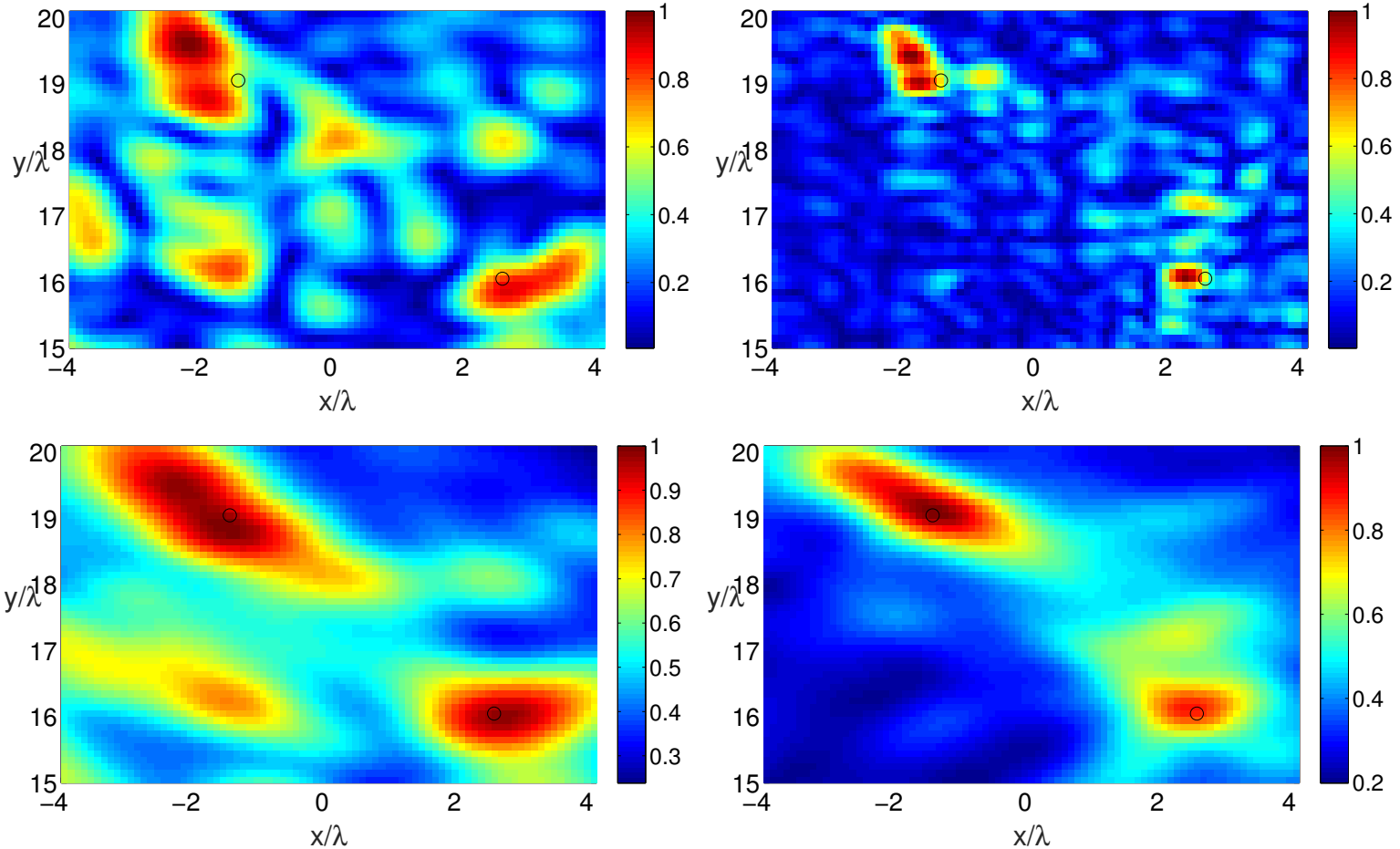
- All the above results hold qualitatively.

One realization of the random medium



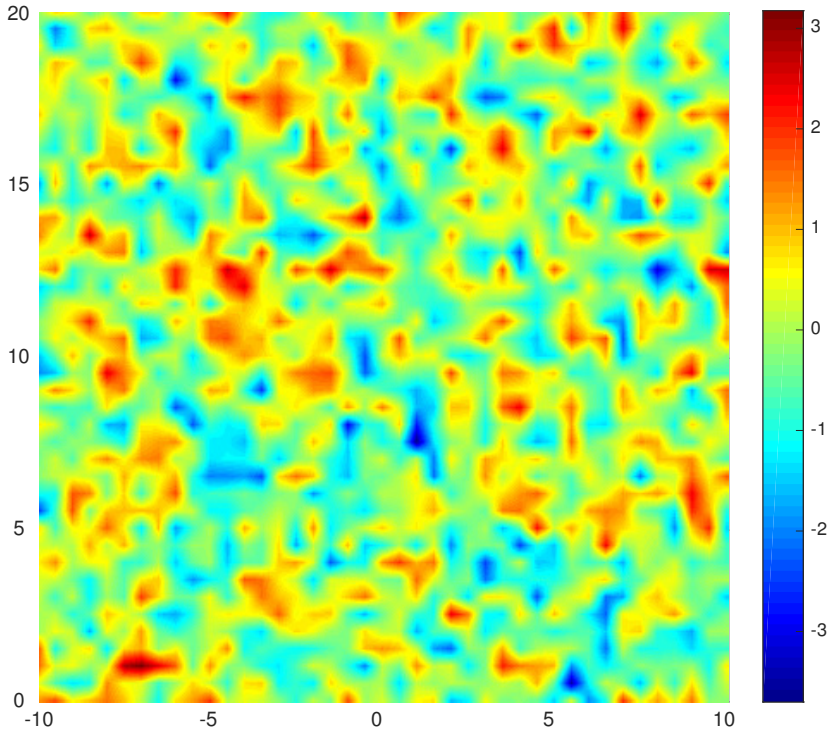
First realization of η.

Images



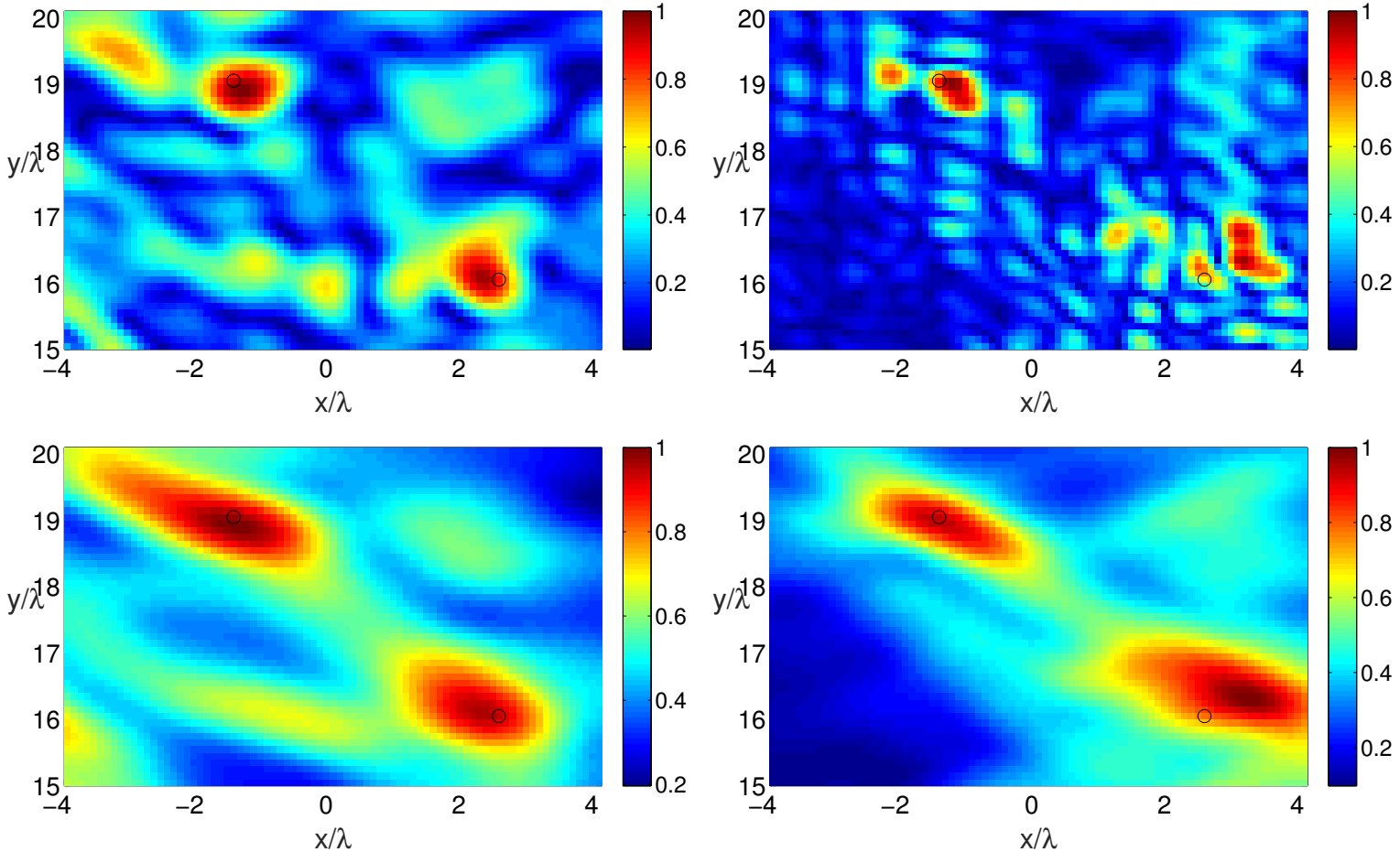
The Migration (top) and CINT (bottom) images of η_1 (left) and η_2 (right).

Another realization of the random medium



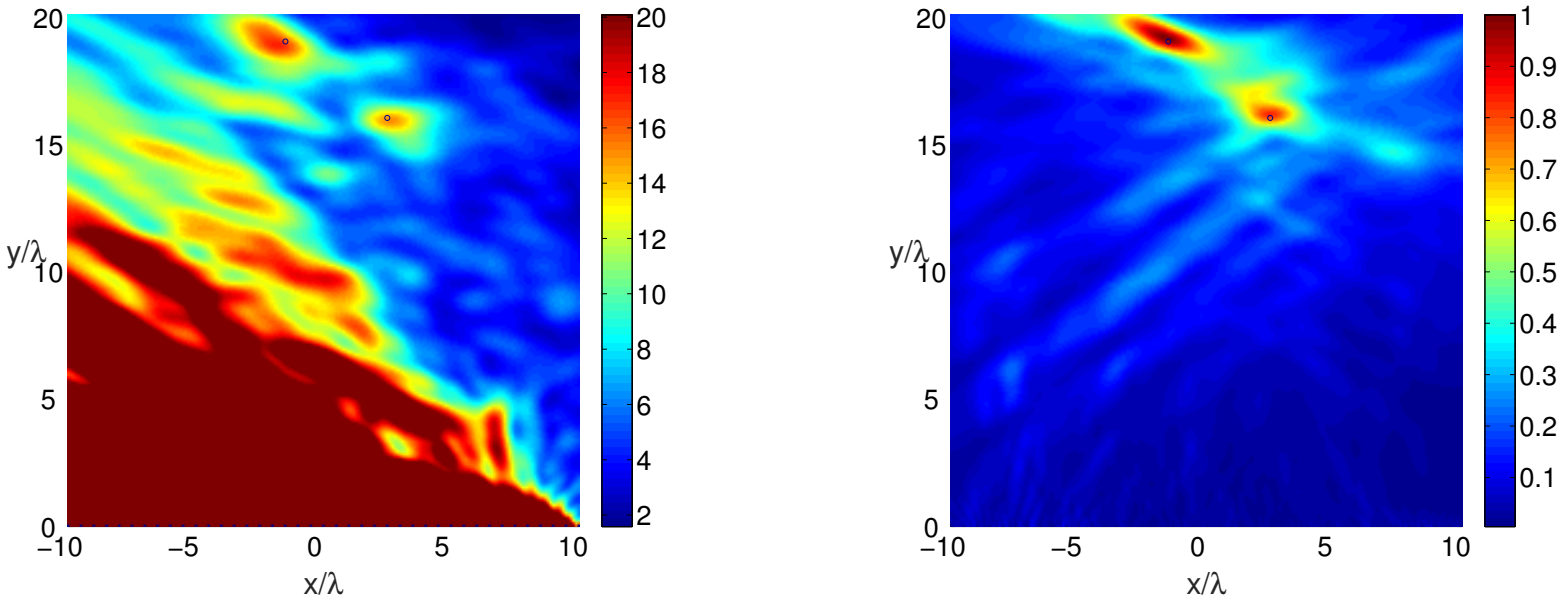
Another realization of η.

Images



The Migration (top) and CINT (bottom) images of η_1 (left) and η_2 (right).

Numerics



The CINT images of η_1 (left, rescaled) and η_2 (right) in the entire domain.

Thank you!